

## Research Article

# Asymptotic and Numerical Methods in Estimating Eigenvalues

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Asymptotic formulas and numerical estimations for eigenvalues of Sturm-Liouville problems having singular potential functions, with Dirichlet boundary conditions, are obtained. This study gives a comparison between the eigenvalues obtained by the asymptotic and the numerical methods.

## 1. Introduction

Let  $L(q)$  be an operator generated in  $L_2[0, 1]$  by the expression

$$L(q) = -y''(x) + q(x)y(x), \quad 0 \leq x \leq 1, \quad (1)$$

and by Dirichlet boundary conditions

$$y(0) = y(1) = 0, \quad (2)$$

where  $q(x)$  is a complex-valued summable function.

In this paper, we consider the small and large eigenvalues of the operator  $L(q)$  when  $q(x)$  has a finite number of singularities. The large eigenvalues are investigated by the asymptotic method given in [1, 2]. Note that in classical investigations in order to obtain the asymptotic formulas of order  $O(n^{-1})$  it is required that  $q(x)$  be  $(l-1)$  times differentiable (see [3–10]). The method of [1] gives the possibility of obtaining the asymptotic formulas of order  $O(n^{-1})$  of eigenvalues and eigenfunctions of  $L(q)$  when  $q(x)$  is an arbitrary summable complex-valued function. The small eigenvalues are investigated by numerical and asymptotic methods. Then, we compare the results with the ones obtained by the other methods.

Expression of differential equations in matrix form and the advances in the field of the computers have led to major developments in numerical methods. Regarding the

numerical solution of the Sturm-Liouville problems, finite difference method is amongst the popular methods (see [11, 12]). Finite difference method can give effective results for the eigenvalues when it is used in connection with asymptotic correction technique. In [13] and [14] the Sturm-Liouville problems with Dirichlet and the general boundary conditions were studied, respectively. Andrew and Paine [15] found the approximate eigenvalues of regular Sturm-Liouville problem by using the finite element method. Chen and Ho [16] used the differential transform method to solve the eigenvalue problems. Ghelardoni [17] named some linear multistep methods as boundary value methods and found the approximate eigenvalues of Sturm-Liouville problem. Ghelardoni and Gheri [18] used the shooting technique for the calculation of the eigenvalues of Sturm-Liouville problem by considering the Prüfer transformation given in [19]. Kumar [20], Kumar and Aziz [21] gave numerical examples to linear or nonlinear boundary value problems by using finite differences method for singular boundary value problems. Kumar and Singh [22] made a study which collected and classified various calculation techniques for the solution of singular boundary value problems.

## 2. Asymptotic Formulas for Eigenvalues

It is well known that (see formulas (47a), (47b) in page 65 of [7]) the eigenvalues of the operator  $L(q)$ , where  $q(x)$  is

a complex-valued summable function, consist of the sequence  $\{\lambda_n\}$  satisfying

$$\lambda_n = (n\pi)^2 + O(1). \quad (3)$$

In [1] (see Theorem 1 of [1]), it is proved that the eigenvalues  $\lambda_n$  satisfy the following formula

$$\lambda_n = (n\pi)^2 + C_0 + F_m + O\left(\left(\frac{\ln |n|}{n}\right)^{m+1}\right), \quad (4)$$

$$\forall m = 0, 1, 2, \dots,$$

where  $F_0 = -C_{2n}$ ,  $C_n = \int_0^1 q(x) \cos n\pi x dx$ ,

$$F_1 = A_1((n\pi)^2) = -C_{2n} + \sum_{\substack{n_1=-\infty \\ n_1 \neq -2n}}^{\infty} \frac{C_{n_1}(C_{n_1} - C_{n_1+2n})}{[(n\pi)^2 - (\pi(n+n_1))^2]}, \quad (5)$$

$$F_k = A_k((n\pi)^2 + F_{k-1}), \quad \forall k = 2, 3, \dots \quad (6)$$

Note that in [1], without loss of generality, it was assumed that  $C_0 = 0$ . Then using (4), the cases  $q(x) = p(x) + c/x^\alpha$  and  $q(x) = a_0/\sqrt{x} + a_1/\sqrt{1-x} + b_0\sqrt{x} + b_1\sqrt{1-x}$  (where  $c, a_0, a_1, b_0, b_1$  are complex numbers) are investigated in detail.

In this paper, we consider the case

$$q(x) = \sum_{k=0}^v \frac{c_k}{|x - t_k|^{\alpha_k}}, \quad 0 < \alpha_k < 1, \quad (7)$$

where  $v$  is a positive integer and  $c_k$  is a complex number. First using (4) we prove the following.

**Theorem 1.** *The eigenvalue  $\lambda_n$  of the operator  $L(q)$  with potential (7) satisfies the asymptotic formula:*

$$\begin{aligned} \lambda_n &= (n\pi)^2 + C_0 \\ &- \sum_{k=0}^v \frac{c_k}{(2n)^{1-\alpha_k}} (\cos 2n\pi t_k (d_{4k} + d_{4k+2}) \\ &- \sin 2n\pi t_k (d_{4k+1} + d_{4k+3})) \\ &+ O\left(\frac{\ln |n|}{n}\right), \end{aligned} \quad (8)$$

where

$$\begin{aligned} d_{4k} &= \int_0^\infty \frac{\cos \pi s}{s^{\alpha_k}} ds, & d_{4k+2} &= \int_{-\infty}^0 \frac{\cos \pi s}{s^{\alpha_k}} ds, \\ d_{4k+1} &= \int_{-\infty}^0 \frac{\sin \pi s}{s^{\alpha_k}} ds, & d_{4k+3} &= \int_0^\infty \frac{\sin \pi s}{s^{\alpha_k}} ds. \end{aligned} \quad (9)$$

*Proof.* At (4) for  $m = 0$ , let us use the formula

$$\lambda_n = (n\pi)^2 + C_0 - C_{2n} + O\left(\frac{\ln |n|}{n}\right), \quad (10)$$

where

$$C_n = \int_0^1 q(x) \cos n\pi x dx = \sum_{k=0}^v \int_0^1 \frac{c_k \cos n\pi x}{(x - t_k)^{\alpha_k}} dx. \quad (11)$$

In the last equality, using the transformations  $t = x - t_k$  and  $nt = s$  we obtain

$$\begin{aligned} C_n &= \sum_{k=0}^v \int_{-t_k}^{1-t_k} \frac{c_k \cos n\pi (t + t_k)}{t_k^{\alpha_k}} dt \\ &= \sum_{k=0}^v \int_{-t_k}^{1-t_k} \frac{c_k \cos n\pi t \cos n\pi t_k}{t_k^{\alpha_k}} dt \\ &- \sum_{k=0}^v \int_{-t_k}^{1-t_k} \frac{c_k \sin n\pi t \sin n\pi t_k}{t_k^{\alpha_k}} dt, \quad (12) \\ C_n &= \sum_{k=0}^v \frac{c_k}{n^{1-\alpha_k}} \left( \cos n\pi t_k \int_{-nt_k}^{n(1-t_k)} \frac{\cos \pi s}{s^{\alpha_k}} ds \right. \\ &- \left. \sin n\pi t_k \int_{-nt_k}^{n(1-t_k)} \frac{\sin \pi s}{s^{\alpha_k}} ds \right). \end{aligned}$$

Let  $n(1 - t_k) = n_k$ . By (9) we have

$$\begin{aligned} C_n &= \sum_{k=0}^v \frac{c_k}{n^{1-\alpha_k}} \left( \cos n\pi t_k \left( d_{4k} - \int_{n_k}^\infty \frac{\cos \pi s}{s^{\alpha_k}} ds \right. \right. \\ &+ d_{4k+2} - \int_{-\infty}^{-nt_k} \frac{\cos \pi s}{s^{\alpha_k}} ds \Big) \\ &- \sin n\pi t_k \left( d_{4k+1} - \int_{-\infty}^{-nt_k} \frac{\sin \pi s}{s^{\alpha_k}} ds \right. \\ &+ d_{4k+3} - \int_{n_k}^\infty \frac{\sin \pi s}{s^{\alpha_k}} ds \Big) \Big). \end{aligned} \quad (13)$$

Arguing as in the proof of (41) of [1] one can readily see that

$$\begin{aligned} \int_{-\infty}^{-nt_k} \frac{\cos \pi s}{s^{\alpha_k}} ds &= O\left(\frac{1}{n_k^{\alpha_k}}\right), & \int_{n_k}^\infty \frac{\cos \pi s}{s^{\alpha_k}} ds &= O\left(\frac{1}{n_k^{\alpha_k}}\right), \\ \int_{-\infty}^{-nt_k} \frac{\sin \pi s}{s^{\alpha_k}} ds &= O\left(\frac{1}{n_k^{\alpha_k}}\right), & \int_{n_k}^\infty \frac{\sin \pi s}{s^{\alpha_k}} ds &= O\left(\frac{1}{n_k^{\alpha_k}}\right). \end{aligned} \quad (14)$$

Therefore

$$C_n = \sum_{k=0}^v \frac{c_k}{n^{1-\alpha_k}} (\cos n\pi t_k (d_{4k} + d_{4k+2}) - \sin n\pi t_k (d_{4k+1} + d_{4k+3})) + O\left(\frac{1}{n}\right). \quad (15)$$

Thus (8) follows from (4) for  $m = 0$ . The theorem is proved.  $\square$

Now assuming that

$$\alpha_1 = \alpha_2 = \dots = \alpha_v = \frac{1}{2} \quad (16)$$

we obtain more precise asymptotic formula by using more subtle estimations.

**Theorem 2.** *If (16) holds, then the eigenvalue  $\lambda_n$  of the operator  $L(q)$  with potential (7) satisfies the asymptotic formula:*

$$\lambda_n = (n\pi)^2 + C_0 - C_{2n} + O\left(\left(\frac{\ln |n|}{n}\right)^2\right). \quad (17)$$

*Proof.* To prove the theorem we use (4) for  $m = 1$ , (5) and prove that

$$\sum_{\substack{n_1=-\infty \\ n_1 \neq 0, -2n}}^{\infty} \frac{C_{n_1}^2}{n_1 (2n + n_1)} = O\left(\left(\frac{\ln |n|}{n}\right)^2\right), \quad (18)$$

$$\sum_{\substack{n_1=-\infty \\ n_1 \neq 0, -2n}}^{\infty} \frac{C_{n_1} C_{n_1+2n}}{n_1 (2n + n_1)} = O\left(\left(\frac{\ln |n|}{n}\right)^2\right).$$

In (15), instead of  $n$  and  $\alpha_k$  taking  $n_1$  and  $1/2$  we get

$$C_{n_1} = \sum_{k=0}^v \frac{c_k}{(n_1)^{1/2}} (\cos n_1 \pi t_k (d_{4k} + d_{4k+2}) - \sin n_1 \pi t_k (d_{4k+1} + d_{4k+3})) + O\left(\frac{1}{n_1}\right). \quad (19)$$

From (19) one can readily see that there exists a constant  $r$  such that

$$|C_{n_1}|^2 < r \frac{1}{n_1} \quad (20)$$

for  $n_1 = 1, 2, \dots$ . Therefore, instead of equation (56) of [1], using (20) and repeating the proof of equation (55) of [1] we get the proof of (18). Thus the proof of the theorem follows from (4), (5), and (18). The theorem is proved.  $\square$

### 3. Numerical Approximation

Now, we consider the small eigenvalues of the  $L(q)$  operator by a numerical method.

For the finite difference method [11, 19] take an equally spaced mesh ( $m \geq 2$ )

$$0 = x_0 < x_1 < \dots < x_{m+1} = 1, \quad (21)$$

where

$$x_j = jh, \quad h = \frac{1}{m+1}. \quad (22)$$

Writing  $y(x_j)$  as  $y_j$ ,  $q(x_j)$  as  $q_j$ , and  $y''(x_j)$  as  $y_j''$ , we use the centered difference approximation

$$-y_j'' \approx \frac{-y_{j-1} + 2y_j - y_{j+1}}{h^2}. \quad (23)$$

Substituting in (1) we obtain the approximating scheme

$$\frac{-Y_{j-1} + 2Y_j - Y_{j+1}}{h^2} + q_j Y_j = \Lambda Y_j, \quad j = 1, 2, \dots, m. \quad (24)$$

Incorporating the boundary conditions, we get

$$Y_0 = 0, \quad Y_{m+1} = 0. \quad (25)$$

This can be written in matrix form as

$$TY = \Lambda Y, \quad (26)$$

where

$$T = \frac{1}{h^2} K + Q^*, \quad (27)$$

$T$  is a tridiagonal matrix and

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix},$$

$$K = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}, \quad (28)$$

$$Q^* = \begin{pmatrix} q_1 & & & & \\ & q_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & q_m \end{pmatrix}.$$

The eigenvalues of (1), (2) are approximated by the eigenvalues of matrix  $T$ .

In the previous section, the asymptotic formulas for eigenvalues of the operator  $L(q)$  (1), (2) with the potential (7) are investigated. In this section, we will find the eigenvalues of the operator  $L(q)$  by using the finite difference method when  $c_k = 1$ ,  $\alpha_k = 1/2$  for  $k = 0, 1, \dots, \nu$ , and  $t_k = k/\nu$ . Let us introduce the notation

$$q_k(x) = \frac{1}{|x - t_k|^{1/2}} \quad (29)$$

and denote the  $n$ th eigenvalue of the operator  $L(q_k)$  by  $\lambda_n^k$ . The  $n$ th eigenvalue of the operator  $L(Q_\nu)$ , where

$$Q_\nu(x) = \sum_{k=0}^{\nu} q_k(x), \quad (30)$$

is denoted by  $\Lambda_n^\nu$ .

In order to be able to apply the Finite Difference method, the nodes should not coincide with the singular points. Let  $m_\nu = \nu(m+1)$  and  $x_j$  nodal points be

$$x_j = \frac{j}{m_\nu}, \quad j = 1, 2, \dots, m_\nu - 1; \quad j \neq k(m+1). \quad (31)$$

Then  $x_j \neq t_k$ .

The approximate eigenvalues of the operators  $L(q_k)$  and  $L(Q_\nu)$  obtained by the numerical method are denoted  $s_n^k$  and  $S_n^\nu$ , respectively.

*Example 3.* In this example we find the eigenvalues of the following boundary value problem

$$\begin{aligned} -y'' + \sum_{k=0}^5 \frac{1}{|(x - t_k)|^{1/2}} y &= \lambda y, \\ y(0) &= y(1) = 0 \end{aligned} \quad (32)$$

for  $m_\nu = 150$ ,  $t_k = k/\nu$ , and  $\nu = 5$  by using Finite Difference method. In Table 1 an example of the computation of the eigenvalues of the operators  $L(q_k)$  and  $L(Q_\nu)$  is given.

One can see from Table 1 that for  $n \geq 30$  the eigenvalues of the operators  $L(q_k)$  and  $L(Q_\nu)$  are close to each other. This shows that the effect of the potential to the large eigenvalues is small. Moreover the eigenvalues in first, second, and third columns coincide with the eigenvalues in the sixth, fifth, and fourth columns, respectively, since the potential  $q_k(x)$  can be reduced to  $q_{\nu-k}(x)$  by using the transformation  $x = 1 - z$ .

#### 4. Comparison of the Asymptotic and Numerical Methods

In this section we compare the estimations obtained by numerical and asymptotic methods of the eigenvalues  $\Lambda_n^2$  of the operator  $L(Q_2)$ , where  $Q_2$  is defined by (30) and (29). The  $n$ th eigenvalue of the operator  $L(0)$  is  $(n\pi)^2$ . The effect of the potential  $q_k$  on the  $n$ th eigenvalue  $\lambda_n^k$  of the operator  $L(q_k)$ ,

that is, the perturbation of the  $n$ th eigenvalue when  $L(0)$  is perturbed by  $q_k$  is

$$p_n^k = \lambda_n^k - (n\pi)^2. \quad (33)$$

Similarly, the effect of  $Q_2$  on the  $n$ th eigenvalue  $\Lambda_n^2$ , that is, the perturbation of the  $n$ th eigenvalue when  $L(0)$  is perturbed by  $Q_2$  is

$$P_n^2 = \Lambda_n^2 - (n\pi)^2. \quad (34)$$

The perturbations  $P_n^2$ ,  $p_n^k$  evaluated by the numerical and asymptotic methods are denoted by  $P_n^2(s)$ ,  $p_n^k(s)$ ,  $P_n^2(a)$ , and  $p_n^k(a)$ , respectively.

According to Theorem 2 we define the approximate eigenvalues, denoted by  $a_n^k$  and  $A_n^2$ , of the operators  $L(q_k)$  and  $L(Q_2)$  obtained by the asymptotic method as follows

$$a_n^k = (n\pi)^2 + \int_0^1 q_k(x) dx - \int_0^1 q_k(x) \cos 2\pi n x dx, \quad (35)$$

$$A_n^2 = (n\pi)^2 + \int_0^1 Q_2(x) dx - \int_0^1 Q_2(x) \cos 2\pi n x dx. \quad (36)$$

Therefore it is natural to define  $P_n^2(a)$  and  $p_n^k(a)$  by

$$\begin{aligned} P_n^2(a) &= A_n^2 - (n\pi)^2 = \int_0^1 Q_2(x) dx - \int_0^1 Q_2(x) \cos 2\pi n x dx, \\ p_n^k(a) &= a_n^k - (n\pi)^2 = \int_0^1 q_k(x) dx - \int_0^1 q_k(x) \cos 2\pi n x dx. \end{aligned} \quad (37)$$

It readily follows from formulas (37) and (30) that

$$P_n^2(a) = \sum_{k=0}^2 p_n^k(a). \quad (38)$$

It means that for the large eigenvalues the effect of  $Q_2$  is asymptotically equal to the sum of the effects of the potentials  $q_k$ .

The perturbations  $P_n^2(s)$  and  $p_n^k(s)$  evaluated via the finite difference method are given in Table 2. In order to see the effect of the singular points, the number of subintervals is taken as  $m_\nu = 20000$ .

Table 2 shows that the effect of  $Q_2$  is approximately within the value range of  $5 \cdot 10^{-5}$  and  $2 \cdot 10^{-2}$ , equal to the sum of the effects of the potentials  $q_k$ . Thus the perturbation estimations by the numerical methods validate the naturality of (38).

TABLE 1

| $n$ | $s_n^0$ | $s_n^1$  | $s_n^2$ | $s_n^3$ | $s_n^4$  | $s_n^5$ | $s_n^5$ |
|-----|---------|----------|---------|---------|----------|---------|---------|
| 1   | 11,3346 | 12,0617  | 13,6112 | 13,6112 | 12,0617  | 11,3346 | 24,8946 |
| 2   | 41,0082 | 42,3680  | 42,5655 | 42,5655 | 42,3680  | 41,0082 | 54,4909 |
| 3   | 90,3543 | 92,1439  | 91,4556 | 91,4556 | 92,1439  | 90,3543 | 103,912 |
| 4   | 159,389 | 161,282  | 161,330 | 161,330 | 161,282  | 159,389 | 174,822 |
| 5   | 248,089 | 249,774  | 249,278 | 249,278 | 249,774  | 248,089 | 261,654 |
| 6   | 356,418 | 357,740  | 357,883 | 357,883 | 357,740  | 356,418 | 369,832 |
| 7   | 484,332 | 485,347  | 486,023 | 486,023 | 485,347  | 484,332 | 497,594 |
| 8   | 631,775 | 632,690  | 632,941 | 632,941 | 632,690  | 631,775 | 643,821 |
| 9   | 789,683 | 799,727  | 800,334 | 800,334 | 799,727  | 789,683 | 811,972 |
| 10  | 984,984 | 986,288  | 986,440 | 986,440 | 986,288  | 984,984 | 998,400 |
| 20  | 3892,79 | 3864,330 | 3894,39 | 3894,39 | 3864,330 | 3892,79 | 3906,97 |
| 30  | 8599,57 | 8600,85  | 8601,04 | 8601,04 | 8600,85  | 8599,57 | 8612,95 |
| 40  | 14902,3 | 14903,5  | 14903,7 | 14903,7 | 14903,5  | 14902,3 | 14915,3 |
| 50  | 22529,2 | 22530,6  | 22530,7 | 22530,7 | 22530,6  | 22529,2 | 22542,7 |
| 60  | 31151,4 | 31152,8  | 31152,8 | 31152,8 | 31152,8  | 31151,4 | 31165,1 |
| 70  | 40396,8 | 40398    | 40398,3 | 40398,3 | 40398    | 40396,8 | 40410,1 |
| 80  | 49866,8 | 49867,9  | 49868,2 | 49868,2 | 49867,9  | 49866,8 | 49879,9 |
| 90  | 59152,9 | 59154,3  | 59154,3 | 59154,3 | 59154,3  | 59152,9 | 59166,4 |
| 100 | 67854,6 | 67856,1  | 67856,1 | 67856,1 | 67856,1  | 67854,6 | 67868,3 |

TABLE 2

| $n$ | $P_n^2(s)$  | $p_n^0(s)$  | $p_n^1(s)$  | $p_n^2(s)$  | $\sum_{k=0}^2 p_n^k(s)$ | $ p_n^2(s) - \sum_{k=0}^2 p_n^k(s) $ |
|-----|-------------|-------------|-------------|-------------|-------------------------|--------------------------------------|
| 1   | 6,853010948 | 1,507680075 | 3,819522987 | 1,507680075 | 6,834883138             | 0,018127810                          |
| 2   | 5,436436039 | 1,649044169 | 2,135885787 | 1,649044169 | 5,433974124             | 0,002461915                          |
| 3   | 6,833127542 | 1,712792234 | 3,412323071 | 1,712792234 | 6,837907540             | 0,004779998                          |
| 4   | 5,833596917 | 1,750983257 | 2,332560213 | 1,750983257 | 5,834526726             | 0,000929810                          |
| 5   | 6,821354115 | 1,777106646 | 3,269640370 | 1,777106646 | 6,823853662             | 0,002499546                          |
| 6   | 6,014222685 | 1,796412736 | 2,422163800 | 1,796412736 | 6,014989272             | 0,000766587                          |
| 7   | 6,817074514 | 1,811422205 | 3,195628783 | 1,811422205 | 6,818473194             | 0,001398680                          |
| 8   | 6,122590551 | 1,823514738 | 2,476005854 | 1,823514738 | 6,123035330             | 0,000444779                          |
| 9   | 6,814993899 | 1,833516325 | 3,148677685 | 1,833516325 | 6,815710336             | 0,000716437                          |
| 10  | 6,196675434 | 1,841954226 | 2,512821873 | 1,841954226 | 6,196730326             | 0,000054891                          |
| 20  | 6,378247333 | 1,885045278 | 2,601825385 | 1,885045278 | 6,371915940             | 0,006331392                          |

It is well known that if we consider the Sturm-Liouville operator

$$L(\varepsilon Q_2) = -\frac{d}{dx^2} + \varepsilon Q_2(x), \quad (39)$$

$$y(0) = y(1) = 0,$$

where  $\varepsilon$  is a small positive parameter, then the asymptotic methods can be applied more successfully. The  $n$ th eigenvalue of the operators  $L(\varepsilon Q_2)$  is denoted by  $\Lambda_n^\nu(\varepsilon)$ . The approximate eigenvalues obtained by the asymptotic and numerical methods are denoted by  $A_n^\nu(\varepsilon)$  and  $S_n^\nu(\varepsilon)$ , respectively.

It follows from Theorem 2 and formulas (36), (30) that

$$\Lambda_n^2(\varepsilon) = A_n^2(\varepsilon) + O\left(\left(\frac{\varepsilon \ln |n|}{n}\right)^2\right), \quad (40)$$

where

$$A_n^2(\varepsilon) = (\pi n)^2 + \varepsilon \int_0^1 Q_2(x) dx - \varepsilon C_{2n}$$

$$= (\pi n)^2 + \varepsilon \int_0^1 \left( \frac{1}{\sqrt{|x-1|}} + \frac{1}{\sqrt{|x-1/2|}} + \frac{1}{\sqrt{|x|}} \right) dx$$

$$- \varepsilon \int_0^1 \left( \frac{1}{\sqrt{|x-1|}} + \frac{1}{\sqrt{|x-1/2|}} + \frac{1}{\sqrt{|x|}} \right) \cos 2n\pi x dx. \quad (41)$$

In Tables 3, 4, and 5 the approximate eigenvalues  $A_n^2(\varepsilon)$  obtained by the asymptotic method and their comparison

TABLE 3

| $n$ | $\lambda_n(0)$ | $A_n^2(1)$  | $S_n^2(1)$ | $ A_n^2(1) - S_n^2(1) $ | $ A_n^2(1) - \lambda_n(0) $ | $ S_n^2(1) - \lambda_n(0) $ |
|-----|----------------|-------------|------------|-------------------------|-----------------------------|-----------------------------|
| 1   | 9,8696         | 16,779308   | 16,7201    | 0,059208                | 6,909708                    | 6,8505                      |
| 2   | 39,4784        | 44,915438   | 44,9149    | 0,000538                | 5,437038                    | 5,4365                      |
| 3   | 88,8264        | 95,665322   | 95,6569    | 0,008422                | 6,838922                    | 6,8305                      |
| 4   | 157,9137       | 163,748052  | 163,7473   | 0,000752                | 5,834352                    | 5,8336                      |
| 5   | 246,7401       | 253,572371  | 253,5588   | 0,013571                | 6,832271                    | 6,8187                      |
| 6   | 355,3058       | 361,320361  | 361,3199   | 0,000461                | 6,014561                    | 6,0141                      |
| 7   | 483,6106       | 490,44101   | 490,4249   | 0,01611                 | 6,83041                     | 6,8143                      |
| 8   | 631,6547       | 637,77751   | 637,7770   | 0,00051                 | 6,12281                     | 6,1223                      |
| 9   | 799,4379       | 806,267576  | 806,2502   | 0,017376                | 6,829676                    | 6,8123                      |
| 10  | 986,9604       | 993,157379  | 993,1570   | 0,000379                | 6,196979                    | 6,1966                      |
| 20  | 3947,8418      | 3954,223216 | 3954,2175  | 0,005216                | 6,381416                    | 6,3757                      |
| 30  | 8882,6440      | 8889,107347 | 8889,0782  | 0,029347                | 6,463347                    | 6,4342                      |
| 40  | 15791,3670     | 15797,8793  | 15797,7870 | 0,0893                  | 6,51236                     | 6,42                        |
| 50  | 24674,0110     | 24680,55663 | 24680,3312 | 0,22663                 | 6,54563                     | 6,3202                      |
| 60  | 35530,5758     | 35537,1461  | 35536,6786 | 0,4661                  | 6,5703                      | 6,1028                      |
| 70  | 48361,0616     | 48367,65097 | 48366,7849 | 0,87097                 | 6,58937                     | 5,7233                      |
| 80  | 63165,4682     | 63172,073   | 63170,5955 | 1,473                   | 6,6048                      | 5,1273                      |
| 90  | 79943,7956     | 79950,41327 | 79948,0466 | 2,36327                 | 6,61767                     | 4,251                       |
| 100 | 98696,0440     | 98702,67245 | 98699,0652 | 3,60245                 | 6,62845                     | 3,0212                      |

TABLE 4

| $n$ | $\lambda_n(0)$ | $A_n^2(0, 1)$ | $S_n^2(0, 1)$ | $ A_n^2(0, 1) - S_n^2(0, 1) $ | $ A_n^2(0, 1) - \lambda_n(0) $ | $ S_n^2(0, 1) - \lambda_n(0) $ |
|-----|----------------|---------------|---------------|-------------------------------|--------------------------------|--------------------------------|
| 1   | 9,8696         | 10,5605748    | 10,5582       | 0,0023748                     | 0,6909748                      | 0,6886                         |
| 2   | 39,4784        | 40,0221196    | 40,0221       | $1,96 \times 10^{-5}$         | 0,5437196                      | 0,5437                         |
| 3   | 88,8264        | 89,5103278    | 89,5085       | 0,0018278                     | 0,6839278                      | 0,6821                         |
| 4   | 157,9137       | 158,4971086   | 158,4971      | $8,6 \times 10^{-6}$          | 0,5834086                      | 0,5834                         |
| 5   | 246,7401       | 247,4233362   | 247,4214      | 0,0019362                     | 0,6832362                      | 0,6813                         |
| 6   | 355,3058       | 355,9072187   | 355,9072      | $1,87 \times 10^{-5}$         | 0,6014187                      | 0,6014                         |
| 7   | 483,6106       | 484,2936551   | 484,2916      | 0,0020551                     | 0,6830551                      | 0,681                          |
| 8   | 631,6547       | 632,2669645   | 632,2668      | 0,0001645                     | 0,6122645                      | 0,6121                         |
| 9   | 799,4379       | 800,1209185   | 800,1187      | 0,0022185                     | 0,6830185                      | 0,6808                         |
| 10  | 986,9604       | 987,580134    | 987,5798      | 0,000334                      | 0,619734                       | 0,6194                         |
| 20  | 3947,8418      | 3948,479906   | 3948,4741     | 0,005806                      | 0,638106                       | 0,6323                         |
| 30  | 8882,6440      | 8883,2903     | 8883,2611     | 0,0291996                     | 0,6463                         | 0,6171                         |
| 40  | 15791,3670     | 15792,01827   | 15791,9259    | 0,0923677                     | 0,65127                        | 0,5589                         |
| 50  | 24674,0110     | 24674,66557   | 24674,4401    | 0,225465                      | 0,65457                        | 0,4291                         |
| 60  | 35530,5758     | 35531,23287   | 35530,7654    | 0,4674694                     | 0,65707                        | 0,1896                         |
| 70  | 48361,0616     | 48361,72051   | 48360,8544    | 0,8661055                     | 0,65891                        | 0,2072                         |
| 80  | 63165,4682     | 63166,12865   | 63164,6511    | 1,4775505                     | 0,66045                        | 0,8171                         |
| 90  | 79943,7956     | 79944,45741   | 79942,0907    | 2,3667109                     | 0,66181                        | 1,7049                         |
| 100 | 98696,0440     | 98696,70685   | 98693,0996    | 3,6072546                     | 0,66285                        | 2,9444                         |

with  $S_n^2(\varepsilon)$  and nonperturbated eigenvalues  $\lambda_n(0) = (n\pi)^2$  for  $\varepsilon = 1$ ,  $\varepsilon = 0, 1$ , and  $\varepsilon = 0, 01$ , respectively, are given.

Table 3 shows the eigenvalues of  $L(\varepsilon Q_2)$  operator obtained by asymptotic method and finite difference method, respectively, for  $\varepsilon = 1$ . Here the number of subintervals is taken as  $m_y = 15000$ .

Table 4 shows the eigenvalues of  $L(\varepsilon Q_2)$  operator obtained by asymptotic method and finite difference method, respectively, for  $\varepsilon = 0, 1$ . Here the number of subintervals is taken as  $m_y = 15000$ .

Table 5 shows the eigenvalues of  $L(\varepsilon Q_2)$  operator obtained by asymptotic method and finite difference method, respectively, for  $\varepsilon = 0, 01$ . Here the number of subintervals is taken as  $m_y = 15000$ .

## 5. Conclusion

It is natural and well known that for small values of the parameter  $\varepsilon$  and for large eigenvalues the asymptotic method

TABLE 5

| $n$ | $\lambda_n(0)$ | $A_n^2(0, 01)$ | $S_n^2(0, 01)$ | $ A_n^2(0, 01) - S_n^2(0, 01) $ | $ A_n^2(0, 01) - \lambda_n(0) $ | $ S_n^2(0, 01) - \lambda_n(0) $ |
|-----|----------------|----------------|----------------|---------------------------------|---------------------------------|---------------------------------|
| 1   | 9,8696         | 9,93870144     | 9,9385         | 0,00020144                      | 0,06910144                      | 0,0689                          |
| 2   | 39,4784        | 39,53278781    | 39,5328        | $1,219 \times 10^{-5}$          | 0,05438781                      | 0,0544                          |
| 3   | 88,8264        | 88,89482843    | 88,8946        | 0,00022843                      | 0,06842843                      | 0,0682                          |
| 4   | 157,9137       | 157,9720142    | 157,9720       | $1,423 \times 10^{-5}$          | 0,0583142                       | 0,0583                          |
| 5   | 246,7401       | 246,8084326    | 246,8082       | 0,00023264                      | 0,0683326                       | 0,0681                          |
| 6   | 355,3058       | 355,3659045    | 355,3659       | $4,46 \times 10^{-6}$           | 0,0601045                       | 0,0601                          |
| 7   | 483,6106       | 483,6789196    | 483,6786       | 0,0003196                       | 0,0683196                       | 0,068                           |
| 8   | 631,6547       | 631,71591      | 631,7158       | 0,00010995                      | 0,06121                         | 0,0611                          |
| 9   | 799,4379       | 799,5062527    | 799,5058       | 0,00045269                      | 0,0683527                       | 0,0679                          |
| 10  | 986,9604       | 987,0224095    | 987,0220       | 0,00040949                      | 0,0620095                       | 0,0616                          |
| 20  | 3947,8418      | 3947,905575    | 3947,8998      | 0,00577499                      | 0,063775                        | 0,058                           |
| 30  | 8882,6440      | 8882,708595    | 8882,6794      | 0,02919485                      | 0,064595                        | 0,0354                          |
| 40  | 15791,3670     | 15791,43216    | 15791,3398     | 0,09236434                      | 0,06516                         | 0,0272                          |
| 50  | 24674,0110     | 24674,07646    | 24673,8510     | 0,22545896                      | 0,06546                         | 0,16                            |
| 60  | 35530,5758     | 35530,64155    | 35530,1740     | 0,46754647                      | 0,06575                         | 0,4018                          |
| 70  | 48361,0616     | 48361,12746    | 48360,2614     | 0,86605935                      | 0,06586                         | 0,8002                          |
| 80  | 63165,4682     | 63165,53422    | 63164,0567     | 1,47751533                      | 0,06602                         | 1,4115                          |
| 90  | 79943,7956     | 79943,86183    | 79941,4951     | 2,36672503                      | 0,06623                         | 2,3005                          |
| 100 | 98696,0440     | 98696,1103     | 98692,5031     | 3,60719526                      | 0,0663                          | 3,5409                          |

gives us approximations with smaller errors. The numerical method, in general, gives better results for smaller eigenvalues. The tables show that the results of the asymptotic method also give quiet acceptable results for small eigenvalues, since  $A_n^2(\varepsilon) - S_n^2(\varepsilon)$  is small.

Therefore we can easily observe that both of two methods give high-precision results for the calculation of the small eigenvalues. Additionally while the perturbation parameter tends to zero both of the methods are enhanced for smaller eigenvalues, but while this fact is limited to  $n = 10$  for the numerical approximation, the enhancement continues for the asymptotic method applied to higher eigenvalues. Thus we can conclude that the asymptotic method coupled with a perturbation parameter near to zero provides us a better approximation quality in calculating eigenvalues.

In Tables 3–5 there are two observations to be considered: the first observation is that for small eigenvalues the perturbed results by numerical and asymptotic methods are close to each other for all  $\varepsilon$ . The second observation is that for the large eigenvalues the perturbed results obtained by asymptotic methods decrease linearly with respect to small  $\varepsilon$ , while the perturbed results obtained by numerical methods are almost the same for all values of  $\varepsilon$ . This shows that for small values of the perturbation parameter  $\varepsilon$  the asymptotic method is preferable.

## Conflict of Interests

The authors of the paper do not have any direct or indirect financial relation with the commercial identities mentioned in the paper.

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