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On the existence of an almost generalized weakly-symmetric Sasakian manifold admitting quarter symmetric metric connection

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ABSTRACT
The object of the present work is to study an almost generalized weakly symmetric Sasakian manifold admitting quarter symmetric metric connection with a non-trivial example.

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1. Introduction

The notion of a weakly symmetric Riemannian manifold was initiated by Tamássy and Binh [1]. Thereafter, it becomes focus of interest for many geometers. For details, we refer to [2–13] and the references there in. In analogy to [14], a weakly symmetric Riemannian manifold $(M^n, g)(n > 2)$, is said to be an almost weakly pseudo symmetric manifold, if its curvature tensor $\bar{R}$ of type $(0, 4)$ is not identically zero and satisfies the identity

$$\begin{aligned}(\nabla_X\bar{R})(Y,U,V,W) = {} & [A_1(X)+B_1(X)]\bar{R}(Y,U,V,W)\\ & + C_1(Y)\bar{R}(X,U,V,W)\\ & + C_1(U)\bar{R}(Y,X,V,W)\\ & + D_1(V)\bar{R}(Y,U,X,W)\\ & + D_1(W)\bar{R}(Y,U,V,X), \qquad (1)\end{aligned}$$

where A_1, B_1, C_1 & D_1 are non-zero 1-forms defined by $A_1(X) = g(X, \sigma_1)$, $B_1(X) = g(X, \varrho_1)$, $C_1(X) = g(X, \pi_1)$ and $D_1(X) = g(X, \partial_1)$, for all X and $\bar{R}(Y,U,V,W) = g(R(Y,U)V,W)$, ∇ being the operator of the covariant differentiation with respect to the metric tensor g. An n-dimensional Riemannian manifold of this kind is denoted by $A(WPS)_n$-manifold.

Keeping in tune with Dubey [15], the author in [16] has recently introduced the notion of an almost generalized weakly symmetric manifold (which is abbreviated hereafter as $A(GWS)_n$-manifold) if it admits the equation

$$\begin{aligned}(\nabla_X\bar{R})(Y,U,V,W) = {} & [A_1(X)+B_1(X)]\bar{R}(Y,U,V,W)\\ & + C_1(Y)\bar{R}(X,U,V,W)\\ & + C_1(U)\bar{R}(Y,X,V,W)\\ & + D_1(V)\bar{R}(Y,U,X,W)\\ & + D_1(W)\bar{R}(Y,U,V,X)\\ & + [A_2(X)+B_2(X)]\bar{G}(Y,U,V,W)\\ & + C_2(Y)\bar{G}(X,U,V,W)\\ & + C_2(U)\bar{G}(Y,X,V,W)\\ & + D_2(V)\bar{G}(Y,U,X,W)\\ & + D_2(W)\bar{G}(Y,U,V,X) \qquad (2)\end{aligned}$$

where

$$\bar{G}(Y,U,V,W) = \left[g(U,V)g(Y,W) - g(Y,V)g(U,W)\right] \qquad (3)$$

and A_i, B_i, C_i & D_i are non-zero 1-forms defined by $A_i(X) = g(X, \sigma_i)$, $B_i(X) = g(X, \varrho_i)$, $C_i(X) = g(X, \pi_i)$ and $D_i(X) = g(X, \partial_i)$, for $i = 1, 2$. The beauty of such $A(GWS)_n$-manifold is that it has the flavour of

(i) locally symmetric space [17], (for $A_i = B_i = C_i = D_i = 0$),
(ii) recurrent space [18], R_n (for $A_1 \neq 0$, $B_i = C_i = D_i = 0$),

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(iii) generalized recurrent space [15], $(GR)_n$ ($A_i \neq 0$ and $B_i = C_i = D_i = 0$),
(iv) pseudo symmetric space [19], $(PS)_n$ (for $A_1 = B_1 = C_1 = D_1 \neq 0$ and $A_2 = B_2 = C_2 = D_2 = 0$),
(v) semi-pseudo symmetric space [20], $(SPS)_n$ (for $A_1 = -B_1, C_1 = D_1$ and $A_2 = B_2 = C_2 = D_2 = 0$),
(vi) generalized semi-pseudo symmetric space [21], $(GSPS)_n$ (for $A_1 = -B_1, C_1 = D_1$ and $A_2 = -B_2$, $C_2 = D_2$),
(vii) generalized pseudo symmetric space [22], $(GPS)_n$ (for $A_i = B_i = C_i = D_i \neq 0$),
(viii) almost pseudo symmetric space [14], $A(PS)_n$ (for $B_1 \neq 0,\ A_1 = C_1 = D_1 \neq 0$ and $A_2 = B_2 = C_2 = D_2 = 0$),
(ix) almost generalized pseudo symmetric space [16], $A(GPS)_n$ (for $B_i \neq 0, A_i = C_i = D_i \neq 0$) and
(x) weakly symmetric space [1], $(WS)_n$ (for $A_2 = B_2 = C_2 = D_2 = 0$).

In [23], Golab defined and studied quarter-symmetric connection in a differentiable manifold with affine connection, which generalizes the thought of semi-symmetric connection. After Golab quarter symmetric connection has been studied by many geometers like as Mondal and De [24], Rastogi [25, 26], Mishra and Pandey [27], Yano and Imai [28] and others. A linear connection $\bar{\nabla}$ on an n-dimensional Riemannian manifold (M, g) is called a quarter-symmetric connection [23] if its torsion tensor T of the connection $\bar{\nabla}$

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where η is a 1-form and ϕ is a (1, 1) tensor field.

In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [29, 30]. Thus the notion of quarter-symmetric connection generalizes that of the semi-symmetric connection.

Furthermore, if a quarter-symmetric connection $\bar{\nabla}$ admits the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0$$

then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection [31].

Our work is structured as follows. Section 2 is concerned with Sasakian manifolds and some known results. In Section 3, we have investigated almost generalized weakly symmetric Sasakian manifolds admitting a quarter-symmetric metric connection $\bar{\nabla}$, which will be abbreviated by $[A(GPS)_n, \bar{\nabla}]$. It is observed that in a Sasakian manifold a necessary condition (i) for each of $[R_n, \bar{\nabla}]$, $[(GR)_n, \bar{\nabla}]$, $[(PS)_n, \bar{\nabla}]$ and $[(GPS)_n, \bar{\nabla}]$ to be, respectively, $[R_n, \nabla]$, $[(GR)_n, \nabla]$, $[(PS)_n, \nabla]$ and $[(GPS)_n, \nabla]$ is $A_1(\xi) = 0$; (ii) for each of $[(SPS)_n, \bar{\nabla}]$ and $[(GSPS)_n, \bar{\nabla}]$ to be , respectively, $[(SPS)_n, \nabla]$ and $[(GSPS)_n, \nabla]$ is $C_1(\xi) = 0$ & (iii) for $[(WS)_n, \bar{\nabla}]$ to be $[(WS)_n, \nabla]$ is $A_1(\xi) + C_1(\xi) + D_1(\xi) = 0$. Finally, we have constructed a non-trivial example of an $[A(GPS)_n, \bar{\nabla}]$.

2. Sasakian manifold and some known results

Let M be an $n = (2m + 1)$-dimensional almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a (1, 1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g. Then

$$\phi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,$$

$$\phi\xi = 0,\ \eta(\phi X) = 0, \quad (4)$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \forall X, Y \in TM. \quad (5)$$

From (4) and (5), it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X,\xi) = \eta(X), \forall X, Y \in TM. \quad (6)$$

An almost contact metric manifold M is said to be (a) a contact metric manifold if

$$g(X, \phi Y) = d\eta(X,Y), \forall X, Y \in TM; \quad (7)$$

(b) a K-contact manifold if the vector field ξ is Killing equivalently

$$\nabla_X \xi = -\phi X, \quad (8)$$

where ∇ is Riemannian connection and

(c) a Sasakian manifold if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \forall X, Y \in TM. \quad (9)$$

A K-contact manifold is a contact metric manifold, while the converse is true if the Lie derivative of ϕ in the characteristic direction ξ vanishes identically. A Sasakian manifold is always a K-contact manifold. A 3-dimensional K-contact manifold is a Sasakian manifold.

It is well known that a contact metric manifold is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \forall X, Y \in TM. \quad (10)$$

In a Sasakian manifold equipped with the structure (ϕ, ξ, η, g), the following relations also hold [32–34]:

$$(\nabla_X \eta)Y = g(X, \phi Y), \quad (11)$$

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \quad (12)$$

$$S(X,\xi) = (n-1)\eta(X), \quad (13)$$

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \quad (14)$$

for all $X, Y, Z \in TM$, where S is the Ricci tensor.

The relation between the quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ of

(M^n, g) has been obtained by Yano and Imai [28], which is given by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\phi X; \tag{15}$$

given by.

A relation between the curvature tensor R^{∇} of M with respect to the quarter-symmetric metric connection $\bar{\nabla}$ and R of M with respect to the the Riemannian connection ∇, is given by

$$\begin{aligned} R^{\nabla}(X,Y)Z &= R(X,Y)Z - 2g(X,\phi Y)\phi Z + \eta(X)g(Y,Z)\xi \\ &\quad - \eta(Y)g(X,Z)\xi + \eta(Z)\{\eta(Y)X - \eta(X)Y\}. \end{aligned} \tag{16}$$

which yields

$$S^{\nabla}(Y,Z) = S(Y,Z) - g(Y,Z) + n\eta(Y)\eta(Z), \tag{17}$$

where S^{∇} and S are the Ricci tensors of the connections $\bar{\nabla}$ and ∇ , respectively.

3. Sasakian manifold with $[A(GPS)_n, \bar{\nabla}]$

For an $[A(GPS)_n, \bar{\nabla}]$, we have

$$\begin{aligned} (\nabla_X \bar{R}^{\nabla})(Y,U,V,W) &= [A_1(X) + B_1(X)]\bar{R}^{\nabla}(Y,U,V,W) \\ &\quad + C_1(Y)\bar{R}^{\nabla}(X,U,V,W) + C_1(U)\bar{R}^{\nabla}(Y,X,V,W) \\ &\quad + D_1(V)\bar{R}^{\nabla}(Y,U,X,W) + D_1(W)\bar{R}^{\nabla}(Y,U,V,X) \\ &[A_2(X) + B_2(X)]G(Y,U,V,W) \\ &\quad + C_2(Y)G(X,U,V,W) + C_2(U)G(Y,X,V,W) \\ &\quad + D_2(V)G(Y,U,X,W) + D_2(W)G(Y,U,V,X) \end{aligned} \tag{18}$$

for all X, Y, Z, U. Making use of (15), we can find

$$\begin{aligned} (\tilde{\nabla}_X \tilde{R}^{\nabla})(Y,U,V,W) &= (\nabla_X \tilde{R}^{\nabla})(Y,U,V,W) \\ &\quad - \eta(\tilde{R}^{\nabla}(Y,U)V)g(\phi X,W) \\ &\quad + \eta(Y)\tilde{R}^{\nabla}(\phi X,U,V,W) \\ &\quad + \eta(U)\tilde{R}^{\nabla}(Y,\phi X,V,W) \\ &\quad + \eta(V)\tilde{R}^{\nabla}(Y,U,\phi X,W) \\ &\quad + \eta(W)\tilde{R}^{\nabla}(Y,U,V,\phi X). \end{aligned} \tag{19}$$

Now, using (16) in the foregoing equation, we have

$$\begin{aligned} (\tilde{\nabla}_X \tilde{R}^{\nabla})(Y,U,V,W) &= (\nabla_X \bar{R})(Y,U,V,W) \\ &\quad - 2(\nabla_X g)(Y,\phi U)g(\phi V,W) \\ &\quad - 2g(Y,\phi U)(\nabla_X g)(\phi V,W) \\ &\quad + (\nabla_X \eta)(Y)g(U,V)\eta(W) \\ &\quad + \eta(Y)g(U,V)(\nabla_X \eta)(W) \\ &\quad - (\nabla_X \eta)(U)g(Y,V)\eta(W) \\ &\quad - \eta(U)g(Y,V)(\nabla_X \eta)(W) \\ &\quad + (\nabla_X \eta)(V)\{g(Y,W)\eta(U) \\ &\quad - g(U,W)\eta(Y)\} \\ &\quad + \eta(V)\{g(Y,W)(\nabla_X \eta)(U) \\ &\quad - g(U,W)(\nabla_X \eta)(Y)\} \\ &\quad - [\eta(R(Y,U)V) + g(U,V)\eta(Y) \\ &\quad - g(Y,V)\eta(U)]g(\phi X,W) \\ &\quad + \eta(Y)[R(\phi X,U,V,W) \\ &\quad - 2g(\phi X,\phi U)g(\phi V,W) \\ &\quad - g(\phi X,V)\eta(U)\eta(W) \\ &\quad + g(\phi X,W)\eta(U)\eta(V)] \\ &\quad + \eta(U)[R(Y,\phi X,V,W) \\ &\quad + 2g(\phi Y,\phi X)g(\phi V,W) \\ &\quad + g(\phi X,V)\eta(Y)\eta(W) \\ &\quad - g(\phi X,W)\eta(Y)\eta(V)] \\ &\quad + \eta(V)[R(Y,U,\phi X,W) \\ &\quad + 2g(Y,\phi U)g(\phi X,\phi W) \\ &\quad + g(\phi X,U)\eta(Y)\eta(W) \\ &\quad - g(\phi X,Y)\eta(U)\eta(W)] \\ &\quad + \eta(W)[R(Y,U,V,\phi X) \\ &\quad - 2g(Y,\phi U)g(\phi V,\phi X) \\ &\quad + g(Y,\phi X)\eta(U)\eta(V) \\ &\quad - g(\phi X,U)\eta(Y)\eta(V)]. \end{aligned} \tag{20}$$

Theorem 3.1: *An $[A(GPS)_n, \bar{\nabla}]$ is an $[A(GPS)_n, \nabla]$, if the 1-forms are related by the following relation*

$$A_1(\xi) + B_1(\xi) + C_1(\xi) + D_1(\xi) = 0. \tag{21}$$

Proof: As a direct consequence of (16), (18) and (20) one can say that an almost generalized weakly symmetric Sasakian manifold admitting quarter symmetric connection $\bar{\nabla}$ reduces to an almost generalized weakly symmetric Sasakian manifold admitting Riemannian metric connection ∇, if the following relation holds

$$\begin{aligned} &- 2(\nabla_X g)(Y,\phi U)g(\phi V,W) - 2g(Y,\phi U)(\nabla_X g)(\phi V,W) \\ &\quad + (\nabla_X \eta)(Y)g(U,V)\eta(W) + \eta(Y)g(U,V)(\nabla_X \eta)(W) \\ &\quad - (\nabla_X \eta)(U)g(Y,V)\eta(W) - \eta(U)g(Y,V)(\nabla_X \eta)(W) \\ &\quad + (\nabla_X \eta)(V)\{g(Y,W)\eta(U) - g(U,W)\eta(Y)\} \\ &\quad + \eta(V)\{g(Y,W)(\nabla_X \eta)(U) - g(U,W)(\nabla_X \eta)(Y)\} \\ &\quad - [\eta(R(Y,U)V) + g(U,V)\eta(Y) \\ &\quad - g(Y,V)\eta(U)]g(\phi X,W) + \eta(Y)[R(\phi X,U,V,W) \\ &\quad - 2g(\phi X,\phi U)g(\phi V,W) - g(\phi X,V)\eta(U)\eta(W) \\ &\quad + g(\phi X,W)\eta(U)\eta(V)] + \eta(U)[R(Y,\phi X,V,W) \\ &\quad + 2g(\phi Y,\phi X)g(\phi V,W) + g(\phi X,V)\eta(Y)\eta(W) \\ &\quad - g(\phi X,W)\eta(Y)\eta(V)] + \eta(V)[R(Y,U,\phi X,W) \\ &\quad + 2g(Y,\phi U)g(\phi X,\phi W) + g(\phi X,U)\eta(Y)\eta(W) \end{aligned}$$

$$- g(\phi X, Y)\eta(U)\eta(W)] + \eta(W)[R(Y, U, V, \phi X) - 2g(Y, \phi U)g(\phi V, \phi X) + g(Y, \phi X)\eta(U)\eta(V) - g(\phi X, U)\eta(Y)\eta(V)] + [A_1(X) + B_1(X)] \times [2g(Y, \phi U)g(\phi V, W) - g(U, V)\eta(Y)\eta(W) + g(Y, V)\eta(U)\eta(W) - \{g(Y, W)\eta(U) - g(U, W)\eta(Y)\}\eta(V)] + C_1(Y)[2g(X, \phi U)g(\phi V, W) - g(U, V)\eta(X)\eta(W) + g(X, V)\eta(U)\eta(W) - \{g(X, W)\eta(U) - g(U, W)\eta(X)\}\eta(V)] + C_1(U)[2g(Y, \phi X)g(\phi V, W) - g(X, V)\eta(Y)\eta(W) + g(Y, V)\eta(X)\eta(W) - \{g(Y, W)\eta(X) - g(X, W)\eta(Y)\}\eta(V)] + D_1(V)[2g(Y, \phi U)g(\phi X, W) - g(U, X)\eta(Y)\eta(W) + g(Y, X)\eta(U)\eta(W) - \{g(Y, W)\eta(U) - g(U, W)\eta(Y)\}\eta(X)] + D_1(W)[2g(Y, \phi U)g(\phi V, X) - g(U, V)\eta(Y)\eta(X) + g(Y, V)\eta(U)\eta(X) - \{g(Y, X)\eta(U) - g(U, X)\eta(Y)\}\eta(V)] = 0 \tag{22}$$

which yields

$$A_1(\xi) + B_1(\xi) + C_1(\xi) + D_1(\xi) = 0$$

for $X = U = V = \xi$. ■

From, the above one can state the followings

Claim 3.2: *In a Sasakian manifold, a necessary condition for each of $[R_n, \bar{\nabla}]$, $[(GR)_n, \bar{\nabla}]$, $[(PS)_n, \bar{\nabla}]$ and $[(GPS)_n, \bar{\nabla}]$ to be $[R_n, \nabla]$, $[(GR)_n, \nabla]$, $[(PS)_n, \nabla]$ and $[(GPS)_n, \nabla]$ is $A_1(\xi) = 0$.*

Claim 3.3: *In a Sasakian manifold, a necessary condition for each of $[(SPS)_n, \bar{\nabla}]$ and $[(GSPS)_n, \bar{\nabla}]$, to be $[(SPS)_n, \nabla]$ and $[(GSPS)_n, \nabla]$ is $C_1(\xi) = 0$.*

Claim 3.4: *In a Sasakian manifold, a necessary condition for $[(WS)_n, \bar{\nabla}]$ to be $[(WS)_n, \nabla]$ is $A_1(\xi) + C_1(\xi) + D_1(\xi) = 0$.*

Again, contracting Equation (18), we have

$$(\bar{\nabla}_X S^{\nabla})(U, V) = [A_1(X) + B_1(X)]S^{\nabla}(U, V) + C_1(R^{\nabla}(X, U)V) + C_1(U)S^{\nabla}(X, V) + D_1(V)S^{\nabla}(U, X) - D_1(R^{\nabla}(V, X)U) + [A_2(X) + B_2(X)](n-1)g(U, V) + [C_2(X)g(U, V) - C_2(U)g(V, X)] + (n-1)C_2(U)g(X, V) + (n-1)D_2(V)g(U, X) + [D_2(X)g(U, V) - D_2(V)g(U, X)]. \tag{23}$$

Replacing V by ξ in the above equation and then using the relations (16), (17), (10), (13) and (14), we get

$$0 = 2(n-1)[A_1(X) + B_1(X)] + 2[C_1(X) + D_1(X)] + 2(n-2)[C_1(\xi) + D_1(\xi)]\eta(X) + (n-1)[A_2(X) + B_2(X)] + C_2(X) + D_2(X) + (n-2)[C_2(\xi) + D_2(\xi)]\eta(X). \tag{24}$$

Thus we can state the followings

Theorem 3.5: *In a $[(AGPS)_n, \bar{\nabla}]$, the 1-forms are related by (24).*

4. Example of a Sasakian manifold with $[A(GPS)_n, \bar{\nabla}]$

Chose a 3-dimensional manifold spanned by a set of vector fields $\{e_1, e_2, e_3\}$ defined by

$$e_1 = x_1\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) - 2x_2\frac{\partial}{\partial x_3}, e_2 = \frac{\partial}{\partial x^2}, e_3 = \xi = \frac{\partial}{\partial x^3},$$

where $\{x^1; x^2; x^3\}$ is a standard coordinates in R^3. Define 1-form η, characteristic vector field ξ, Riemannian metric g and (1-1) tensor ϕ by $\eta(Z) = g(Z, e_3)$, $\xi = \frac{\partial}{\partial x_3}$, $g(e_i, e_j) = \delta_{ij}$ and $\phi e_1 = -e_2, \phi e_2 = e_1$ and $\phi e_3 = 0$. Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g. Then we have $[e_1, e_2] = 2e_3$, $[e_1, e_3] = 0$, $[e_2, e_3] = 0$. Thus, $M(\phi, \xi, \eta, g)$ defines a Sasakian manifold.

The Levi-Civita connection ∇ of the metric tensor g can be obtained by using Koszul's formula which are as follows:

$$\nabla_{e_1}e_3 = -e_2, \quad \nabla_{e_1}e_2 = e_3, \quad \nabla_{e_1}e_1 = 0,$$
$$\nabla_{e_2}e_3 = e_1, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_1 = -e_3,$$
$$\nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_2 = e_1, \quad \nabla_{e_3}e_1 = -e_2.$$

In view of the above, one can easily obtain the following:

$$\bar{\nabla}_{e_1}e_3 = 0, \quad \bar{\nabla}_{e_1}e_2 = e_3, \quad \bar{\nabla}_{e_1}e_1 = 0,$$
$$\bar{\nabla}_{e_2}e_3 = 0, \quad \bar{\nabla}_{e_2}e_2 = 0, \quad \bar{\nabla}_{e_2}e_1 = -e_3,$$
$$\bar{\nabla}_{e_3}e_3 = 0, \quad \bar{\nabla}_{e_3}e_2 = e_1, \quad \bar{\nabla}_{e_3}e_1 = -e_2.$$

By virtue of the above results, we can easily obtain the non-vanishing components of the curvature tensors as follows:

$$R(e_1, e_2)e_2 = -3e_1, \ R(e_1, e_3)e_3 = e_1, \ R(e_2, e_3)e_3 = e_2;$$

and

$$\tilde{R}(e_1, e_2)e_2 = -5e_1, \tilde{R}(e_1, e_3)e_3 = 2e_1,$$
$$\tilde{R}(e_2, e_3)e_3 = 2e_2.$$

Since $\{e_1, e_2, e_3\}$ forms a basis of the Sasakian manifold, any vector field $Y, U, V, W \in \chi(M)$ can be written as

$$Y = a_1e_1 + b_1e_2 + c_1e_3; U = a_2e_1 + b_2e_2 + c_2e_3;$$
$$V = a_3e_1 + b_3e_2 + c_3e_3 \quad W = a_4e_1 + b_4e_2 + c_4e_3;$$

where $a_i, b_i, c_i \in R^+$ (the set of all positive real numbers). Then

$$\begin{aligned}
\bar{G}(Y,U,V,W) &= (a_2a_3 + b_2b_3 + c_2c_3) \\
&\quad \times (a_1a_4 + b_1b_4 + c_1c_4) \\
&\quad - (a_1a_3 + b_1b_3 + c_1c_3) \\
&\quad \times (a_2a_4 + b_2b_4 + c_2c_4) = \theta(\text{say}) \\
\bar{G}(e_1,U,V,W) &= a_4(a_2a_3 + b_2b_3 + c_2c_3) \\
&\quad - a_3(a_2a_4 + b_2b_4 + c_2c_4) = \theta_1(\text{say}) \\
\bar{G}(e_2,U,V,W) &= b_4(a_2a_3 + b_2b_3 + c_2c_3) \\
&\quad - b_3(a_2a_4 + b_2b_4 + c_2c_4) = \theta_2(\text{say}) \\
\bar{G}(e_3,U,V,W) &= c_4(a_2a_3 + b_2b_3 + c_2c_3) \\
&\quad - c_3(a_2a_4 + b_2b_4 + c_2c_4) = \theta_3(\text{say}) \\
\bar{G}(Y,e_1,V,W) &= a_3(a_1a_4 + b_1b_4 + c_1c_4) \\
&\quad - a_4(a_1a_3 + b_1b_3 + c_1c_3) = \theta_4(\text{say}) \\
\bar{G}(Y,e_2,V,W) &= b_3(a_1a_4 + b_1b_4 + c_1c_4) \\
&\quad - b_4(a_1a_3 + b_1b_3 + c_1c_3) = \theta_5(\text{say}) \\
\bar{G}(Y,e_3,V,W) &= c_3(a_1a_4 + b_1b_4 + c_1c_4) \\
&\quad - c_4(a_1a_3 + b_1b_3 + c_1c_3) = \theta_6(\text{say}) \\
\bar{G}(Y,U,e_1,W) &= a_2(a_1a_4 + b_1b_4 + c_1c_4) \\
&\quad - a_1(a_2a_4 + b_2b_4 + c_2c_4) = \theta_7(\text{say}) \\
\bar{G}(Y,U,e_2,W) &= b_2(a_1a_4 + b_1b_4 + c_1c_4) \\
&\quad - b_1(a_2a_4 + b_2b_4 + c_2c_4) = \theta_8(\text{say}) \\
\bar{G}(Y,U,e_3,W) &= c_2(a_1a_4 + b_1b_4 + c_1c_4) \\
&\quad - c_1(a_2a_4 + b_2b_4 + c_2c_4) = \theta_9(\text{say}) \\
\bar{G}(Y,U,V,e_1) &= a_1(a_2a_3 + b_2b_3 + c_2c_3) \\
&\quad - a_1(a_1a_3 + b_1b_3 + c_1c_3) = \theta_{10}(\text{say}) \\
\bar{G}(Y,U,V,e_2) &= b_1(a_2a_3 + b_2b_3 + c_2c_3) \\
&\quad - b_1(a_1a_3 + b_1b_3 + c_1c_3) = \theta_{11}(\text{say}) \\
\bar{G}(Y,U,V,e_3) &= c_1(a_2a_3 + b_2b_3 + c_2c_3) \\
&\quad - c_1(a_1a_3 + b_1b_3 + c_1c_3) = \theta_{12}(\text{say})
\end{aligned}$$

$$\begin{aligned}
R^\nabla(Y,U,V,W) &= -5(a_1b_2 - a_2b_1)(a_3b_4 - a_4b_3) \\
&\quad + 2(a_1c_2 - a_2c_1)(a_3c_4 - a_4c_3) \\
&\quad + 2(b_1c_2 - b_2c_1)(b_3c_4 - b_4c_3) \\
&= \lambda(\text{say}) \\
R^\nabla(e_1,U,V,W) &= -5b_2(a_3b_4 - a_4b_3) + 2c_2(a_3c_4 \\
&\quad - a_4c_3) = \lambda_1(\text{say}) \\
R^\nabla(e_2,U,V,W) &= -5a_2(a_3b_4 - a_4b_3) \\
&\quad + 2c_2(b_3c_4 - b_4c_3) = \lambda_2(\text{say}) \\
R^\nabla(e_3,U,V,W) &= 2c_2(a_3c_4 - a_4c_3) - 2b_2(b_3c_4 - b_4c_3) \\
&= \lambda_3(\text{say}) \\
R^\nabla(Y,e_1,V,W) &= 5b_1(a_3b_4 - a_4b_3) - 2c_1(a_3c_4 - a_4c_3) \\
&= \lambda_4(\text{say}) \\
R^\nabla(Y,e_2,V,W) &= -5a_1(a_3b_4 - a_4b_3) \\
&\quad + 2c_1(b_3c_4 - b_4c_3) = \lambda_5(\text{say}) \\
R^\nabla(Y,e_3,V,W) &= 2c_1(a_3c_4 - a_4c_3) - 2b_1(b_3c_4 - b_4c_3) \\
&= \lambda_6(\text{say}) \\
R^\nabla(Y,U,e_1,W) &= -5b_4(a_1b_2 - a_2b_1) \\
&\quad + 2c_4(a_1c_2 - a_2c_1) = \lambda_7(\text{say}) \\
R^\nabla(Y,U,e_2,W) &= -5a_4(a_1b_2 - a_2b_1) \\
&\quad + 2c_4(b_1c_2 - b_2c_1) = \lambda_8(\text{say}) \\
R^\nabla(Y,U,e_3,W) &= 2c_4(a_1c_2 - a_2c_1) - 2b_4(b_1c_2 - b_2c_1) \\
&= \lambda_9(\text{say}) \\
R^\nabla(Y,U,V,e_1) &= -5b_3(a_1b_2 - a_2b_1) \\
&\quad + 2c_3(a_1c_2 - a_2c_1) = \lambda_{10}(\text{say}) \\
R^\nabla(Y,U,V,e_2) &= -5a_3(a_1b_2 - a_2b_1) \\
&\quad + 2c_3(b_1c_2 - b_2c_1) = \lambda_{11}(\text{say}) \\
R^\nabla(Y,U,V,e_3) &= 2c_3(a_1c_2 - a_2c_1) - 2b_3(b_1c_2 - b_2c_1) \\
&= \lambda_{12}(\text{say}).
\end{aligned}$$

Making use of the above results we obtain the covariant derivative as follows:

$$\begin{aligned}
(\bar{\nabla}_{e_1}R^\nabla)(Y,U,V,W) &= -[b_1\lambda_3 + b_2\lambda_6 + b_3\lambda_9 + b_4\lambda_{12}] \\
(\bar{\nabla}_{e_2}R^\nabla)(Y,U,V,W) &= [a_1\lambda_3 + a_2\lambda_6 + a_3\lambda_9 + a_4\lambda_{12}] \\
(\bar{\nabla}_{e_2}R^\nabla)(Y,U,V,W) &= [a_1\lambda_2 + a_2\lambda_5 + a_3\lambda_8 + a_4\lambda_{11}] \\
&\quad - [b_1\lambda_1 + b_2\lambda_4 + b_3\lambda_7 + b_4\lambda_{10}]
\end{aligned}$$

where $R^\nabla(Y,U,V,W) = g(\bar{R}(Y,U)V,W)$. For the following choice of the one forms

$$\begin{aligned}
A_1(e_1) &= -\frac{b_1\lambda_3}{T_1}, B_1(e_1) = -\frac{b_2\lambda_6}{T_1}, \\
A_2(e_1) &= -\frac{b_3\lambda_9}{T_2}, B_2(e_1) = -\frac{b_4\lambda_{12}}{T_2}, \\
A_1(e_2) &= \frac{a_1\lambda_3}{T_1}, B_1(e_2) = \frac{a_2\lambda_6}{T_1}, \\
A_2(e_2) &= \frac{a_3\lambda_9}{T_2}, B_2(e_2) = \frac{a_4\lambda_{12}}{T_2}, \\
A_1(e_3) &= \frac{a_3\lambda_8 + a_4\lambda_{11}}{T_1}, B_1(e_3) = \frac{a_1\lambda_2 + a_2\lambda_5}{T_1}, \\
C_1(e_3) &= \frac{1}{a_3\lambda_3 + b_3\lambda_6}, C_2(e_3) = \frac{1}{a_3\theta_3 + b_3\theta_6},
\end{aligned}$$

$$D_1(e_3) = -\frac{1}{c_3\lambda_9 + d_3\lambda_{12}}, D_2(e_3) = -\frac{1}{c_3\theta_9 + d_3\theta_{12}},$$
$$A_2(e_3) = -\frac{b_1\lambda_1 + b_2\lambda_4}{T_2}, B_2(e_3) = -\frac{b_3\lambda_7 + b_4\lambda_{10}}{T_2},$$

one can easily verify the relations

$$\begin{aligned}(\bar{\nabla}_{e_i}R^{\nabla})(X,Y,U,V) = {} & [A_1(e_i) + B_1(e_i)]R^{\nabla}(X,Y,U,V)\\ & + C_1(X)R^{\nabla}(e_i,Y,U,V)\\ & + C_1(Y)R^{\nabla}(X,e_i,U,V)\\ & + D_1(U)R^{\nabla}(X,Y,e_i,V)\\ & + D_1(V)R^{\nabla}(X,Y,U,e_i)\\ & + [A_2(e_i) + B_2(e_i)]\bar{G}(X,Y,U,V)\\ & + C_2(X)\bar{G}(e_i,Y,U,V)\\ & + C_2(Y)\bar{G}(X,e_i,U,V)\\ & + D_2(U)\bar{G}(X,Y,e_i,V)\\ & + D_2(V)\bar{G}(X,Y,U,e_i)\end{aligned}$$

for 1, 2, 3. From the above, we can state that

Theorem 4.1: *There exists a Sasakian manifold* (M^3, g) *which is an* $[A(GPS)_n, \bar{\nabla}]$.

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