Pergamon

Mathl. Comput. Modelling Vol. 20, No. 2, pp. 9–17, 1994

0895-7177/94 $7.00 + 0.00

0895-7177(94)00100-6

Two-Machine Ordered Flowshop Scheduling Under Random Breakdowns

A. ALLAHVERDI
Department of Industrial Engineering
Marmara University, Istanbul, Turkey

J. MITTENTHAL
Department of Decision Sciences and Engineering Systems
Rensselaer Polytechnic Institute, Troy, New York 12180, U.S.A.

Abstract—The problem of scheduling on a two-machine ordered flowshop, where machines suffer random breakdowns, is addressed with respect to the individual objectives of makespan and mean flowtime. An optimal sequence with makespan objective is obtained when only one of the two machines suffers random breakdowns. An optimal sequence with respect to mean flowtime is obtained when both machines are subject to random breakdowns.

Keywords—Stochastic scheduling, Flowshops, Machine breakdowns, Mean flowtime.

1. INTRODUCTION

In the classical flowshop scheduling problem with n jobs and m machines, each job has m operations and each operation has to be done on a different machine. All jobs are ready at time zero, and setup times of the operations are sequence independent and are included in processing times. An operation on a machine must be finished before the machine can start another operation. The operations of a job cannot be in process simultaneously, and a machine cannot process more than one job at a time. Machines are assumed to be continuously available.

For $m = 2$, Johnson [1] provided an optimal sequence with respect to a makespan objective. Garey, Johnson, and Sethi [2] proved that this problem is unary NP-complete when $m = 3$. And Gonzalez and Sahni [3] showed that the problem for $m = 2$ with a mean flowtime objective is NP-hard.

Smith, Panwalkar, and Dudek [4] defined a subcategory of the classical flowshop problem where the processing times of the jobs have the following two properties:

(i) if one job has a smaller processing time than another job on any machine, then this relationship will hold on all other machines,
(ii) for $k = 1, 2, \dots, m$ the machine with the k^{th} smallest processing time for a particular job will also have the k^{th} smallest processing time for all other jobs.

A flowshop with these two characteristics is called an ordered flowshop. Smith *et al.* [4] discussed the practical basis for ordered flowshop problems.

An ordered flowshop where the largest processing time for every job occurs on the first machine is called Type A. Similarly, if the largest processing time for every job occurs on the last machine it is called Type B. Smith *et al.* [4] established that LPT (Longest Processing Time) and SPT (Shortest Processing Time) job orderings minimize makespan, respectively, for Types A and B m-machine ordered flowshops. Later, Panwalkar and Khan [5] showed that in an m-machine ordered flowshop, SPT sequence minimizes mean flowtime.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Allahverdi and Mittenthal [6] discussed the problem of minimizing makespan on a two-machine flowshop where machines suffer stochastic breakdowns with respect to makespan. Under certain conditions on breakdown distributions, they showed that Johnson's method stochastically minimizes makespan.

In this paper, we address a two-machine ordered flowshop problem where machines are subject to random breakdowns. Without making any assumptions on breakdown distributions, we provide a sequence which minimizes makespan with probability 1 (w.p.1) when only one of the machines suffers breakdowns. Under certain conditions on breakdown distributions, we also provide a sequence that minimizes mean flowtime stochastically for the same problem when both machines suffer breakdowns.

2. ASSUMPTIONS AND DEFINITIONS

Since the machines are subject to random breakdowns, the machines alternate between being available for a random length of time, $U_{rk}(r = 1, 2, \dots,$ and $k = 1, 2)$, and being unavailable for a random length of time, $D_{rk}(r = 1, 2, \dots,$ and $k = 1, 2)$. U_{rk} denotes the r^{th} uptime of Machine k, i.e., the length of time between the $(r-1)^{\text{th}}$ and r^{th} breakdowns on Machine k. D_{rk} represents the r^{th} downtime of Machine k, i.e., the duration of the r^{th} downtime of Machine k. The breakdown process of Machine k is denoted by a sequence of finite-valued, positive random vectors $\{U_{rk}, D_{rk}\}$, for $r = 1, 2, \dots$. Let $\{N_k(t) : t \geq 0\}$ be a counting process associated with the sequence $\{U_{rk}\}$ $(r = 1, 2, \dots, \infty)$ for each $k = 1, 2$. $N_k(t)$ denotes the number of breakdowns occurring on Machine k up to, and including, time t. Note that t is "clocked" only while the given machine is processing jobs. Specifically, t remains unchanged for Machine k when either that machine is available but idle, or the machine is down as a result of a breakdown.

We assume that machines do not break down while they are idle. This is not a restrictive assumption, since it is very unlikely that machines break down when the machines are idle.

We also assume that if a job is being processed when a breakdown occurs, the work done on a job is not lost, i.e., a preempt-resume model as opposed to preempt-restart model where the work done is lost.

There are two strategies to be considered when scheduling on multiple machines subject to random breakdowns. In the literature, these strategies are called simple recourse and general recourse. In this paper, we are concerned with simple recourse. Under a simple recourse strategy, the sequence of job completion times in a schedule is fixed, but the completion times may be pushed back as a result of one or more machine breakdowns.

Let Ω represent the set of all realizations of uptimes and downtimes, and ω denote one realization of uptimes and downtimes, i.e., $\omega = \{U_{rk}, D_{rk}\}$ $(k = 1, 2$ and $r = 1, 2, \dots, \infty)$. Let X and Y be two random variables defined on the same sample space Ω. X is said to be smaller than Y with probability 1 (w.p.1) if, for all realizations $\omega \in \Omega - \Gamma$, $X(\omega) \leq Y(\omega)$ where Γ satisfies $\Pr(\Gamma) = 0$. X is said to be stochastically smaller than $Y (X \leq_s Y)$ if, for all t, $\Pr(X \leq t) \geq \Pr(Y \leq t)$. The makespan of a sequence is a minimum w.p.1, if the makespan of that sequence is smaller than that of any other sequence for all realizations $\omega \in \Omega - \Gamma$. The mean flowtime of a sequence is minimized stochastically if the mean flowtime of that sequence is stochastically smaller than that of any other sequence.

The objectives considered in this paper are makespan and mean flowtime. Let $TD_k(a, b)$ denote total breakdown duration on Machine k over the interval $[a, b)$, i.e.,

$$TD_k(a, b) = \sum_{r=N_k(a)}^{N_k(b)} D_{rk}.$$

Also let $t_{[pk]}$ denote processing time of the job in position p on Machine k, and K_{pk} represent the total processing time of the jobs in positions $1, 2, \dots, p$ on Machine k, i.e., $K_{pk} = t_{[1k]} + \dots + t_{[pk]}$.

Allahverdi and Mittenthal [6] have shown that total idle time on the second machine, caused by processing and breakdowns on the first machine, until the job in position p is completed, M_p, is

$$M_p = \max\{\Delta_1, \Delta_2, \dots, \Delta_p\}, \tag{1}$$

where

$$\Delta_p = [K_{p1} + TD_1(0, K_{p1})] - [K_{p-1,2} + TD_2(0, K_{p-1,2})], \tag{2}$$

and $K_{02} = 0$.

Let R_p be the completion time of the job in position p, then

$$R_p = K_{p2} + TD_2(0, K_{p2}) + M_p. \tag{3}$$

Therefore, makespan ($R_{\max}$) is

$$R_{\max} = K_{n2} + TD_2(0, K_{n2}) + M_n. \tag{4}$$

If TF denotes total flowtime, then it follows from (3) that

$$TF = \sum_{p=1}^{n} R_p. \tag{5}$$

3. MAKESPAN

In this section, we are concerned only with the makespan objective. Allahverdi and Mittenthal [6] have provided an elimination criterion for the generic two-machine flowshop problem when both machines suffer breakdowns. However, notice that to develop an algorithm to minimize $E(R_{\max})$ it is necessary to be able to evaluate $E(M_n)$. Such an evaluation is quite difficult since M_n is the maximum of n dependent terms each of which includes the difference of two random sums of random variables. Hence, they were not able to develop an optimal solution procedure based on the established elimination criterion. By making certain assumptions on the breakdown processes, they have shown that Johnson's method stochastically minimizes makespan. These assumptions are that both machines have the same breakdown duration distributions and identical counting processes.

The elimination criterion they have provided states that in a sequence which minimizes makespan w.p.1, job i should precede job j if $t_{i1} \le t_{j1}$ and $t_{j2} \le t_{i2}$, where t_{rk} denote processing time of job r on Machine k. However, for ordered flowshop problems if $t_{i1} < t_{j1}$ then $t_{i2} \le t_{j2}$ and so this elimination criterion is no longer applicable.

Under the assumptions in [6], Johnson's method of course still stochastically minimizes makespan for ordered flowshops. In fact, for two-machine ordered flowshops, Johnson's method produces an LPT sequence for Type A and an SPT sequence for Type B ordered flowshops. Note that a two-machine ordered flowshop is either Type A or Type B.

Our goal here is to determine an optimal solution to the problem in environments for which the above assumptions do not hold. Hence, we only consider the cases when just one of the machines suffers breakdowns. Obtaining an optimal solution may not be possible even when only one of the machines is subject to breakdowns. For instance, consider a two-machine, Type A ordered flowshop when only the second machine suffers breakdowns. Since the ordered flowshop is of Type A, job processing times on Machine 1 are larger than those on the second machine. It is impossible to guarantee that the time between the start and completion of a job on the second machine is smaller than that of the job on the first machine. Thus, the resulting problem is much like the generic two-machine flowshop discussed above in that analyzing M_n is quite difficult. Therefore, we do not consider Type A ordered flowshops when the second machine suffers breakdowns. Similarly, we do not consider Type B ordered flowshops when the first machine is subject to breakdowns. Thus, there remain only the following two cases: Type A when Machine 1 suffers breakdowns, and Type B when Machine 2 suffers breakdowns.

In this section, we show that for two-machine ordered flowshop problems, the sequence LPT minimizes makespan w.p.1 for Type A problems when the first machine is subject to breakdowns whereas when the second machine is subject to random breakdowns the sequence SPT minimizes makespan w.p.1 for the Type B problems.

Note first that the term K_{n2} in (4) is a constant. Note also that the second term in (4), $TD_2(0, K_{n2})$, is independent of the sequence. Therefore, the sequence that minimizes M_n w.p.1 also minimizes $R_{\max}$ w.p.1.

THEOREM 1. *Suppose that only Machine 1 is subject to breakdowns. For Type A ordered flowshop problems, sequencing the jobs in LPT sequence minimizes makespan w.p.1.*

PROOF. It follows from (2) that $\Delta_p = [K_{p1} + TD_1(0, K_{p1})] - K_{p-1,2}$, since Machine 2 is reliable, or

$$\Delta_p = K_{p-1,1} + t_{[p1]} - K_{p-1,2} + TD_1(0, K_{p1}),$$

since $K_{p1} = K_{p-1,1} + t_{[p1]}$. Consider two job sequences:

$$\sigma_1 : \dots, j, i, \dots \quad \text{and} \quad \sigma_2 : \dots, i, j, \dots,$$

where σ_1 is an optimal sequence in which job j is in position $\bar{p}$ and job i is in position $\bar{p}+1$, sequence σ_2 is obtained from sequence σ_1 by interchanging the jobs in the positions $\bar{p}$ and $\bar{p}+1$, and $t_{j1} < t_{i1}$. We shall prove that sequence σ_2 is no worse than sequence σ_1, by showing that $M_n(\sigma_2) \le M_n(\sigma_1)$ w.p.1, and hence, LPT minimizes makespan w.p.1. Let ω be a given realization of uptimes and downtimes.

Observe that for $p = 1, \dots, \bar{p}-1$, $\Delta_p(\sigma_2, \omega) = \Delta_p(\sigma_1, \omega)$, since both sequences have the same jobs in these positions. Observe also that $\Delta_p(\sigma_2, \omega) = \Delta_p(\sigma_1, \omega)$ for $p = \bar{p}+2, \bar{p}+3, \dots, n$, as well. This follows from the fact that both sequences have the same jobs in Positions $1, \dots, \bar{p}-1, \bar{p}+2, \bar{p}+3, \dots, n$. Furthermore, $\Delta_p(p \ge \bar{p}+2)$ for both sequences include the sum of processing times of the jobs (including breakdowns during these periods) in positions $\bar{p}$ and $\bar{p}+1$, and hence, the interchange does not affect the sum. Therefore, in order to show that $M_n(\sigma_2, \omega) \le M_n(\sigma_1, \omega)$ we only need to show that

$$\max\{\Delta_{\bar{p}}(\sigma_2, \omega), \Delta_{\bar{p}+1}(\sigma_2, \omega)\} \le \max\{\Delta_{\bar{p}}(\sigma_1, \omega), \Delta_{\bar{p}+1}(\sigma_1, \omega)\}.$$

Note that

$$\begin{aligned}
\Delta_{\bar{p}}(\sigma_1, \omega) &= K_{\bar{p}-1,1} + t_{j1} - K_{\bar{p}-1,2} + TD_1(0, K_{\bar{p}-1,1} + t_{j1}), \\
\Delta_{\bar{p}+1}(\sigma_1, \omega) &= K_{\bar{p}-1,1} + t_{j1} + t_{i1} - K_{\bar{p}-1,2} - t_{j2} + TD_1(0, K_{\bar{p}-1,1} + t_{j1} + t_{i1}), \\
\Delta_{\bar{p}}(\sigma_2, \omega) &= K_{\bar{p}-1,1} + t_{i1} - K_{\bar{p}-1,2} + TD_1(0, K_{\bar{p}-1,1} + t_{i1}), \quad \text{and} \\
\Delta_{\bar{p}+1}(\sigma_2, \omega) &= K_{\bar{p}-1,1} + t_{i1} + t_{j1} - K_{\bar{p}-1,2} - t_{i2} + TD_1(0, K_{\bar{p}-1,1} + t_{i1} + t_{j1}).
\end{aligned}$$

Notice that

$$\Delta_{\bar{p}}(\sigma_2, \omega) - \Delta_{\bar{p}+1}(\sigma_1, \omega) = t_{j2} - t_{j1} + TD_1(0, K_{\bar{p}-1,1} + t_{i1}) - TD_1(0, K_{\bar{p}-1,1} + t_{j1} + t_{i1}) < 0,$$

since $t_{j2} < t_{j1}$, and hence,

$$\Delta_{\bar{p}}(\sigma_2, \omega) < \Delta_{\bar{p}+1}(\sigma_1, \omega). \tag{6}$$

Also notice that $\Delta_{\bar{p}+1}(\sigma_2, \omega) - \Delta_{\bar{p}+1}(\sigma_1, \omega) = t_{j2} - t_{i2} \le 0$, since $t_{j2} \le t_{i2}$ which follows from the fact that $t_{j1} < t_{i1}$. Therefore,

$$\Delta_{\bar{p}+1}(\sigma_2, \omega) \le \Delta_{\bar{p}+1}(\sigma_1, \omega). \tag{7}$$

As a result of (6) and (7),

$$\max\{\Delta_{\bar{p}}(\sigma_2,\omega),\Delta_{\bar{p}+1}(\sigma_2,\omega)\} \le \max\{\Delta_{\bar{p}}(\sigma_1,\omega),\Delta_{\bar{p}+1}(\sigma_1,\omega)\}.$$

Thus, $M_n(\sigma_2,\omega) \le M_n(\sigma_1,\omega)$. Since this argument is valid for any ω, $M_n(\sigma_2) \le M_n(\sigma_1)$ w.p.1. Hence, σ_2 is also an optimal sequence and is closer to LPT than σ_1. Repeat this argument until an LPT sequence is achieved. Then, LPT minimizes the makespan w.p.1. ∎

THEOREM 2. *Assume that only Machine 2 is subject to breakdowns. For Type B ordered flowshop problems, sequencing the jobs in SPT sequence minimizes makespan w.p.1.*

The proof of this theorem is very similar to that of Theorem 1 (see the Appendix).

4. MEAN FLOWTIME

In this section, we consider a two-machine ordered flowshop with respect to mean flowtime, where both machines are subject to random breakdowns.

LEMMA 1. *Assume that $\{D_{r1}\}$ and $\{D_{r2}\}$ have identical distributions, and $N_1(t)$ and $N_2(t)$ have the same distribution with stationary increments. If $t_1 \ge t_2$, then*

$$TD_1(T_1, T_1 + t_1) \ge TD_2(T_2, T_2 + t_2),$$

for any times T_1, $T_2 \ge 0$.

For the proof of this lemma refer to [6].

Under the assumptions in Lemma 1, we shall show that an SPT sequence stochastically minimizes mean flowtime for two-machine ordered flowshops. The two-machine ordered flowshop problem is either Type A or Type B. In the following Theorems 3a and 3b, we will show that the SPT sequence stochastically minimizes makespan, respectively, for Type A and B ordered flowshops. Therefore, the SPT sequence is optimal for a two-machine ordered flowshop with respect to the mean flowtime objective. The following discussion presents the background common to both proofs.

Consider the sequences σ_1 and σ_2 as described earlier where now $t_{j1} > t_{i1}$. Let $TF(\sigma_r)$ denote total flow time of the sequence $\sigma_r (r = 1, 2)$. In the following Theorems 3a and 3b, we will show that for both Types A and B ordered flowshops $TF(\sigma_2) \le_s TF(\sigma_1)$. Note that stochastically minimizing mean flowtime is equivalent to stochastically minimizing total flowtime. Therefore, the SPT sequence is optimal with respect to the former objective.

Let ω be a realization of uptimes and downtimes. Note that

$$\Delta_{\bar{p}}(\sigma_2,\omega) = K_{\bar{p}-1,1} + t_{i1} + TD_1\left(0, K_{\bar{p}-1,1} + t_{i1}\right) - K_{\bar{p}-1,2} - TD_2\left(0, K_{\bar{p}-1,2}\right), \tag{8}$$

$$\begin{aligned}\Delta_{\bar{p}+1}(\sigma_2,\omega) = {} & K_{\bar{p}-1,1} + t_{i1} + t_{j1} + TD_1\left(0, K_{\bar{p}-1,1} + t_{i1} + t_{j1}\right) - K_{\bar{p}-1,2} - t_{i2} \\ & - TD_2\left(0, K_{\bar{p}-1,2} + t_{i2}\right),\end{aligned} \tag{9}$$

$$\Delta_{\bar{p}}(\sigma_1,\omega) = K_{\bar{p}-1,1} + t_{j1} + TD_1\left(0, K_{\bar{p}-1,1} + t_{j1}\right) - K_{\bar{p}-1,2} - TD_2\left(0, K_{\bar{p}-1,2}\right), \tag{10}$$

$$\begin{aligned}\Delta\bar{p} + 1(\sigma_1,\omega) = {} & K_{\bar{p}-1,1} + t_{j1} + t_{i1} + TD_1\left(0, K_{\bar{p}-1,1} + t_{j1} + t_{i1}\right) - K_{\bar{p}-1,2} - t_{j2} \\ & - TD_2\left(0, K_{\bar{p}-1,2} + t_{j2}\right).\end{aligned} \tag{11}$$

Observe that since $t_{j1} > t_{i1}$, from (8) and (10), we have

$$\Delta_{\bar{p}}(\sigma_2,\omega) < \Delta_{\bar{p}}(\sigma_1,\omega). \tag{12}$$

Observe also that since in either sequence the jobs in Positions $1, \dots, \bar{p} - 1$ are the same,

$$M_j(\sigma_2,\omega) = M_j(\sigma_1,\omega), \qquad \text{for } j = 1, \dots, \bar{p} - 1. \tag{13}$$

THEOREM 3A. *Under the assumptions in Lemma 1, SPT sequence stochastically minimizes mean flowtime for two-machine, Type A, ordered flowshops when both machines suffer random breakdowns.*

PROOF. Let $TF_{p1,p2}(\sigma_r)$ denote the total flowtime of jobs in positions $p_1, p_1+1, \dots, p_2-1, p_2 (p_1 < p_2)$ in sequence σ_r, and $F_p(\sigma_r)$ represent the flowtime of the job in position p in sequence σ_r, $r = 1, 2$. Since both sequences have the same jobs in positions $1, 2, \dots, \bar{p}-1$, it follows that

$$TF_{1,\bar{p}-1}(\sigma_2,\omega) = TF_{1,\bar{p}-1}(\sigma_1,\omega).$$

Furthermore, for Type A ordered flowshops, it can easily be shown that

$$TF_{\bar{p}+2,n}(\sigma_2,\omega) = {}_s TF_{\bar{p}+2,n}(\sigma_1,\omega).$$

Hence, in order to show that $TF(\sigma_2,\omega) \le_s TF(\sigma_1,\omega)$, it suffices to show that $TF_{\bar{p},\bar{p}+1}(\sigma_2,\omega) \le_s TF_{\bar{p},\bar{p}+1}(\sigma_1,\omega)$.

Note that

$$\begin{aligned} TF_{\bar{p},\bar{p}+1}(\sigma_2,\omega) - TF_{\bar{p},\bar{p}+1}(\sigma_1,\omega) &=_s [F_{\bar{p}}(\sigma_2,\omega) + F_{\bar{p}+1}(\sigma_2,\omega)] - [F_{\bar{p}}(\sigma_1,\omega) + F_{\bar{p}+1}(\sigma_1,\omega)] \\ &= [t_{i2} + TD_2\,(K_{\bar{p}-1,2}, K_{\bar{p}-1,2} + t_{i2}) \\ &\quad + \max\{M_{\bar{p}-1}(\sigma_2,\omega), \Delta_{\bar{p}}(\sigma_2,\omega)\} \\ &\quad + \max\{M_{\bar{p}-1}(\sigma_2,\omega), \Delta_{\bar{p}}(\sigma_2,\omega), \Delta_{\bar{p}+1}(\sigma_2,\omega)\}] \\ &\quad - [t_{j2} + TD_2\,(K_{\bar{p}-1,2}, K_{\bar{p}-1,2} + t_{j2}) \\ &\quad + \max\{M_{\bar{p}-1}(\sigma_1,\omega), \Delta_{\bar{p}}(\sigma_1,\omega)\} \\ &\quad + \max\{M_{\bar{p}-1}(\sigma_1,\omega), \Delta_{\bar{p}}(\sigma_1,\omega), \Delta_{\bar{p}+1}(\sigma_1,\omega)\}]. \end{aligned}$$

By (12) and (13), $\max\{M_{\bar{p}-1}(\sigma_2,\omega), \Delta_{\bar{p}}(\sigma_2,\omega)\} \le \max\{M_{\bar{p}-1}(\sigma_1,\omega), \Delta_{\bar{p}}(\sigma_1,\omega)\}$. Thus,

$$\begin{aligned} TF_{\bar{p},\bar{p}+1}(\sigma_2,\omega) - TF_{\bar{p},\bar{p}+1}(\sigma_1,\omega) &\le_s [t_{i2} + TD_2\,(K_{\bar{p}-1,2}, K_{\bar{p}-1,2} + t_{i2}) \\ &\quad + \max\{M_{\bar{p}-1}(\sigma_2,\omega), \Delta_{\bar{p}}(\sigma_2,\omega), \Delta_{\bar{p}+1}(\sigma_2,\omega)\}] \\ &\quad - [t_{j2} + TD_2\,(K_{\bar{p}-1,2}, K_{\bar{p}-1,2} + t_{j2}) \\ &\quad + \max\{M_{\bar{p}-1}(\sigma_1,\omega), \Delta_{\bar{p}}(\sigma_1,\omega), \Delta_{\bar{p}+1}(\sigma_1,\omega)\}]\,. \end{aligned} \tag{14}$$

Now there are three cases to consider. Each case corresponds to the term $\max\{M_{\bar{p}-1}(\sigma_2,\omega), \Delta_{\bar{p}}(\sigma_2,\omega), \Delta_{\bar{p}+1}(\sigma_2,\omega)\}$ taking one of its possible values.

CASE 1. Suppose that $\max\{M_{\bar{p}-1}(\sigma_2,\omega), \Delta_{\bar{p}}(\sigma_2,\omega), \Delta_{\bar{p}+1}(\sigma_2,\omega)\} = M_{\bar{p}-1}(\sigma_2,\omega)$. However, $M_{\bar{p}-1}(\sigma_2,\omega) \le \max\{M_{\bar{p}-1}(\sigma_1,\omega), \Delta_{\bar{p}}(\sigma_1,\omega), \Delta_{\bar{p}+1}(\sigma_1,\omega)\}$, since $M_{\bar{p}-1}(\sigma_2,\omega) = M_{\bar{p}-1}(\sigma_1,\omega)$. Furthermore, $t_{i2} \le t_{j2}$ by the first property of ordered flowshops. Therefore, it follows from (14) that $TF_{\bar{p},\bar{p}+1}(\sigma_2,\omega) \le_s TF_{\bar{p},\bar{p}+1}(\sigma_1,\omega)$.

CASE 2. Suppose that $\max\{M_{\bar{p}-1}(\sigma_2,\omega), \Delta_{\bar{p}}(\sigma_2,\omega), \Delta_{\bar{p}+1}(\sigma_2,\omega)\} = \Delta_{\bar{p}}(\sigma_2,\omega)$. Again, by the first property of ordered flowshops, $t_{i2} \le t_{j2}$. Note that

$$\Delta_{\bar{p}}(\sigma_2,\omega) \le \max\{M_{\bar{p}-1}(\sigma_1,\omega), \Delta_{\bar{p}}(\sigma_1,\omega), \Delta_{\bar{p}+1}(\sigma_1,\omega)\}\,,$$

since by (12), $\Delta_{\bar{p}}(\sigma_2,\omega) \le \Delta_{\bar{p}}(\sigma_1,\omega)$. Thus, by (14), we have $TF_{\bar{p},\bar{p}+1}\,(\sigma_2,\omega) \le {}_s TF_{\bar{p},\bar{p}+1}(\sigma_1,\omega)$.

CASE 3. Suppose that $\max\{M_{\bar{p}-1}(\sigma_2,\omega), \Delta_{\bar{p}}(\sigma_2,\omega), \Delta_{\bar{p}+1}(\sigma_2,\omega)\} = \Delta_{\bar{p}-1}(\sigma_2,\omega)$. Then, it follows from (14) that

$$\begin{aligned} TF_{\bar{p},\bar{p}+1}(\sigma_2,\omega) - TF_{\bar{p},\bar{p}+1}(\sigma_1,\omega) &\le_s [t_{i2} + TD_2(K_{\bar{p}-1,2}, K_{\bar{p}-1,2} + t_{i2}) + \Delta_{\bar{p}+1}(\sigma_2,\omega)] \\ &\quad - [t_{j2} + TD_2\,(K_{\bar{p}-1,2}, K_{\bar{p}-1,2} + t_{j2}) \\ &\quad + \max\{M_{\bar{p}-1}(\sigma_1,\omega), \Delta_{\bar{p}}(\sigma_1,\omega), \Delta_{\bar{p}+1}(\sigma_1,\omega)\}] \\ &\le [t_{i2} + TD_2(K_{\bar{p}-1,2}, K_{\bar{p}-1,2} + t_{i2}) + \Delta_{\bar{p}+1}(\sigma_2,\omega)] \\ &\quad - [t_{j2} + TD_2\,(K_{\bar{p}-1,2}, K_{\bar{p}-1,2} + t_{j2}) + \Delta_{\bar{p}+1}(\sigma_1,\omega)]\,. \end{aligned} \tag{15}$$

Now, putting the values of $\Delta_{\bar{p}+1}(\sigma_2,\omega)$ and $\Delta_{\bar{p}+1}(\sigma_1,\omega)$ in (15), we get $TF_{\bar{p},\bar{p}+1}(\sigma_2,\omega) \leq_s TF_{\bar{p},\bar{p}+1}(\sigma_1,\omega)$.

As a result of the three cases considered, we have $TF(\sigma_2,\omega) \leq_s TF(\sigma_1,\omega)$. Since no assumption has been made with regard to realization ω, this result is true for any ω. Hence, $TF(\sigma_2) \leq_s TF(\sigma_1)$. Repeat this argument until SPT is obtained. ∎

THEOREM 3B. *Under the assumptions in Lemma 1, SPT sequence stochastically minimizes mean flowtime for two-machine, Type B, ordered flowshops when both machines suffer random breakdowns.*

PROOF. From (9) and (10), we have

$$\begin{aligned}\Delta_{\bar{p}+1}(\sigma_2,\omega) - \Delta_{\bar{p}}(\sigma_1,\omega) = t_{i1} - t_{i2} + TD_1(K_{\bar{p}-1,1} + t_{j1},\ K_{\bar{p}-1,1} + t_{i1} + t_{j1})\\ - TD_2(K_{\bar{p}-1,2},\ K_{\bar{p}-1,2} + t_{i2}).\end{aligned}$$

Since the ordered flowshop is Type B, it follows that $t_{i1} \leq t_{i2}$. Now by Lemma 1,

$$\Delta_{\bar{p}+1}(\sigma_2,\omega) \leq_s \Delta_{\bar{p}}(\sigma_1,\omega). \tag{16}$$

From (12) and (16) we have,

$$\max\{\Delta_{\bar{p}}(\sigma_2,\omega), \Delta_{\bar{p}+1}(\sigma_2,\omega)\} \leq_s \max\{\Delta_{\bar{p}}(\sigma_1,\omega), \Delta_{\bar{p}+1}(\sigma_1,\omega)\}. \tag{17}$$

Now, notice that by (12) and (13),

$$M_{\bar{p}}(\sigma_2,\omega) \leq M_{\bar{p}}(\sigma_1,\omega) \tag{18}$$

and by (17) and (13),

$$M_{\bar{p}+1}(\sigma_2,\omega) \leq_s M_{\bar{p}+1}(\sigma_1,\omega). \tag{19}$$

It follows from the definition of Δ_p that

$$\Delta_p(\sigma_2,\omega) = \Delta_p(\sigma_1,\omega), \qquad \text{for } p = \bar{p}+2,\dots,n. \tag{20}$$

Now, note that for $p = \bar{p}+2,\dots,n$,

$$\begin{aligned}M_p(\sigma_2,\omega) - M_p(\sigma_1,\omega) =& \max\{M_{\bar{p}-1}(\sigma_2,\omega), \Delta_{\bar{p}}(\sigma_2,\omega), \Delta_{\bar{p}+1}(\sigma_2,\omega),\dots,\Delta_p(\sigma_2,\omega)\}\\ &- \max\{M_{\bar{p}-1}(\sigma_1,\omega), \Delta_{\bar{p}}(\sigma_1,\omega), \Delta_{\bar{p}+1}(\sigma_1,\omega),\dots,\Delta_p(\sigma_1,\omega)\}.\end{aligned}$$

By (13) and (20), we have

$$M_p(\sigma_2,\omega) - M_p(\sigma_1,\omega) = \max\{\Delta_{\bar{p}}(\sigma_2,\omega), \Delta_{\bar{p}+1}(\sigma_2,\omega)\} - \max\{\Delta_{\bar{p}}(\sigma_1,\omega), \Delta_{\bar{p}+1}(\sigma_1,\omega)\}.$$

Now, it follows from (17) that

$$M_p(\sigma_2,\omega) \leq_s M_p(\sigma_1,\omega), \qquad \text{for } p = \bar{p}+2,\dots,n. \tag{21}$$

Notice that by (13)

$$R_p(\sigma_2,\omega) = R_p(\sigma_1,\omega), \qquad \text{for } p = 1,\dots,\bar{p}-1, \tag{22}$$

and, by (21),

$$R_p(\sigma_2,\omega) \leq_s R_p(\sigma_1,\omega), \qquad \text{for } p = \bar{p}+2,\dots,n. \tag{23}$$

It follows from (22) and (23) that

$$\begin{aligned}TF(\sigma_2,\omega) - TF(\sigma_1,\omega) \leq_s& R_{\bar{p}}(\sigma_2,\omega) + R_{\bar{p}+1}(\sigma_2,\omega) - R_{\bar{p}}(\sigma_1,\omega) - R_{\bar{p}+1}(\sigma_1,\omega)\\ =& t_{i2} + TD_2(0, K_{\bar{p}-1,2} + t_{i2}) + M_{\bar{p}}(\sigma_2,\omega) + M_{\bar{p}+1}(\sigma_2,\omega)\\ &- t_{j2} - TD_2(0, K_{\bar{p}-1,2} + t_{j2}) - M_{\bar{p}}(\sigma_1,\omega) - M_{\bar{p}+1}(\sigma_1,\omega)\\ \leq_s& 0,\end{aligned}$$

since $t_{i2} \leq t_{j2}$ and by (18) and (19). Thus, $TF(\sigma_2,\omega) \leq_s TF(\sigma_1,\omega)$. This result holds for any ω since no assumption has been made with regard to realization ω. Therefore, $TF(\sigma_2) \leq_s TF(\sigma_1)$. Repeat this argument until SPT is obtained. ∎

5. CONCLUSIONS

In this paper, we have discussed the two-machine flowshop problem under stochastic machine breakdowns with makespan and mean flowtime objectives. Without any assumptions on breakdown distributions, we have shown that for Types A and B ordered flowshops, LPT and SPT sequences, respectively, minimizes makespan w.p.1 when only the first and the second machine suffers breakdowns. We have also shown that when both machines are subject to random breakdowns, with certain conditions on breakdown distributions, SPT sequence stochastically minimizes mean flowtime. Further research may address the problem with other objective functions such as tardiness. It remains a challenge to extend the stochastic machine problem to a jobshop environment or to a flowshop environment with more than two machines.

APPENDIX

PROOF OF THEOREM 2

Since the first machine is not subject to breakdowns, it follows from (2) that

$$\Delta_p = K_{p-1,1} + t_{[p1]} - K_{p-1,2} - TD_2\left(0, K_{p-1,2}\right).$$

Consider the two job sequences σ_1 and σ_2 described in the proof of Theorem 1 where now $t_{j1} > t_{i1}$. We shall prove that SPT minimizes M_n w.p.1 by showing that sequence σ_2 is no worse than sequence σ_2, i.e., $M_n(\sigma_2) \leq M_n(\sigma_2)$ w.p.1. Let ω be a given realization of uptimes and downtimes. For the same argument made in he proof of Theorem 1, in order to show that $M_n(\sigma_2, \omega) \leq M_n(\sigma_1, \omega)$ it is sufficient to show that $\max\left\{\Delta_{\bar{p}}(\sigma_2, \omega), \Delta_{\bar{p}+1}(\sigma_2, \omega)\right\} \leq \max\left\{\Delta_{\bar{p}}(\sigma_1, \omega), \Delta_{\bar{p}+1}(\sigma_1, \omega)\right\}$.

Notice that

$$\begin{aligned}
\Delta_{\bar{p}}\left(\sigma_1, \omega\right) &= K_{\bar{p}-1,1} + t_{j1} - K_{\bar{p}-1,2} - TD_2\left(0, K_{\bar{p}-1,2}\right), \\
\Delta_{\bar{p}+1}(\sigma_1, \omega) &= K_{\bar{p}-1,1} + t_{j1} + t_{i1} - K_{\bar{p}-1,2} - t_{j2} - TD_2\left(0, K_{\bar{p}-1,1} + t_{j2}\right), \\
\Delta_{\bar{p}}\left(\sigma_2, \omega\right) &= K_{\bar{p}-1,1} + t_{i1} - K_{\bar{p}-1,2} - TD_2\left(0, K_{\bar{p}-1,1}\right), \\
\Delta_{\bar{p}+1}\left(\sigma_2, \omega\right) &= K_{\bar{p}-1,1} + t_{i1} + t_{j1} - K_{\bar{p}-1,2} - t_{i2} - TD_1\left(0, K_{\bar{p}-1,1} + t_{i2}\right).
\end{aligned}$$

Notice that

$$\Delta_{\bar{p}}\left(\sigma_2, \omega\right) - \Delta_{\bar{p}}\left(\sigma_1, \omega\right) = t_{i1} - t_{j1} < 0, \quad \text{since } t_{i1} < t_{j1}, \quad \text{hence, } \Delta_{\bar{p}}\left(\sigma_2, \omega\right) < \Delta_{\bar{p}}(\sigma_1, \omega). \tag{24}$$

Also notice that

$$\Delta_{\bar{p}+1}(\sigma_2, \omega) - \Delta_{\bar{p}}(\sigma_1, \omega) = t_{i1} - t_{i2} + TD_2\left(0, K_{\bar{p}-1,2}\right) - TD_1\left(0, K_{\bar{p}-1,1} + t_{i2}\right) \leq 0,$$

since $t_{i1} \leq t_{i2}$, which follows from the fact that it is the Type B, and $TD_2\left(0, K_{\bar{p}-1,2}\right) \leq TD_1\left(0, K_{\bar{p}-1,1} + t_{i2}\right)$, therefore,

$$\Delta_{\bar{p}+1}(\sigma_2, \omega) \leq \Delta_{\bar{p}}(\sigma_1, \omega). \tag{25}$$

By (24) and (25) we have

$$\max\left\{\Delta_{\bar{p}}(\sigma_2, \omega), \Delta_{\bar{p}+1}(\sigma_2, \omega)\right\} \leq \max\left\{\Delta_{\bar{p}}(\sigma_1, \omega), \Delta_{\bar{p}+1}(\sigma_1, \omega)\right\}.$$

Thus $M_n(\sigma_2, \omega) \leq M_n(\sigma_1, \omega)$. Since this argument is valid for any ω, $M_n(\sigma_2) \leq M_n(\sigma_1)$ w.p.1. Therefore, σ_2 is also an optimal sequence and is closer to SPT than σ_1. Repeat this argument until a SPT sequence is achieved. Then, SPT minimizes the makespan w.p.1. ∎

REFERENCES

1. S.M. Johnson, Optimal two and three-stage production schedules with setup times included, *Naval Research Logistics Quarterly* **1**, 61–68 (1954).
2. M.R. Garey, D.S. Johnson, and R. Sethi, The complexity of flowshop and jobshop scheduling, *Mathematics of Operations Research* **1**, 117–129 (1976).
3. T. Gonzalez and S. Sahni, Flow shop and job shop schedules, *Operations Research* **26**, 36–52 (1978).
4. M.L. Smith, S.S. Panwalkar, and R.A. Dudek, Flowshop sequencing problem with ordered processing time matrices, *Management Science* **21**, 544–549 (1975).
5. S.S. Panwalkar and A.W. Khan, An ordered flow-shop sequencing problem with mean completion time criterion, *International Journal of Production Research* **30**, 631–635 (1976).
6. A. Allahverdi and J. Mittenthal, Scheduling on a two-machine flowshop subject to random breakdowns with makespan objective function, *European Journal of Operational Research* (to appear).