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RINGS WITH DIVISIBILITY ON DESCENDING CHAINS OF IDEALS

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Abstract. This paper deals with the rings which satisfy DCC_d condition. This notion has been introduced recently by R. Dastanpour and A. Ghorbani (2017) as a generalization of Artnian rings. It is of interest to investigate more deeply this class of rings. This study focuses on commutative case. In this vein, we present this work in which we examine the transfer of these rings to the trivial, amalgamation and polynomial ring extensions. We also investigate the relationship between this class of rings and the well known ones. Furthermore, many new results are presented in the scope of this paper. For example, there is one which concerns the decomposition of ideals on prime ones and another which investigate the Krull dimension of the ring satisfying DCC_d condition. At the end of this work, we provide a result which concerns the modules over such rings.

Keywords: DCC_d; amalgamation of ring; trivial ring extension; Noetherian ring; Artinian ring; polynomial ring extension

MSC 2020: 13D05, 13D02

1. Introduction

Throughout this paper, all rings are commutative with nonzero identity, and all modules are nonzero unital. In [8], Dastanpour and Ghorbani introduced the concept of DCC_d condition, which is a generalization of DCC condition. These authors say a ring R satisfies DCC_d if for every descending chain $\{I_n\}_{n\in\mathbb{N}}$ of ideals, there exists an integer i such that for every $j \geqslant i$, there exists an element a_j of R verifying $I_{j+1} = a_j I_j$. If we assume that the multiplicity factor between two successive ideals is invertible, the ring which satisfies DCC_d condition is Artinian. The most of the results presented in [8] are concerned with the associative rings. In this paper, we study more deeply these rings focusing on the commutative case. Namely, we examine the transfer of these rings to trivial extension, amalgamation and polynomial ring extensions. We also study the relation between these rings and the Artinian,

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Noetherian and coherent rings. Many other results are presented in this work, for instance we investigate the decomposition of ideals on prime ones and the Krull dimension of the rings which satisfy DCC_d condition.

Let A and B be rings, J an ideal of B and let $f\colon A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$: The amalgamation of A and B along J with respect to f is the subring of $A \times B$ defined by

$$A \bowtie^f J := \{(a, f(a) + j)\colon a \in A, j \in J\}.$$

This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna, Finacchiaro, and Fontana in [3]–[7]). This construction has been studied in the general case and from the different point of view of pullbacks, by D'Anna and Fontana, see [6]. Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation (see [4], Examples 2.5 and 2.6) and other such constructions like the Nagata's idealization, cf. [12], page 2.

Let A be a ring and E an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') := (aa', ae' + a'e)$. For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of R. Recall that, prime (or maximal) ideals of R have the form $P \propto E$, where P is a prime (or maximal) ideal of A, see [1], Theorem 3.2. Suitable background on commutative trivial ring extensions is in [1], [9], [10].

2. Main results

In this section, we present some properties concerning the rings satisfying the descending chain condition. Namely, we study the relationship between this class of rings and the classical ones, we investigate the Krull dimension of the ring satisfying the DCC_d condition and decomposition of its ideals onto prime ones.

Many works proposed generalization of the Noetherian rings. For instance, we recall that Bakkari in [2] defined Q-Noetherian as a ring A for which A/P is Noetherian for every prime ideal P of A. Based on a result presented in [8], we show that each ring satisfying the DCC_d condition is Q-Noetherian. Using this result, we provide the following theorem, which gives a necessary and sufficient condition for a ring satisfying the DCC_d to be Noetherian.

Theorem 2.1. *Let A be a ring satisfying the DCC_d condition. Then the following statements are equivalent.*

1. *All minimal prime ideals of A are finitely generated.*
2. *A is a Noetherian ring.*

To prove this theorem, we need the following lemma.

Lemma 2.2. *Let A be a ring satisfying the DCC_d condition and P a prime ideal of A. The following statements are satisfied.*

1. *A is Q-Noetherian.*
2. *If P is finitely generated, then every ideal containing P is finitely generated.*

Proof. (1) Let A be a ring satisfying the DCC_d condition. Then for every prime ideal P of A, A/P is an integral domain which satisfies DCC_d condition. Therefore A/P is a principal ideal domain, see [8], Corollary 4.6. Hence, A/P is Noetherian, which means that A is Q-Noetherian.

(2) Let A be a ring satisfying the DCC_d condition and P a finitely generated prime ideal. Then according to the previous statement A is Q-Noetherian. So A/P is Noetherian. Hence, for every ideal I containing P, I/P is finitely generated. Since P is finitely generated, then I is also finitely generated. □

Proof of Theorem 2.1. To show that the first assertion implies the second, it suffices to prove that every prime ideal of A is finitely generated. So, let P be a prime ideal. Further, P contains a minimal prime ideal which is finitely generated. According to Lemma 2.2, P is finitely generated.

The second implication is clear because every ideal in a Noetherian ring is finitely generated, in particular the prime ones. □

Recall that for a ring A, the set of all nilpotent elements is denoted by $\mathrm{nil}(A)$ and if $\mathrm{nil}(A) = (0)$, then A is said to be a reduced ring.

Corollary 2.3. *Let A be a reduced semilocal ring satisfying DCC_d and ACC_d condition. Then A is Noetherian.*

Proof. Let A be a reduced ring satisfying DCC_d and ACC_d condition and possess a finite number of maximal ideals. According to [8], Theorem 3.6, the number of its minimal prime ideals is finite. As A is reduced, according to [11], Lemma 1.12, the minimal prime ideals of A are finitely generated. Hence, Theorem 2.1 implies that A is Noetherian as desired. □

Corollary 2.4. *Let A be a semilocal ring satisfying DCC_d and ACC_d. If $\mathrm{nil}(A)$ is finitely generated, then each ideal of A containing $\mathrm{nil}(A)$ is finitely generated.*

Proof. Let A be a ring satisfying the conditions presented in the above corollary. Since A has a finite number of maximal ideals and satisfies DCC_d and ACC_d condition, $A/\mathrm{nil}(A)$ has a finite number of maximal ideals, satisfies DCC_d and ACC_d conditions and it is reduced. So, according to the above corollary, $A/\mathrm{nil}(A)$ is Noetherian. As $\mathrm{nil}(A)$ is finitely generated, every ideal containing $\mathrm{nil}(A)$ is finitely generated. □

Proposition 2.5. *Let A be a ring satisfying the DCC_d condition such that every principal ideal of A contains a prime ideal. Then A is Noetherian.*

Proof. In order to prove that A is Noetherian, we show that each ideal of A is finitely generated. To do this, we consider an arbitrary ideal I of A. Since A satisfies the DCC_d condition, according to [8], Proposition 4.3, I contains a nonzero ideal J such that every ideal that is contained in J is principal. Hence, there is a finitely generated prime ideal of A contained in J. So according to Lemma 2.2, I is finitely generated as desired. □

Theorem 2.6. *Let R be a ring. If R satisfies DCC_d condition, then R satisfies* ACC *condition on prime ideals.*

Proof. We must show that every ascending chain of prime ideals of R is stationary. Let $P_1 \subseteq P_2 \subseteq P_3 \subseteq P_4 \subseteq P_5 \subseteq \ldots$ be a chain of prime ideals of R. We now prove that this chain is stationary. Since P_1 is a prime ideal, Lemma 2.2 shows that R/P_1 is Noetherian. So the ascending chain $P_2/P_1 \subseteq P_3/P_1 \subseteq P_4/P_1 \subseteq P_5/P_1 \subseteq \ldots$ is stationary. Then $P_1 \subseteq P_2 \subseteq P_3 \subseteq P_4 \subseteq P_5 \subseteq \ldots$ is stationary as desired. □

The following result is an immediate consequence of [2], Theorem 2.2.

Proposition 2.7. *Let A be a ring and E be an A-module, and $R := A \propto E$ be the trivial ring extension of A by E. Then the following statements hold:*

1. *If A is a ring satisfying the DCC_d condition, then R is Q-Noetherian.*
2. *If R is a ring satisfying the DCC_d condition, then A is Q-Noetherian.*

Proposition 2.8. *Let A be a ring satisfying the DCC_d condition. The following statements hold:*

1. *Every prime ideal is either minimal or maximal. Consequently,* $\dim(A) \leqslant 1$.
2. *Every ideal strictly containing a prime ideal P is a product of a finite number of maximal ideals strictly containing P.*

Proof. These statements come from the fact that a principal ideal domain is a Dedekind domain and for every prime ideal P of a ring A satisfying DCC_d, the quotient A/P is a principal ideal domain.

(1) Let Q be a prime ideal of A which is not minimal. So there exists a minimal prime ideal P strictly contained in Q. Since A is a ring satisfying the DCC_d condition, the quotient A/P is a principal ideal domain, so it is a Dedekind domain. As Q/P is a nonzero prime ideal of A/P, it is maximal. Therefore, Q is maximal in A. It follows that $\dim(A) \leqslant 1$.

(2) Let I be an ideal of A which strictly contains a prime ideal P. Since A/P is a principal ideal domain, there exist some prime ideals $P_1, P_2, \ldots, P_n$ containing P such that $I/P = (P_1/P)(P_2/P)\ldots(P_n/P) = (P_1P_2\ldots P_n)/P$. Consequently, $I = P_1P_2\ldots P_n$. It is clear that for each $i \in \{1, 2, \ldots n\}$, P_i is a maximal ideal of A as desired. □

Theorem 2.9. *Let A be a ring. Then:*

1. *Assume that (A, M) is local such that $M^2 = 0$. Then A satisfies DCC_d condition if and only if A is Artinian.*
2. *Let P be a prime ideal of A such that $P^2 = 0$. Then A_P satisfies DCC_d condition if and only if A_P is Artinian.*

Proof. (1) Since every Artinian ring satisfies the DCC_d condition, it suffices to show that every local ring (A, M) with $M^2 = 0$ satisfying DCC_d condition is Artinian. We do this as follows:

Let $I_1 \supseteq I_2 \supseteq \ldots I_n \supseteq \ldots$ be a descending chain of nonzero ideals of A. For the sake of simplicity this chain is denoted by (I). Since A satisfies DCC_d, there exists an integer k such that for each $i \geqslant k$, $I_{i+1} = a_iI_i$ for some $a_i \in A$. Using the hypothesis of this proposition, we show that the later property implies that the chain (I) is stationary. Before doing this, we affirm that $a_i \notin M$ for each $i \geqslant k$. Assume that there exists $n \geqslant k$ such that $a_n \in M$. Then $I_{n+1} = a_nI_n \subseteq M^2 = 0$. So $I_{n+1} = 0$. This contradicts the assumption that I_{n+1} is a nonzero ideal. So, for each $i \geqslant k$, a_i is invertible because (A, M) is local and $a_i \notin M$. Then the equality $I_{i+1} = a_iI_i$ means that $I_i = I_{i+1}$ for all $i \geqslant k$. Thus, the chain (I) is stationary. Consequently, the ring A is Artinian as desired.

(2) Let P be a prime ideal of A such that $P^2 = 0$. So (A_P, PA_P) is a local ring such that $(PA_P)^2 = 0$. Hence, according to the first assertion A_P satisfies DCC_d if and only if A_P is *Artinian.* □

Corollary 2.10. *Let A be an integral domain, $K = qf(A)$ be the quotient field of A, E be a K-vector space such that $\dim_K E = \infty$, $R := A \propto E$ the trivial ring extension of A by E. Then R does not satisfy DCC_d condition.*

Proof. Let $P := 0 \propto E$, which is a prime ideal of R. Since $\dim_K E = \infty$, $R_P := K \propto E$ is not Noetherian. So it is not *Artinian.* Hence, R_P does not satisfy DCC_d by the previous theorem. Consequently, R does not satisfy the DCC_d condition. □

Example 2.11. Using Corollary 2.10, we deduce $\mathbb{Z} \propto \mathbb{R}$ does not satisfy DCC_d condition.

Recall that an R-module M is said to be coherent if M is finitely generated and every finitely generated submodule of M is finitely presented. In particular, a ring R is said to be a coherent ring if it is a coherent module over itself.

Theorem 2.12. *Let A be a ring satisfying the DCC_d condition. If there exists in A a prime ideal which is finitely generated, then A is coherent.*

Proof. Let A be a ring satisfying the DCC_d condition and P be a prime ideal which is finitely generated. Then A/P is a domain which satisfies DCC_d. So A/P is a principal ideal domain and so it is Noetherian. Hence, A is coherent. ☐

Proposition 2.13. *Let A be a ring satisfying the DCC_d condition and M be an A-module such that* $\operatorname{Ann}(M)$ *contains a prime ideal. Then M is finitely generated if only if M is Noetherian.*

Proof. Let M be a finitely generated module over a ring A satisfying the DCC_d condition such that $\operatorname{Ann}(M)$ contains a prime ideal P. We show that M is Noetherian. Indeed, since $P \subseteq \operatorname{Ann}(M)$, M is an A/P-module. Since A satisfies DCC_d so does A/P. Hence, A/P is a principal ideal domain. Therefore, A/P is Noetherian. Then M is Noetherian as an A/P-module. It follows that M is Noetherian as an A-module. ☐

3. TRANSFER OF DCC_d CONDITION

In this section, we examine the transfer of the DCC_d condition to the polynomial and trivial ring extension. The following result shows that the polynomial ring extension of the ring satisfying DCC_d does not necessarily satisfy DCC_d condition.

Theorem 3.1. *Let A be an integral domain that satisfies DCC_d condition. Then $A[X]$ satisfies DCC_d condition if only if A is a field.*

Proof. Assume that A is a ring such that $A[X]$ satisfies the DCC_d condition and prove that A is a field. Indeed, those conditions imply that $A[X]$ is a domain which satisfies the DCC_d condition. So, according to [8], Corollary 4.6, $A[X]$ is principal. Thus, A is a field. To show the converse, it suffices to recall that if A is a field, then $A[X]$ is integral and principal. Hence, it satisfies the DCC_d condition. ☐

Example 3.2. $\mathbb{Z}[X]$ does not satisfy DCC_d condition because $\mathbb{Z}$ is an integral domain, which is not a field.

Remark 3.3. $A[X,Y]$ does not satisfy DCC_d condition if A is a domain because $A[X,Y]=A[X][Y]$ and $A[X]$ is not a field.

In the following theorem, we study the transfer of DCC_d condition to the trivial extension.

Theorem 3.4. *Let A be a ring and E be an A-module, and let $R := A \propto E$ be the trivial ring extension of A by E. Then:*

1. *If R satisfies DCC_d condition, then so does A.*
2. *Assume that (A, M) is local and E a nonzero A-module with $ME = 0$. If R satisfies DCC_d condition, then A is Artinian.*

Proof. (1) It suffices to apply the properties: $R/(0 \propto E) \cong A$ and the homomorphic image of a ring satisfying the DCC_d condition also satisfies DCC_d condition.

(2) We assume that R satisfies DCC_d and we prove that A is *Artinian.* To achieve this aim, we show that every descending chain is stationary. So, let $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ be a descending chain of ideals of A. Since R satisfies DCC_d and $I_1 \propto E \supseteq I_2 \propto E \supseteq \ldots \supseteq I_n \propto E \supseteq \ldots$ is a descending chain of R, there exists an integer k such that for each $i \geqslant k$, there exists an element (a_i, e_i) of R satisfying the following relation: $I_{i+1} \propto E = (a_i, e_i)(I_i \propto E)$. One can easily see that $I_{i+1} = a_i I_i$. Now, we show that a_i is invertible. Assume that a_i is not invertible. Then $a_i \in M$. Let $e \in E$. Since $(0, e) \in I_{i+1} \propto E$, we can write $(0, e) = (a_i, e_i)(x_i, y_i)$. This gives $e = a_i y_i + x_i e_i$. Since $ME = 0$ and $y_i \in I_i \subseteq M$, we obtain $e = 0$. Then we have $E = 0$, which is a contradiction. □

Remark 3.5. To see that the converse of the second assertion of Theorem 3.4 is not true, it suffices to use Corollary 2.10.

Theorem 3.6. *Let $f\colon A \to B$ be a ring homomorphism and let J be an ideal of B. Then:*

1. *if $A \bowtie^f J$ satisfies the DCC_d condition, so does A;*
2. *if $A \bowtie^f J$ satisfies the DCC_d condition and $J \subseteq \operatorname{ann}(\operatorname{Im} f)$, then $J^2 = J$.*

Proof. Let $f\colon A \to B$ be a ring homomorphism and J be an ideal of B.

(1) Suppose that $A \bowtie^f J$ satisfies the DCC_d condition. We show that A also satisfies the DCC_d condition. Since $A \bowtie^f J$ satisfies the DCC_d condition and according to [8], Proposition 2.3, $(A \bowtie^f J)/(0 \bowtie^f J)$ satisfies the DCC_d condition. As $(A \bowtie^f J)/(0 \bowtie^f J) \cong A$, A satisfies the DCC_d condition.

(2) Let $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ be a descending chain of ideals of A. It is obvious that $I_1 \bowtie^f J \supseteq I_2 \bowtie^f J \supseteq \ldots \supseteq I_n \bowtie^f J \supseteq \ldots$ is a descending chain of nonzero ideals of $A \bowtie^f J$. Hence, there exists an integer k such that

for each $i \geqslant k$, $I_{i+1} \bowtie^f J = (b_i, f(b_i) + l_i)I_i \bowtie^f J$ for an element $(b_i, f(b_i) + l_i)$ of $A \bowtie^f J$. So, for each a_{i+1} in I_{i+1} and j_{i+1} in J, there exist a_i in I_i and j_i in J such that $(a_{i+1}, f((a_{i+1}) + j_{i+1}) = (b_i, f(b_i) + l_i)(a_i, f((a_i) + j_i)$. Then $a_{i+1} = b_i a_i$ and $f(a_{i+1}) + j_{i+1} = (f(b_i) + l_i)(f(a_i) + j_i)$. This leads to the following equations: $a_{i+1} = b_i a_i$ and $f(a_{i+1}) + j_{i+1} = f(b_i)f(a_i) + l_i f(a_i) + j_i f(b_i) + j_i l_i$. Using the assumption $J \subseteq \operatorname{ann}(\mathrm{Imf})$, we obtain $j_{i+1} = l_i j_i$. So $J^2 = J$ as desired. ☐

Example 3.7. $f\colon A \to B$ such that $A = \mathbb{Z}$, $B = \mathbb{Z}/8\mathbb{Z}$ and $f(n) = \overline{2n}$. $\mathrm{Imf} = 2\mathbb{Z}/8\mathbb{Z}$ and $\operatorname{ann}(\mathrm{Imf}) = 4\mathbb{Z}/8\mathbb{Z}$. Let $J = 4\mathbb{Z}/8\mathbb{Z}$. We have $J^2 = 0$ and $J \neq 0$. So $J^2 \neq J$. Consequently, $\mathbb{Z} \bowtie^f \mathbb{Z}/8\mathbb{Z}$ does not satisfy the DCC_d condition.

Remark 3.8. Let A be a ring satisfying the DCC_d condition. Even if $A \propto E$ and $A \bowtie^f J$ do not satisfy DCC_d condition, they can inherit some properties from A. This is also true for a ring satisfying the ACC_d condition. For instance, we give the following proposition.

Proposition 3.9. *Let $f\colon A \to B$ be a ring homomorphism and let J be an ideal of B. Let A be a ring satisfying the DCC_d condition and P be a prime ideal of A. Then the following statement holds:*

(1) *$A \propto E/P \propto E$ and $A \bowtie^f J/P \bowtie^f J$ are principal ideal domains.*

(2) *If P is finitely generated and E is a finitely generated A-module, then each ideal of $A \propto E$ containing $P \propto E$ is finitely generated.*

(3) *If P is finitely generated and J is a finitely generated $f(A)$-module, then each ideal of $A \bowtie^f J$ containing $P \bowtie^f J$ is finitely generated.*

Proof. (1) Assume that A is a ring satisfying DCC_d condition. So, according to [8], Corollary 4.6, A/P is a principal ideal domain. Moreover, $A \propto E/P \propto E \cong A/P$ and $A \bowtie^f J/P \bowtie^f J \cong A/P$. Consequently, $A \propto E/P \propto E$ and $A \bowtie^f J/P \bowtie^f J$ are principal ideal domains.

(2) Since P and E are finitely generated, $P \propto E$ is also finitely generated. According to the first statement, $A \propto E/P \propto E$ is a principal ideal domain. So, each ideal of $A \propto E$ containing $P \propto E$, is finitely generated.

(3) Since P is finitely generated and J is a finitely generated $f(A)$-module, $P \bowtie^f J$ is also finitely generated. According to the first statement, $A \bowtie^f J/P \bowtie^f J$ is a principal ideal domain. So, each ideal of $A \bowtie^f J$ containing $P \bowtie^f J$ is finitely generated. ☐

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