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ON QUASI MAXIMAL IDEALS OF COMMUTATIVE RINGS

Murat Alan, Mesut Kılıç, Suat Koç*, Ünsal Tekir**

Dedicated to the memory of Professor Kar Ping Shum

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Abstract

Let R be a commutative ring with $1 \neq 0$. A proper ideal I of R is said to be a quasi maximal ideal if for every $a \in R - I$, either $I + Ra = R$ or $I + Ra$ is a maximal ideal of R. This class of ideals lies between 2-absorbing ideals and maximal ideals which is different from prime ideals. In addition to give fundamental properties of quasi maximal ideals, we characterize principal ideal UN-rings with $\sqrt{0}^2 = (0)$, direct product of two fields, and Noetherian zero dimensional modules in terms of quasi maximal ideals.

Key words: maximal ideal, quasi-maximal ideal, prime ideal, primary ideal, 2-absorbing ideal

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1. Introduction. Throughout this article, we focus only on commutative rings with nonzero identity and nonzero unital modules. Let R always denote such a ring and M denote such an R-module. $\mathrm{Max}(R)$ and $\mathrm{Spec}(R)$ denote the set of all maximal ideals and prime ideals, respectively. The notion of prime ideals and its generalizations have a distinguished place in commutative algebra since they are used in the characterization of many important classes of rings and also they have some applications to other areas such as general topology, algebraic geometry, graph theory, cryptology, etc. (see, [1–3]). One of the well-known generalizations of prime ideal is primary ideal: a proper ideal I of R is said to be a *primary ideal* if whenever $ab \in I$ for some $a, b \in R$, then $a \in I$ or $b^n \in I$ for some $n \in \mathbb{N}$ [4]. In this case, $\sqrt{I} = \{x \in R \colon \exists\ n \in \mathbb{N},\ x^n \in I\}$ is a prime ideal of R.

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Lasker–Noether theorem says that in a Noetherian ring (i.e. a ring R whose every ideal is finitely generated) every proper ideal is an intersection of finitely many primary ideals. In fact, this is the natural extension of fundamental theorem of arithmetic. In 2007, Badawi [2] defined the concept of 2-absorbing ideal as follows: a proper ideal I of R is said to be a 2-*absorbing ideal* if whenever $abc \in I$ for some $a, b, c \in R$, then $ab \in I$ or $ac \in I$ or $bc \in I$. It is clear that every prime ideal of a ring R is also 2-absorbing. However, the converse is not true in general. For instance, if k is a field, then (XY) is a 2-absorbing ideal of $k[X, Y]$ which is not prime. Badawi in his paper [2] gave many properties of 2-absorbing ideals and characterized Dedekind domains in terms of 2-absorbing ideals. Afterwards, the concept of 2-absorbing ideals has attracted the attention of many authors and is generalized in many papers (see [5–7]). Our aim in this paper is to introduce the concept of quasi maximal ideals in commutative rings which is an intermediate class of ideals between maximal ideals and 2-absorbing ideals, and to use them in the characterization of some important classes of rings/modules. A proper ideal I of R is said to be a *quasi maximal ideal* if $I + Ra = R$ or $I + Ra \in \operatorname{Max}(R)$ for every $a \in R - I$. It is easy to see that every maximal ideal is also a quasi maximal ideal. But the converse is not true in general. For instance, if k is a field, then (X^2) is a quasi maximal ideal in the ring $k[[X]]$ of formal power series over k which is not maximal. The set of all quasi maximal ideals of a ring R will be denoted by $q\operatorname{Max}(R)$. Thus, we obtain $\operatorname{Max}(R) \subseteq q\operatorname{Max}(R)$. Among other results in this paper, we show that maximal ideal $\Rightarrow$ quasi maximal ideal $\Rightarrow$ 2-absorbing ideal and the reverse implications are not true in general (see Example 1, Example 2, Theorem 1). Also, in Proposition 1, we show that every quasi maximal ideal in a local ring R is either prime or primary (in general ring, this may not be true, see Example 2). It is known that every maximal ideal I of a ring R is also prime, but the converse is not true in general. However, in Theorem 2, we prove that the converse is true provided that I is quasi maximal ideal, that is, $\operatorname{Spec}(R) \cap q\operatorname{Max}(R) = \operatorname{Max}(R)$. By using this result, we also characterize fields and π-regular rings (sometimes called zero dimensional rings) in terms of quasi maximal ideals (see Corollary 1). We also investigate the stability of quasi maximal ideals in homomorphism of rings, in factor rings, in the ring of formal power series and in the idealization of a module (see Proposition 2, Corollary 2, Theorem 3, Theorem 6). Furthermore, we characterize a subclass of UN-rings (a ring R is said to be a *UN-ring* if every nonunit element of R is a product of unit and nilpotent [8]) in terms of quasi maximal ideals (see, Theorem 4 and Corollary 3). Finally, in Theorem 5, we show that in a finitely generated multiplication R-module M, if $(0 : M) = \{r \in R \colon rM = 0\}$ is a quasi maximal ideal of R, then M is Noetherian module and every prime submodule is maximal. In this case, either (i) M is simple or (ii) M has three submodules $\{(0), K, M\}$, where all of them are cyclic and (0) is not prime, or (iii) $M = K \oplus N$ for some maximal submodules K, N of M (i.e. M is semisimple).

For general background and any undefined terminology the reader may consult [4,9,10].

2. Main results.

Definition 1. A proper ideal I of R is said to be a quasi maximal ideal if for every $a \in R - I$, either $I + Ra = R$ or $I + Ra \in \operatorname{Max}(R)$.

Example 1. Let p be a prime number and $R = \mathbb{Z}_{p^2}$. Consider zero ideal $I = (\overline{0})$ of R. Then I is not a maximal ideal of R. Let $\overline{a} \in R - I$. If $\overline{a}$ is a unit of R, then $I + R\overline{a} = R$. So assume that $\overline{a}$ is nonzero nonunit of R. Then we can write $a = mp$, where m is an integer and $\gcd(m, p) = 1$. Then $R\overline{a} = R\overline{mp} = R\overline{p}$ and $I + R\overline{a} = R\overline{p} \in \operatorname{Max}(R)$. Thus I is a quasi maximal ideal of R.

Example 2. Let R be a PID which is not a field and I a nonzero proper ideal of R. Then $I = Rp_1p_2 \cdots p_k$ for some prime elements $p_1, p_2, \ldots, p_k$ of R. Assume that I is a quasi maximal ideal of R. Now, we will show that $k \leq 2$. Assume that $k > 3$. Since $p_1p_2 \in R - I$ and $I + Rp_1p_2 = Rp_1p_2 \notin \operatorname{Max}(R)$, I cannot be a quasi maximal ideal of R. If I is a quasi maximal ideal of R, then $k \leq 2$. If $k = 1$, that is $I = Rp_1$, then I is a maximal ideal. Thus I is a quasi maximal ideal of R. Now, assume that $k = 2$, that is, $I = Rp_1p_2$. Let $a \in R - I$ be a nonunit element of R. Then either $\gcd(a, p_1p_2) = p_1$, p_2 or 1. Since R is a Bezout domain, we have $I + Ra = Rp_1$, Rp_2 or R. Thus $I + Ra = R$ or $I + Ra \in \operatorname{Max}(R)$. Then $q\operatorname{Max}(R) = \{Rp, Rpq \colon p, q \text{ are prime elements of } R\}$.

Theorem 1. *Every quasi maximal ideal is a* 2*-absorbing ideal.*

Proof. Let I be a quasi maximal ideal of R and $abc \in I$ for some $a, b, c \in R$. Assume that $bc \notin I$. Since I is a quasi maximal ideal, $I + Rbc \in \operatorname{Max}(R)$ or $I + Rbc = R$. If $I + Rbc = R$, then $i + rbc = 1$ for some $i \in I$ and $r \in R$. Then we obtain $a = ai + r(abc) \in I$ and so $ac, ab \in I$. Now, assume that $I + Rbc \in \operatorname{Max}(R)$. Since $I + Rbc \subseteq I + Rb$, we have either $I + Rbc = I + Rb$ or $I + Rb = R$.

Case 1. Let $I + Rbc = I + Rb$. Then we can write $b = j + sbc$ for some $j \in I$ and $s \in R$. Thus, we have $ab = aj + s(abc) \in I$.

Case 2. Let $I + Rb = R$. Then we can write $1 = x + yb$ for some $x \in I$ and $y \in R$. This gives $ac = xac + y(abc) \in I$. Thus, I is a 2-absorbing ideal of R. □

The converse of previous theorem is not true in general. For instance, consider the ring $\mathbb{Z}$ of integers and take $I = (0)$. Then I is a 2-absorbing ideal which is not a quasi maximal ideal of $\mathbb{Z}$. We also note that prime ideals and quasi maximal ideals are different concepts (see Example 2).

Remark 1. Let I be a quasi maximal ideal of R and $I \subsetneq M$, where M is an ideal of R. Then $M = R$ or M is a maximal ideal of R.

If I is a primary ideal of a ring R, then we know that $\sqrt{I} = P$ is a prime ideal and in this case I is called *P-primary.* Also, we say that R is a quasi local ring if $|\operatorname{Max}(R)| = 1$ [4].

Proposition 1. *Let R be a quasi local ring with unique maximal ideal M. If I is a quasi maximal ideal, then it is prime or M-primary.*

Proof. Let I be a quasi maximal ideal of R but not prime. Then there exists $a, b \notin I$ such that $ab \in I$. Since I is quasi maximal, we have four cases about $I + Ra$ and $I + Rb$.

Case 1. Let $I+Ra = R = I+Rb$. In this case we have $R = (I+Ra)(I+Rb) \subseteq I$ which implies that $I = R$, a contradiction.

Case 2. Let $I+Ra \in \operatorname{Max}(R)$ and $I+Rb = R$. Then we can write $1 = i+xb$ for some $i \in I$ and $x \in R$. Then we have $a = ai+x(ab) \in I$, again a contradiction.

Case 3. Let $I + Rb \in \operatorname{Max}(R)$ and $I + Ra = R$. Similar argument in Case 2 gives a contradiction.

Case 4. Let $I + Ra, I + Rb \in \operatorname{Max}(R)$. Since R is quasi local, we have $I + Ra = M = I + Rb$. This implies that $M^2 = (I + Ra)(I + Rb) \subseteq I \subseteq M$. Thus we have $\sqrt{I} = M$ and so I is M-primary. □

Theorem 2. *Let I be a proper ideal of R. Then I is a maximal ideal if and only if I is prime and quasi maximal ideal of R, that is,* $\operatorname{Max}(R) = \operatorname{Spec}(R) \cap q\operatorname{Max}(R)$.

Proof. (⇒): obvious.

(⇐): Assume that I is prime and quasi maximal ideal of R. Let $a \in R - I$. Since I is prime, we have $a^2 \notin I$. By the fact that I is quasi maximal ideal, we have $I + Ra^2 = R$ or $I + Ra^2 \in \operatorname{Max}(R)$. If $I + Ra^2 = R$, then $I + Ra = R$. Assume that $I + Ra^2 \in \operatorname{Max}(R)$. Since $I + Ra^2 \subseteq I + Ra$, we have $I + Ra = R$ or $I + Ra^2 = I + Ra$. We may assume that $I + Ra^2 = I + Ra$. This implies $a = i + ra^2$ for some $i \in I$ and $r \in R$. Then $a - ra^2 = a(1 - ra) = i \in I$. Since I is prime and $a \notin I$, we conclude that $1 - ra \in I$ which implies that $I + Ra = R$. Hence I is a maximal ideal of R. The rest is clear. □

Recall from [11] that a ring R is said to be a *π-regular* if for each $a \in R$ there exists $n \in \mathbb{N}$ such that $Ra^n = Ra^{n+1}$, that is, $a^n = a^{n+1}x$ for some $x \in R$. It is known that a ring R is a π-regular ring if and only if every prime ideal is maximal, or equivalently, $\dim(R) = 0$ [12]. As a consequence of Theorem 2 we have the following explicit results.

Corollary 1.

(i) *Let I be a proper ideal of R. Then R/I is a field if and only if I is prime and quasi maximal.*

(ii) *R is a field if and only if R is an integral domain and* (0) *is quasi maximal.*

(iii) *R is a π-regular ring if and only if its every prime ideal is quasi maximal.*

Proposition 2. *Let $f\colon R \to S$ be a surjective homomorphism of rings. The following statements are satisfied:*

(i) *If I is a quasi maximal ideal of R containing* $\operatorname{Ker}(f)$, *then $f(I)$ is a quasi maximal ideal of S.*

(ii) *If* I^* *is a quasi maximal ideal of* S*, then* $f^{-1}(I^*)$ *is a quasi maximal ideal of* R*.*

Proof. (i) Let $b \in S - f(I)$. Since f is surjective, there exists $r \in R$ such that $f(r) = b$. Then we have $r \notin I$. Since I is quasi maximal ideal, we have $I + Rr = R$ or $I + Rr$ is a maximal ideal of R. Thus we have $f(I + Rr) = f(I) + Sb = S$ or $f(I) + Sb$ is a maximal ideal of S.

(ii) Let $a \in R - f^{-1}(I^*)$. Then $f(a) \in S - I^*$. Since I is a quasi maximal ideal, we have $I^* + Sf(a) = S$ or $I^* + Sf(a)$ is a maximal ideal of S. Assume that $I^* + Sf(a) = S$. Let $J = f^{-1}(I^*) + Ra$. Then $f(J) = I^* + Sf(a) = S$ and so $J = f^{-1}(f(J)) = R$. Now assume that $I^* + Sf(a)$ is a maximal ideal of S. Put $J = f^{-1}(I^*) + Ra$ and choose an ideal K of R containing J. Then $f(J) = I^* + Sf(a) \subseteq f(K)$ and so $f(K) = f(J)$ or $f(K) = S$ and this implies that $J = K$ or $K = R$. Hence J is a maximal ideal of R. □

As an immediate consequences of previous proposition and Remark 1, we have the following explicit results.

Corollary 2. *Let* J *be a proper ideal of* R*. The following statements are satisfied:*

(i) *Quasi maximal ideals of quotient ring* R/J *has the form* I/J*, where* I *is a quasi maximal ideal of* R *containing* J*.*

(ii) *If* J *is a quasi maximal ideal of* R*, then every proper ideal of* R/J *is a quasi maximal ideal.*

For any ring R, $R[[X]]$ denotes the ring of formal power series over R. It is known that there is one to one correspondence between maximal ideals of $R[[X]]$ containing (X) and the set of all maximal ideals of R. The following theorem proves a similar result for quasi maximal ideals.

Theorem 3. *Let* I *be a proper ideal of* R*. The following statements are satisfied:*

(i) *If* I *is a quasi maximal ideal of* R*, then* $(I, X) = \left\{\sum_{i=0}^{\infty} a_i X^i \in R[[X]]\colon a_0 \in I\right\}$ *is a quasi maximal ideal of* $R[[X]]$*.*

(ii) *If* I^* *is a quasi maximal ideal of* $R[[X]]$*, then* $\left\{a_0 \in R\colon \sum_{i=0}^{\infty} a_i X^i \in I^*\right\}$ *is a quasi maximal ideal of* R*.*

Proof. (i) Consider the surjective homomorphism $\pi\colon R[[X]] \to R$ defined by $\pi\left(\sum_{i=0}^{\infty} a_i X^i\right) = a_0$. Assume that I is a quasi maximal ideal of R. Then, by Proposition 2, (I, X) is a quasi maximal ideal of $R[[X]]$.

(ii) Let I^* be a quasi maximal ideal of $R[[X]]$ and $I := \left\{a_0 \in R\colon \sum_{i=0}^{\infty} a_i X^i \in I^*\right\}$. Now we will show that I is a quasi maximal ideal of R. Let $a \in R - I$. Then the constant function a is not in I^*. Since I^* is a quasi maximal ideal of $R[[X]]$, we have either $I^* + R[[X]]a = R[[X]]$ or $I^* + R[[X]]a = M^* \in \mathrm{Max}(R[[X]])$. Assume that $I^*+R[[X]]a=R[[X]]$. Then we have $\sum_{i=0}^{\infty} s_iX^i+\left(\sum_{i=0}^{\infty} b_iX^i\right)a=1$ for some $\sum_{i=0}^{\infty} s_iX^i\in I^*$ and $\sum_{i=0}^{\infty} b_iX^i\in R[[X]]$. This implies that $1-\left(\sum_{i=0}^{\infty} b_iX^i\right)a=\sum_{i=0}^{\infty} s_iX^i\in I^*$ and so $1-b_0a\in I$ and this yields that $I+Ra=R$. Now assume that $I^*+R[[X]]a=M^*\in \mathrm{Max}(R[[X]])$. Put $M=\left\{a_0\in R\colon \sum_{i=0}^{\infty} a_iX^i\in M^*\right\}$. Let $r\in M$. Then there exists $f(X)=\sum_{i=0}^{\infty} a_iX^i\in M^*$ such that $f(0)=r$. This implies that $\sum_{i=0}^{\infty} a_iX^i=\sum_{i=0}^{\infty} s_i^*X^i+\left(\sum_{i=0}^{\infty} b_i^*X^i\right)a$ and so $\sum_{i=0}^{\infty} a_iX^i-\left(\sum_{i=0}^{\infty} b_i^*X^i\right)a\in I^*$ and this yields that $r-b_0^*a\in I$ and so that $r\in I+Ra$. Thus we conclude that $M\subseteq I+Ra$. Note that M is maximal ideal of R and so $I+Ra=R$ or $I+Ra=M$ which completes the proof. □

Theorem 4. *Let R be a ring. Then every proper ideal is a quasi maximal ideal if and only if one of the following conditions holds*:

(i) *R is a principal ideal UN-ring with $\sqrt{0}^2=0$.*

(ii) *R is a direct product of two fields.*

Proof. ($\Rightarrow$): Let R be a ring in which every proper ideal is a quasi maximal ideal. Then every prime ideal is a quasi maximal, so by Corollary 1, R is zero dimensional, i.e., every prime is maximal ideal. Now we will show that R has at most two maximal ideals. Assume that M_1, M_2, M_3 are different maximal ideals of R. Put $I=M_1\cap M_2\cap M_3$. Then by assumption, I is a quasi maximal ideal of R. Take $a\in (M_1\cap M_2)-M_3$. Then $a\notin I$. Since I is a quasi maximal ideal, we conclude that $I+Ra=R$ or $I+Ra\in \mathrm{Max}(R)$. Since $I+Ra\subseteq M_1\cap M_2$, we have neither $I+Ra=R$ nor $I+Ra\in \mathrm{Max}(R)$. Thus R has at most two maximal ideals. Assume that R has two maximal ideals, say M_1, M_2. Now we will show that $M_1\cap M_2=(0)$. Take $0\neq a\in M_1\cap M_2$. Put $K=(0)$. Since K is a quasi maximal ideal and $a\notin K$, we have either $K+Ra=R$ or $K+Ra\in \mathrm{Max}(R)$. Since $K+Ra=Ra\subseteq M_1\cap M_2$, we have a contradiction. So that $M_1\cap M_2=0$. By Chinese remainder theorem, we conclude that $R=R/M_1\cap M_2\cong R/M_1\times R/M_2$. Assume that R has one maximal ideal, say P. Since R is zero dimensional, we have that $\sqrt{0}=P$ is a maximal ideal and so unique prime ideal. By Proposition 2 (3) in [8], R is a UN ring. Now we will show that $P^2=0$. Assume that $P^2\neq 0$. Then there exists $a,b\in P$ such that $ab\neq 0$. Since zero ideal is quasi maximal and ab is not unit, we have that $Rab\subseteq P$ implies $Rab=P\subseteq P^2$ and so $P=Rab=R(ab)^2$ which implies that $P=eR$, where e is an idempotent element of R. Note that $e\in Jac(R)$, so we have $e=0$ and thus $P=P^2=eR=0$, a contradiction. Thus we have $P^2=\sqrt{0}^2=0$. Now we will show that P is principal ideal. Let $0\neq a$ be a nonunit element of R. Since zero ideal is quasi maximal, we have Ra is a maximal ideal and so $P=\sqrt{0}=Ra$. By Proposition 4 in [13], R is principal ideal ring and every element of R is of the form ua^k, where $k\in\{0,1,2\}$ and u is unit of R. Thus all ideals of R are (0), Ra, R.

($\Leftarrow$): Assume that R is a principal ideal UN-ring with $\sqrt{0}^2=0$. By Proposition 4 in [13], $L(R)=\{(0),Ra,R\}$, where $L(R)$ is the set of all ideals of R and

$Ra = \sqrt{0}$. Thus, clearly every ideal is a quasi maximal. If R is a direct product of two fields, then it is easily seen that every proper ideal is quasi maximal ideal. □

Recall from [4] that a ring R is an *indecomposable* if it cannot be written as $R \cong A \times B$ for some nonzero rings A and B. Otherwise, R is called a decomposable ring.

Corollary 3.

(i) *Let R be an indecomposable ring. Then every proper ideal is a quasi maximal ideal if and only if R is a principal ideal UN-ring with $\sqrt{0}^2 = 0$.*

(ii) *Let R be an indecomposable reduced ring (e.g. R is an integral domain). Then every proper ideal is a quasi maximal ideal if and only if R is a field.*

An R-module M is said to be a *Noetherian module* if every submodule of M is finitely generated [4]. M is called a *simple module* if the only submodules of M are (0) and M [4]. Also, M is said to be a *semisimple module* if M is a direct sum of its simple submodules [9]. Recall from [4] that M is said to be a *cyclic module* if $M = Rm$ for some $m \in M$. Also, M is said to be a *multiplication module* if its each submodule N of M has the form IM for some ideal I of R [14]. It is clear that all cyclic modules are multiplication.

Theorem 5. *Let M be a finitely generated multiplication R-module and $(0 : M)$ be a quasi maximal ideal of R. Then M is Noetherian module and every prime submodule is maximal, that is, M is zero dimensional. In this case, one of the following statements are satisfied:*

(i) *M is simple module.*

(ii) *M is cyclic module whose all submodules are cyclic with $L(M) = \{(0), K, M\}$ and (0) is not prime.*

(iii) *$M = K \oplus N$ for some maximal submodules K, N of M. In this case, M is semisimple.*

Proof. Let M be a finitely generated R-module such that $(0 : M)$ is a quasi maximal ideal of R. To prove that M is Noetherian, it is enough to show that every prime submodule is finitely generated. Let P be a prime submodule. If $(0 : M) = (P : M)$, then $P = 0$. Assume that $(P : M)$ properly contains $(0 : M)$. Then there exists $a \in (P : M) - (0 : M)$. Since $(0 : M)$ is a quasi maximal ideal, then $(0 : M) + Ra$ is a maximal ideal and so that $(0 : M) + Ra \subseteq (P : M)$ implies $(0 : M) + Ra = (P : M)$. This yields that $aM = P$. Since M is finitely generated, we have $M = Rm_1 + Rm_2 + \cdots + Rm_n$ for some $m_1, m_2, \ldots, m_n \in M$ and thus $aM = Ram_1 + \cdots + Ram_n = P$. Hence P is finitely generated, as needed. Thus M is a Noetherian module. Let P be a nonzero prime submodule of M. Then

$(0:M) \subsetneq (P:M)$, by Remark 1, $(P:M)$ is a maximal ideal of R. Since M is multiplication module, it can be easily seen that P is a maximal submodule of M. Let $P=(0)$ be a prime submodule of M. Then $(P:M)=(0:M)$ is prime and quasi maximal, by Theorem 2, $(P:M)$ is maximal. Thus $P=(0)$ is maximal submodule. In this case, M is simple module. Thus M is zero dimensional module. If $P=(0)$ is a prime submodule, then by above argument, M is simple module, that is, (i) holds. Now, assume that zero submodule is not prime. Let K be a nonzero proper submodule of M. Then $(0:M) \subsetneq (K:M)$, by Remark 1, $(K:M)$ is a maximal ideal which implies that K is a maximal submodule of M. Assume that K is unique nonzero proper submodule of M, that is, $L(M)=\{(0),K,M\}$, where $L(M)$ is the set of all submodules of M. Now, we will show that K, M are cyclic. Let $m \in M-K$. Put $N=Rm$. Since $L(M)=\{(0),K,M\}$ and $N \neq K$, we have $N=Rm=M$ which implies that M is cyclic. By taking $0 \neq m \in K$, one can similarly show that $K=Rm$ is cyclic. Thus (ii) holds. Now assume that there exists a (nonzero) maximal submodule N of M such that $N \neq K$. Now we will show that $N \cap K=(0)$. Assume that $N \cap K \neq (0)$. Then $(0:M) \subsetneq (N \cap K:M)$ which implies that $N \cap K$ is a maximal submodule and this gives $N \cap K=N=K$, a contradiction. Thus $N \cap K=(0)$. Since $N+K=M$, we have $M=K \oplus N$ which is semisimple module. □

Proposition 3. *Let I be a quasi maximal ideal of R. Then one of the following statements holds:*

(i) *I is a maximal ideal.*

(ii) *I is an intersection of two maximal ideals.*

(iii) *I is a primary ideal of R such that $\sqrt{I}=M \in \mathrm{Max}(R)$ and $M^2 \subseteq I \subseteq M$.*

Proof. Assume that I is a quasi maximal ideal of R. Then by Theorem 1, I is a 2-absorbing ideal of R. By [2], there exist at most two minimal prime ideals over I and so $\sqrt{I}=P$ or $\sqrt{I}=P \cap P^*$, where P, P^* are prime ideal minimal over I. Suppose that $\sqrt{I}=I$. By Remark 1, $I=M \in \mathrm{Max}(R)$ or $I=M \cap M^*$, where M, M^* are maximal ideals of R. Suppose that $I \neq \sqrt{I}$. By $I \subsetneq \sqrt{I}$, we have $\sqrt{I}=M \in \mathrm{Max}(R)$. Since $I \neq \sqrt{I}$, I is not prime and so there exists $a,b \notin I$ such that $ab \in I$. Since I is quasi maximal ideal, $I+Ra=M=I+Rb$ and this yields that $M^2=I^2+IRa+IRb+Rab \subseteq I \subseteq M$. □

Remark 2. (i) Let $I=M \cap M^*$ for some maximal ideals M and M^* of R. Assume that $a \notin M \cup M^*$. Then $M+Ra=R=M^*+Ra$ and so $R=M \cap M^*+Ra=I+Ra$. Now assume that $a \in M-M^*$. Then $R=M^*+Ra$ and so $M=M^*M+RaM=M \cap M^*+RaM \subseteq I+Ra$ so that either $I+Ra=M$ or $I+Ra=R$. Thus I is a quasi maximal ideal of R.

(ii) In Proposition 3, condition (iii) need not imply that I is a quasi maximal ideal. Consider $R=k[X,Y]$, where k is a field and X, Y are indeterminates over

k. Put $I = (X^2, XY, Y^2) = M^2$, where $M = (X, Y)$ is maximal ideal of R. Then I is M-primary and $M^2 \subseteq I \subseteq M$. Since $X \notin I$ and $I + RX = (X, Y^2)$ is neither a maximal ideal nor R. Thus I is not a quasi maximal ideal.

Let M be an R-module and $R(+)M = R \oplus M$ denote the *idealization* of R-module M. Then $R(+)M$ is a commutative ring with componentwise addition and multiplication $(a, m)(b, n) = (ab, an + bm)$ for all $a, b \in R$, $m, n \in M$ [10]. If I is an ideal of R and N is a submodule of M, then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$ [15]. Also each maximal ideal of $R(+)M$ is of the form $P(+)M$, where $P \in \mathrm{Max}(R)$.

Theorem 6.

(i) *Let I be an ideal of R. Then $I(+)M$ is a quasi maximal ideal of $R(+)M$ if and only if I is a quasi maximal ideal of R.*

(ii) *Let I be an ideal of R and N a proper submodule of M. If $I(+)N$ is a quasi maximal ideal of R, then N is a maximal submodule of M, I is a maximal ideal of R and $I(+)N = (N : M)(+)N$.*

Proof. (i) Assume that $I(+)M$ is a quasi maximal ideal of $R(+)M$. Let $a \in R - I$. Then $(a, 0) \notin I(+)M$ and so $I(+)M + \langle (a, 0) \rangle = I(+)M + Ra(+)aM = (I + Ra)(+)M = R(+)M$ or $(I + Ra)(+)M \in \mathrm{Max}(R(+)M)$. This implies that $I + Ra = R$ or $I + Ra \in \mathrm{Max}(R)$. Thus I is a quasi maximal ideal of R. Conversely, assume that I is a quasi maximal ideal of R. Let $(a, m) \notin I(+)M$. Then we have $a \notin I$. This implies that $I + Ra = R$ or $I + Ra \in \mathrm{Max}(R)$. It is easy to see that $I(+)M + \langle (a, m) \rangle = (I + Ra)(+)M$ and so $I(+)M + \langle (a, m) \rangle = R(+)M$ or $I(+)M + \langle (a, m) \rangle \in Max(R(+)M)$.

(ii) Now, assume that $N \neq M$ and $I(+)N$ is a quasi maximal ideal of $R(+)M$. Let $m \notin N$. Then $(0, m) \notin I(+)N$ and so $I(+)N + \langle (0, m) \rangle = I(+)N + 0(+)Rm = I(+)(N + Rm) = R(+)M$ or it is a maximal ideal. Since I is proper ideal, we conclude that $I(+)(N + Rm)$ is a maximal ideal of $R(+)M$ which implies that I is a maximal ideal and also $N + Rm = M$ so that N is a maximal submodule of M. Furthermore, $I = (N : M)$. □

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Department of Mathematics
Yildiz Technical University
Davutpasa Campus, Esenler
34210 Istanbul, Turkey
e-mails: alan@yildiz.edu.tr
mestklc@gmail.com

**Department of Mathematics*
Istanbul Medeniyet University
Usküdar
34700 Istanbul, Turkey
e-mail: suat.koc@medeniyet.edu.tr

***Department of Mathematics*
Marmara University, Ziverbey
34722 Istanbul, Turkey
e-mail: utekir@marmara.edu.tr