Target Capturing in an Ellipsoidal Region for a Swarm of Double Integrator Agents

Antonio Bono, *Graduate Student Member, IEEE*, Luigi D'Alfonso, Giuseppe Fedele, *Member, IEEE*, and Veysel Gazi, *Senior Member, IEEE*

***Abstract*—In this paper we focus on the target capturing problem for a swarm of agents modelled as double integrators in any finite space dimension. Each agent knows the relative position of the target and has only an estimation of its velocity and acceleration. Given that the estimation errors are bounded by some *known* values, it is possible to design a control law that ensures that agents enter a user-defined ellipsoidal ring around the moving target. Agents know the relative position of the other members whose distance is smaller than a common detection radius. Finally, in the case of no uncertainty about target data and homogeneous agents, we show how the swarm can reach a static configuration around the moving target. Some simulations are reported to show the effectiveness of the proposed strategy.**

***Index Terms*—Agents-based systems, cooperative control, swarms, target-capturing.**

I. Introduction

STUDIES on natural and engineering swarms have attracted many researchers in recent years. The collective dynamics of swarm systems is an interesting natural phenomenon, but also has wide potential engineering applications [1], [2]. As reported in [3], in many scenarios such as surveillance, security, rescue and so forth, it is required that the swarm members enclose a target object or a prescribed region by using neighbor information.

Some relevant literature contributions are here summarized. One common approach to this task involves a cyclic-pursuit where agent i simply pursues agent $i+1$ modulo n while moving around the target. The paper in [4], for example, considers the target capturing where each agent is modelled as single integrator model and it is able to sense the distance from the target and from the agent to be pursued. More recently, in [5], the cyclic-pursuit approach is managed to be applied in the underwater domain, using a sliding mode control under the hypothesis that one designated agent knows the position of the target object in the inertial frame and broadcasts the target coordinates information to the remaining members of the formation. Other solutions are based on artificial potential functions that constitute the subject of many studies (see [6] and references therein). The method is exploited in [7] by considering multiple circular trajectories around the known moving target. In [8], a control law based on a gradient descent method has been developed using some local data among the agents and the target position in order to encircle and grasp the target. Similarly, in [9] a potential functions and sliding mode control based strategy is developed for capturing/enclosing a moving target by a swarm composed of agents with fully actuated dynamics containing model uncertainties. In the above works, all vehicles require the information of the target in terms of its position or trajectory. Moreover, in those based on the cyclic pursuit, the network topology among the vehicles is limited to cycle graphs and then enclosing the target objects cannot be achieved with other graph configurations. The problem of the partial knowledge on the target has been faced in [10] and [11] where at least one vehicle can observe the target which is included in the connection graph topology. Reference [10] solves the consensus of the agents on a circle around the target in the case of mobile vehicles in Cartesian coordinates. The target position together with its velocity is exactly known by at least one vehicle. These constraints are partially relaxed in [11] where a fleet of UAVs and the evader flying at a constant altitude is considered. In particular the target-centric formation problem is addressed where each UAV should maintain a constant distance from the target at a constant angle. The main limitation of this strategy is the use of the dynamic characteristics of the unmanned aerial vehicles, which makes it difficult to generalise the method in other frameworks. Indeed the uncertainty is managed by using a sliding mode controller to mitigate the partial information available to the capturing vehicles.

In this paper, we consider a swarm of agents modelled by double integrator dynamics and propose a control law that ensures the enclosing of a moving **uncertain** target without imposing a consensus on prescribed positions around it. The choice of modelling agents as double integrator systems also stems from the fact that overall dynamics of many different agents such as fully actuated robots, drones, or even manipulators can be represented by double integrator models. For example, a general form for a fully-actuated holonomic dynamics model can be represented as [12]

Manuscript received November 18, 2021; accepted December 23, 2021. Recommended by Associate Editor Yanjun Liu. *(Corresponding author: Giuseppe Fedele.)*

Citation: A. Bono, L. D'Alfonso, G. Fedele, and V. Gazi, "Target capturing in an ellipsoidal region for a swarm of double integrator agents," *IEEE/CAA J. Autom. Sinica*, vol. 9, no. 5, pp. 801–811, May 2022.

A. Bono, L. D'Alfonso, and G. Fedele are with the Department of Informatics, Modeling, Electronics and Systems Engineering, University of Calabria, Rende 87036, Italy (e-mail: antonio.bono@unical.it; luigi.dalfonso@unical.it; giuseppe.fedele@unical.it).

V. Gazi was with the Dept. Electrical and Electronics Eng., Faculty of Engineering, Marmara University, Istanbul 34722, and he is now with the Department of Control and Automation Engineering, Faculty of Electrical and Electronics Engineering, Yildiz Technical University, Esenler 34220, Turkey (e-mail: vgazi@yildiz.edu.tr).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/JAS.2022.105551

$$M(p_i)\ddot{p}_i + f_i(p_i, \dot{p}_i) = u_i \tag{1}$$

where $p_i, \dot{p}_i, \ddot{p}_i, u_i \in \mathbb{R}^d$, and $M(p_i) \in \mathbb{R}^{d\times d}$ denote the position, velocity, acceleration, control input, and mass (inertia) matrix of agent/robot i, respectively. The term $f_i(p_i, \dot{p}_i)$ represents the other effects (such as centripetal, Coriolis, gravitational effects, and additive disturbances). If f_i and M_i are known, a control input

$$u_i = f_i(p_i, \dot{p}_i) + M(p_i)\bar{u}_i \tag{2}$$

reduces the model (1) to the point mass model

$$\ddot{p}_i = \bar{u}_i. \tag{3}$$

Similarly, the overall dynamics of other types of agents can be captured by the double integrator model. This is one of the reasons for extensive utilization of this model in the literature. Moreover, although in other kinds of cooperative task more complex agents dynamics have been considered [13]–[16], the single or double integrator models can serve as high-level path generator for systems with particular vehicle dynamics [2], [17]. In the simulation section, an example of this approach is described for a swarm of differential drive mobile robots.

In order to enclose the moving uncertain target, the agents arrange themselves in an ellipsoidal ring that is defined by two regions centered on the target: a **distancing region** that is forbidden to the agents and represents the free space allowed to the target and a **containment region** that agents enter in steady-state and they cannot leave. Ellipsoid is a shape which requires different level attractions/repulsions on different semi-axes. It is worth mentioning that, differently from many other approaches, the ellipsoidal ring constitutes a non-trivial geometric constraint to address because of its non-convexity.

The swarm is assumed to know the target position but to have only uncertain information about its velocity and acceleration. The estimation errors are bounded by some values *a priori* known. Moreover, while agents have a continuous, even if uncertain, knowledge of the target, they can sense the other agents through local proximity sensors (for collision avoidance). Consequently, the topology of connections is time variant.

At first, there seems to be a contradiction in the ability of the agent to know the position of the target at each time instant and instead the position of other agents only when they are within its detection range. However, it is worth mentioning that in the robotic context there are usually different types of sensors on board. In the scenario presented, for example, the perception of the leader and of the neighbors agents could be provided by two different modules: one, based on computer vision and therefore capable of detecting objects at a considerable distance, devoted to the perception of the leader; another one, based on proximity sensors (ultrasonic, infrared, etc.), devoted to the perception of the closest agents. The benefit of such a setup would be to use computer vision, which is computationally expensive, only for the leader and to use the simpler and faster proximity sensors for the neighbours agents. Regardless of the possible scenarios, the assumption of target position knowledge is not limiting. In the early stages of the tracking, in fact, we can always use an estimate of the target's position in place of the real data. This estimate will be replaced as soon as the agent is able to locate the target thanks to its perceptive abilities. Different strategies have been proposed in this regard. If at least one vehicle in the swarm has target information also with uncertainties (only the upper and lower bounds on the exact target motion are available), and the corresponding communication graph is connected, then a target-centric formation can be maintained using [11]. Many systems, however, have some kind of sensors to measure the relative position and/or velocity of the target with possibly bounded measuring errors. For example, [18] uses the two-step weighted least squares as well as time of arrival measurements to estimate the position of moving targets. These targets only passively receive signals from the multi-agent network, which then broadcasts the related information to the agents. An estimator of the target location is proposed in [19] based on the position of the agents and the bearing angle to the target. The papers in [20], [21] extend the results in [19] in the case of multiple targets and provide a method for elliptical circumnavigation around them. A bearing-based coordinated circumnavigation control scheme is also proposed in [22], based on the networked agents' coordinated estimation algorithm for the velocity of the target as well as the distance towards the target. In particular, an estimation algorithm is designed to estimate the target's velocity and the distance between the agent and the target, based on the local velocity and bearing information.

The proposed strategy relies on the works in [23]–[26] where multi-agent systems based on kinematic models have been studied in the context of finite-time reference tracking and rotational aggregation around a given reference. Although in all these previous works the swarm model benefits of similar interaction terms among the agents, here the control law uses *ad hoc* attractive and repulsion terms that ensure agents to enclose the target without colliding with it.

Finally, the main contributions of the paper can be summarized as follows:

i) Cooperative enclosing of a moving target, with uncertain information about velocity and acceleration available, is achieved by spreading the follower agents in an ellipsoidal ring.

ii) The ring is formed by two ellipsoidal regions centered on the target: a containment region where agents converge and remain inside during the tracking and a distancing region that is forbidden to the swarm.

iii) In case of exact information, at steady-state, the agents can achieve a static configuration such that the moving target lies in the convex-hull formed by their positions.

iv) The proposed strategy holds for any finite space dimension ($d \geq 2$).

The remainder of this paper is organized as follows: Section II defines the capturing problem that the paper aims at solving; the proposed model and its main properties are described in Sections III and IV, respectively; Section V shows the applicability of the strategy through examples and a software in the loop (SIL) simulation; Section VI concludes the paper by summarizing the achieved goals and discussing future research directions.

Notation: Let $\mathcal{M}$ be the set of $d \times d$ matrices with real elements. The eigenvalues of the generic matrix $A \in \mathcal{M}$ are denoted with $\lambda_i(A), i=1,2,\ldots,d$. The operator $\|\cdot\|$ denotes the Euclidean norm for both vectors and matrices and $\|v\|_A^2 = v^T A v$ is a weighted square norm of the vector $v \in \mathbb{R}^d$ using the matrix A. Consider $P \in \mathcal{M}, P=P^T > 0$, the shaping matrix of the ellipsoid centered at $\xi \in \mathbb{R}^d$, i.e., $\mathcal{E}(P,\xi) = \{x \in \mathbb{R}^d \text{ s.t.} (x-\xi)^T P(x-\xi) \leq 1\}$. The set of the semi-axes of such an ellipsoid will be expressed by $\mathcal{S}(P) \triangleq \{1/\sqrt{\lambda_1(P)}, 1/\sqrt{\lambda_2(P)}, \ldots, 1/\sqrt{\lambda_d(P)}\}$. C^m denotes the set of functions $f(\cdot): \mathbb{R} \to \mathbb{R}^d$ continuously differentiable up to the m-th order.

II. Problem Formulation

Consider a group of n agents whose dynamics evolve in $d \geq 2$ dimensional Euclidean space based on

$$\dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = \frac{1}{m_i} u_i(t), \quad i = 1,2,\ldots,n \tag{4}$$

where $m_i > 0$ denotes the mass, $x_i(t) \in \mathbb{R}^d$ denotes the position, and $v_i(t) \in \mathbb{R}^d$ denotes the velocity of agent i at time $t \geq 0$. The control (force) input is represented by $u_i(t) \in \mathbb{R}^d$. We assume that the agents need to track and capture/enclose a target whose position and velocity at time t are denoted by $x_T(t)$ and $v_T(t) = \dot{x}_T(t)$, respectively, with $x_T(t) \in C^2$. Let us define the position and velocity error dynamics for agent i

$$\begin{aligned} z_i(t) &= x_i(t) - x_T(t) \\ w_i(t) &= v_i(t) - v_T(t), \quad i = 1,2,\ldots,n. \end{aligned} \tag{5}$$

Under these definitions the agent's error dynamics are expressed as

$$\begin{aligned} \dot{z}_i(t) &= w_i(t) \\ \dot{w}_i(t) &= \frac{1}{m_i} u_i(t) - a_T(t), \quad i = 1,2,\ldots,n \end{aligned} \tag{6}$$

where $a_T(t) \in \mathbb{R}^d$ denotes the acceleration of the target or basically $a_T(t) = \dot{v}_T(t) = \ddot{x}_T(t)$. We assume that each agent i possesses only an estimate of the target velocity and acceleration, namely $\hat{v}_T^i(t), \hat{a}_T^i(t) \in C^0$, and that the errors related to these estimates are bounded, i.e.,

$$\|v_T(t) - \hat{v}_T^i(t)\| \leq b_v, \ \ \|a_T(t) - \hat{a}_T^i(t)\| \leq b_a \tag{7}$$

with *known* bounds $b_v \geq 0$ and $b_a \geq 0$. Note that this assumption is both: reasonable, because the data relative to velocity and acceleration typically come from an estimation process that is error prone; and necessary, since otherwise, if the errors are unbounded, the considered problem would be infeasible. In order to accomplish the capturing task, all agents must surround the target without colliding with it. Moreover, here we require the agents to keep some distance from the target by staying outside a distancing region and at the same time to be close to it by staying inside a larger containment region. Both regions were chosen to be of ellipsoidal shape for this work. Formally, the capturing requirements can be expressed as follows. **Problem: Target capturing problem with uncertain data (TCPUD)**. Given a swarm composed of n agents with the double integrator dynamics expressed in (6), consider a target with known position and uncertain velocity and acceleration. Determine the agent's control law $u_i(t) \in \mathbb{R}^d$, $i = 1,2,\ldots,n$, such that for a given shaping matrix $P = P^T > 0 \in \mathbb{R}^{d \times d}$ the following conditions hold:

1) Distancing: All agents remain outside a distancing region defined by the ellipsoid $\mathcal{E}(P, x_T(t))$, i.e., $\forall i = 1,2,\ldots,n$ and $\forall t > t_0, x_i(t_0) \notin \mathcal{E}(P, x_T(t_0)) \Rightarrow x_i(t) \notin \mathcal{E}(P, x_T(t))$.

2) Containment: All agents reach and remain inside the containment region defined by the ellipsoid $\mathcal{E}\left(P/c^2, x_T(t)\right)$. The parameter c that defines the width of this region, must be greater than a lower bound that takes into account the uncertainty on the target velocity and acceleration. In other words, $\exists \underline{c}(b_v, b_a) \geq 1$ such that, at steady-state, $x_i(t) \in \mathcal{E}(P/c^2, x_T(t))$, $c > \underline{c}(b_v, b_a)$ $\forall i = 1,2,\ldots,n$.

Remark 1: Apparently the definition of a containment region seems a trivial consequence of the stability and cohesiveness of the swarm. However, it is worth noting that, in general, the parameter c, i.e., the width of the containment region where agents should remain in order to solve the TCPUD, depends on the estimation errors bounds (b_v, b_a). In Section IV this dependency is clearly analyzed and it is also shown that the value of c can be arbitrarily chosen in two cases: i) the relative velocity of each agent with respect to the target, namely $w_i(t)$, is properly constrained; ii) no uncertainty on the target data exists.

As can be easily seen from above, both the distancing and the containment regions are described by the same type of ellipsoid with the only difference that semi-axes of the containment region are multiplied by the factor c, which depends on the accuracy of the target's velocity and acceleration estimates. Thus, to meet these requirements, agents have to remain in an ellipsoidal ring around the target as the one shown in Fig. 1.

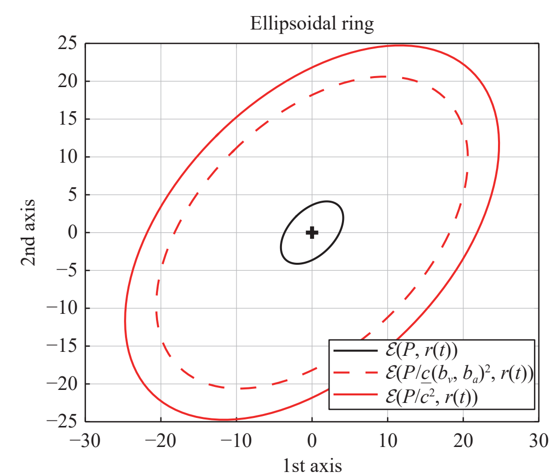


Fig. 1. An example of the ellipsoidal ring where agents have to remain in order to solve the TCPUD. The ellipse drawn with the dashed line represents the limit on the smallest containment region that the user can choose in this example according to the uncertainty on the target's velocity and acceleration.

III. Swarm Model

In order to solve the TCPUD, the control protocol of an individual agent is chosen as

$$u_i(t) = -\alpha_i P z_i(t) + \beta_i h_i(t) + \frac{P z_i(t)}{z_i^T(t) P z_i(t) - 1} - \gamma\left(v_i(t) - \hat{v}_T^i(t)\right) + m_i \hat{a}_T^i(t), \quad i = 1,2,\ldots,n \tag{8}$$

where $\alpha_i \in \mathbb{R}^+$, $\beta_i \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}^+$ are controller parameters to be determined. The term $h_i(t)$ is a swarm interaction term calculated as

$$h_i(t) = \sum_{j \in \mathcal{N}_i(t)} g(\|z_i(t) - z_j(t)\|)(z_i(t) - z_j(t)) \tag{9}$$

where $\mathcal{N}_i(t)$ indicates the set of neighbors of agent i at time instant t. In other words, we have $\mathcal{N}_i(t) = \{j \in \{1,2,\ldots,n\} \setminus \{i\}$ s.t. $\|x_i(t) - x_j(t)\| \le r\}$ where $r \in \mathbb{R}^+$ denotes the common detection radius of the agents; $g : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies

$$\exists l \in \mathbb{R}^+ \text{ s.t. } g(\|q(t)\|)\|q(t)\| \le l \ \ \forall t \tag{10}$$

$$g(\|q(t)\|)q(t) = 0 \quad \text{if} \quad \|q(t)\| \ge r. \tag{11}$$

Remark 2: Agents interact with each other if their mutual distance is less or equal to a common detection radius r. This means that in (9), the interaction term between two agents $g(\|z_i - z_j\|)(z_i - z_j)$ does not exist when the distance is greater than r and instead it is present when the distance is less or equal to r. If $g(\|q\|)\|q\| \neq 0$ when $\|q\| = r$ (agents begin to interact) this would make the swarm interaction term (9) not continuous and consequently also $u_i(t)$, leading to an ODE with a discontinuous right-hand side [27].

The swarm interaction term $h_i(t)$ constitutes the coupling between the otherwise independent individual agent's dynamics. It represents a proximity based collision avoidance and distancing from each other term. Note that $h_i(t)$ satisfies $\sum_{i=1}^n h_i(t) = 0^{d\times 1}$ due to the symmetry of the agent's interactions. This in a sense means that, although the interaction term affects the individual agent's dynamics, its average effect on the overall swarm dynamics is zero. Nonzero $\sum_{i=1}^n h_i(t)$ would mean non-reciprocal interactions resulting in some kind of bias in the overall swarm dynamics. Also, thanks to condition (10) the norm of $h_i(t)$ is bounded by the bound

$$\|h_i(t)\| \le |\mathcal{N}_i(t)|\, l \le nl \tag{12}$$

where $|\mathcal{N}_i(t)|$ is the cardinality of $\mathcal{N}_i(t)$. All the other terms in (8), instead, are related to the influence of the reference on the agents: the first term is responsible for the attraction of each agent to the target position; the third one ensures that no agent enters the distancing region; finally, the fourth and fifth terms take into account the dynamics of the target using velocity and acceleration estimates. The shared control parameter γ, in particular, can be considered as a kind of velocity matching coefficient.

IV. Analysis of Agent's Dynamics

Before showing the capabilities of the swarm under the proposed control law, let us introduce a preliminary result used in the following proofs.

Lemma 1: Let $f(y) = p + py - \ln(y)$ with $p \in \mathbb{R}$, and $y \in \mathbb{R}^+$. If $p > W_0(1/e)$, where $W_0(\cdot)$ is the principal branch of the Lambert W function, then $f(y) > 0$ holds.

Proof: The minimum of the given function is at $y^* = 1/p$ and it is equal to $f(y^*) = 1 + p + \ln(p)$. Such a value is greater than zero if $pe^p > 1/e$ whose solution is given by the principal branch of the Lambert W function [28], i.e., $p > W_0(1/e) \approx 0.28$. ■

A. Boundedness of Agent's Velocities

Here we show that the velocities of the agents relative to the target are bounded.

Lemma 2: Consider the swarm described by the model (6) with the control protocol in (8) where $\alpha_i > W_0(1/e)$. Then, the agent's velocity error $w_i(t)$ is ultimately bounded by

$$\|w_i(t)\| \le \overline{w}_i = b_v + \frac{\beta_i nl + m_i b_a}{\gamma}, \quad i = 1,2,\ldots,n. \tag{13}$$

Proof: Consider the Lyapunov function

$$V_{i1} = \frac{1}{2}\left(\alpha_i z_i^T P z_i - \ln\left(z_i^T P z_i - 1\right) + m_i w_i^T w_i\right) \tag{14}$$

with $\alpha_i > W_0(1/e)$.

Note V_{i1} is positive since the summation of the first two terms is positive due to Lemma 1 and the third term is positive because it is quadratic.

By using (6) and the bounds b_v and b_a, the derivative of V_{i1} can be written as

$$\begin{aligned}\dot{V}_{i1} &= \alpha_i z_i^T P w_i + m_i w_i^T\Big(-\frac{\alpha_i}{m_i} P z_i + \frac{\beta_i}{m_i} h_i + \frac{1}{m_i}\frac{P z_i}{z_i^T P z_i - 1} \\ &\quad - \frac{\gamma}{m_i}(v_i - \hat{v}_T^i) + \hat{a}_T^i - a_T\Big) - \frac{z_i^T P w_i}{z_i^T P z_i - 1} \\ &= \beta_i w_i^T h_i - \gamma\|w_i\|^2 - \gamma w_i^T(v_T - \hat{v}_T^i) + m_i w_i^T(\hat{a}_T^i - a_T) \\ &\le \beta_i nl\|w_i\| - \gamma\|w_i\|^2 + \gamma\|w_i\| b_v + m_i\|w_i\| b_a.\end{aligned} \tag{15}$$

From this inequality one can see that as long as $\|w_i\| > b_v + \frac{\beta_i nl + m_i b_a}{\gamma}$ we will have $\dot{V}_{i1} < 0$ which means that the agents organize so that their velocity error becomes less than $\|w_i(t)\| \le \overline{w}_i = b_v + \frac{\beta_i nl + m_i b_a}{\gamma}$. ■

As expected the bigger the uncertainties about the velocity and acceleration of the target, the bigger the bound. However, one should also note that the bounds are very conservative and in practice the actual bounds turn out to be smaller. Moreover, by increasing the value of the parameter γ the effect on the uncertainty in the acceleration can further be decreased.

B. Solution of the TCPUD

The main result of this work is formalized in the next theorem that indicates how to choose the control parameters to solve the TCPUD.

Theorem 1: Consider the swarm described by the model (6) with the control protocol in (8) and assume the i-th agent starts at t_0 outside a containment region defined by the ellipsoid $\mathcal{E}\left(P/c^2, x_T(t_0)\right)$ with

$$c > \underline{c}(b_v, b_a) = \max_{i=1,2,\ldots,n}\left\{1, \frac{m_i\left\|\sqrt{P}\right\|(m_i b_a + \gamma b_v)}{\gamma^2}\right\}. \tag{16}$$

If the control parameters are chosen as

$$\alpha_i > \max\{\overline{\alpha}_i, W_0(1/e)\} \tag{17}$$

where

$$\overline{\alpha}_i = \frac{\left(c+(c^2-1)\gamma\overline{w}_i\left\|\sqrt{P}^{-1}\right\|\right)\left(c\gamma+m_i\overline{w}_i\left\|\sqrt{P}\right\|\right)}{c(c^2-1)\left(c\gamma-m_i\overline{w}_i\left\|\sqrt{P}\right\|\right)} \tag{18}$$

and

$$\beta_i < \frac{c\gamma^2-m_i\left\|\sqrt{P}\right\|(m_ib_a+\gamma b_v)}{nlm_i\left\|\sqrt{P}\right\|} \tag{19}$$

then, the i -th agent asymptotically enters the containment region and remains indefinitely inside the ellipsoidal ring defined by the containment and distancing regions.

Proof: Consider the Lyapunov function $V_{i2} = \frac{1}{2}(m_iw_i+\gamma z_i)^T(m_iw_i+\gamma z_i)$, the model (6), and the bounds b_v and b_a, then the derivative of V_{i2} is

$$\begin{aligned}
\dot{V}_{i2} &= (m_iw_i+\gamma z_i)^T(m_i\dot{w}_i+\gamma\dot{z}_i)\\
&= (m_iw_i+\gamma z_i)^T\Big(-\alpha_iPz_i+\beta_ih_i+\frac{Pz_i}{z_i^TPz_i-1}\\
&\quad -\gamma(v_i-\hat{v}_T^i+v_T-v_T)+m_i(\hat{a}_T^i-a_T)+\gamma w_i\Big)\\
&= -m_i\alpha_iw_i^TPz_i-\gamma\alpha_iz_i^TPz_i+m_i\beta_iw_i^Th_i+\gamma\beta_iz_i^Th_i\\
&\quad +\frac{m_i\,w_i^TPz_i}{z_i^TPz_i-1}+\frac{\gamma\,z_i^TPz_i}{z_i^TPz_i-1}-m_i\gamma w_i^T(v_T-\hat{v}_T^i)\\
&\quad -\gamma^2z_i^T(v_T-\hat{v}_T^i)+m_i^2w_i^T(\hat{a}_T^i-a_T)+\gamma m_iz_i^T(\hat{a}_T^i-a_T).
\end{aligned} \tag{20}$$

Defining $Y_i=\sqrt{P}z_i$ and $Q_i=\sqrt{P}w_i$ and substituting in the above equation one obtains

$$\begin{aligned}
\dot{V}_{i2} &= -m_i\alpha_iQ_i^TY_i-\gamma\alpha_iY_i^TY_i+m_i\beta_iQ_i^T\sqrt{P}^{-1}h_i\\
&\quad +\gamma\beta_iY_i^T\sqrt{P}^{-1}h_i+\frac{m_i\,Q_i^TY_i}{(Y_i^TY_i-1)}+\frac{\gamma\,Y_i^TY_i}{(Y_i^TY_i-1)}\\
&\quad -(m_i\gamma Q_i^T\sqrt{P}^{-1}+\gamma^2Y_i^T\sqrt{P}^{-1})(v_T-\hat{v}_T^i)\\
&\quad +(m_i^2Q_i^T\sqrt{P}^{-1}+m_i\gamma Y_i^T\sqrt{P}^{-1})(\hat{a}_T^i-a_T).
\end{aligned} \tag{21}$$

Assuming that $\alpha_i > W_0(1/e)$, and using the result of Lemma 2 we have $\|Q_i\| \le \left\|\sqrt{P}\right\|\|w_i\| \le \left\|\sqrt{P}\right\|\overline{w}_i$ and, since $z_i^TPz_i>1$, $\dot{V}_{i2}$ can be upper bounded by

$$\begin{aligned}
\dot{V}_{i2} \le\ & m_i\alpha_i\left\|\sqrt{P}\right\|\overline{w}_i\|Y_i\|-\gamma\alpha_i\|Y_i\|^2\\
&+m_i\beta_i\left\|\sqrt{P}\right\|\overline{w}_i\left\|\sqrt{P}^{-1}\right\|nl+\gamma\beta_i\|Y_i\|\left\|\sqrt{P}^{-1}\right\|nl\\
&+\frac{m_i\left\|\sqrt{P}\right\|\overline{w}_i\|Y_i\|}{\|Y_i\|^2-1}+\frac{\gamma\|Y_i\|^2}{\|Y_i\|^2-1}\\
&+\left(m_i\gamma\left\|\sqrt{P}\right\|\overline{w}_i\left\|\sqrt{P}^{-1}\right\|+\gamma^2\|Y_i\|\left\|\sqrt{P}^{-1}\right\|\right)b_v\\
&+\left(m_i^2\left\|\sqrt{P}\right\|\overline{w}_i\left\|\sqrt{P}^{-1}\right\|+\gamma m_i\|Y_i\|\left\|\sqrt{P}^{-1}\right\|\right)b_a
\end{aligned} \tag{22}$$

where for the third and fourth terms, the bound (12) is used and for the last four terms the estimations bounds (7) are considered. As long as the i-th agent is outside the ellipsoidal region defined by $P/c^2, c>1$ then,

$$z_i^TPz_i-1 \ge c^2-1 \quad\Rightarrow\quad \frac{1}{(z_i^TPz_i-1)} \le \frac{1}{(c^2-1)}. \tag{23}$$

Using this property we have

$$\dot{V}_{i2} \le k_1\|Y_i\|^2+k_2\|Y_i\|+k_3 \tag{24}$$

where

$$k_1 = -\gamma\left(\alpha_i-\frac{1}{(c^2-1)}\right) \tag{25}$$

$$\begin{aligned}
k_2 &= m_i\alpha_i\overline{w}_i\left\|\sqrt{P}\right\|+\gamma\beta_inl\left\|\sqrt{P}^{-1}\right\|+\frac{m_i\overline{w}_i\|P\|}{c^2-1}\\
&\quad +\gamma^2\left\|\sqrt{P}^{-1}\right\|b_v+\gamma m_i\left\|\sqrt{P}^{-1}\right\|b_a
\end{aligned} \tag{26}$$

$$k_3 = m_i\overline{w}_i\left\|\sqrt{P}\right\|\left\|\sqrt{P}^{-1}\right\|(\beta_inl+\gamma b_v+m_ib_a). \tag{27}$$

The right hand side of (24) is a parabola in $\|Y_i\|$. In other words, $\dot{V}_{i2}$ is bounded by a parabola in the variable $\|Y_i\|$ with coefficients k_1, k_2, k_3. Noting that $c>1$ and assuming $\alpha_i > \frac{1}{(c^2-1)}$ we have $k_1<0$ and $k_2, k_3>0$. The right hand side has two real roots, one of which is negative and the other one is positive. Since it is a function of $\|Y_i\|$, only the positive root r_1 is a valid solution. By proper choice of the controller parameters, if we impose that $c>r_1$, then $\dot{V}_{i2}<0$ for $z_i^TPz_i>c^2$ and thus the agents will enter the ellipsoid $\mathcal{E}(P/c^2, x_T(t))$.

To obtain such a behavior we can use the free control parameter α_i. Its admissible values can be easily inferred as $\alpha_i > \overline{\alpha}_i$ with

$$\overline{\alpha}_i = \frac{\left(c+(c^2-1)\gamma\overline{w}_i\left\|\sqrt{P}^{-1}\right\|\right)\left(c\gamma+m_i\overline{w}_i\left\|\sqrt{P}\right\|\right)}{c(c^2-1)\left(c\gamma-m_i\overline{w}_i\left\|\sqrt{P}\right\|\right)}. \tag{28}$$

To ensure that the assumption $\alpha_i > \frac{1}{(c^2-1)}$ is satisfied, β_i must be chosen as

$$\beta_i < \frac{c\gamma^2-m_i\left\|\sqrt{P}\right\|(m_ib_a+\gamma b_v)}{nlm_i\left\|\sqrt{P}\right\|}. \tag{29}$$

At the same time, to meet the hypothesis of a positive β_i, the containment region must be chosen with

$$c > \frac{m_i\left\|\sqrt{P}\right\|(m_ib_a+\gamma b_v)}{\gamma^2}. \tag{30}$$

Finally, to satisfy the requirement $\alpha_i > W_0(1/e)$ we choose $\alpha_i > \max\{\overline{\alpha}_i, W_0(1/e)\}$. Since the radial approach velocities of the agents are finite, the unbounded repulsive effect from the distancing region due to the term $Pz_i(t)/(z_i^T(t)Pz_i(t)-1)$ dominates all the attractive terms and prevents the agents entering the distancing region. Therefore, when the agents enter the containment region then they remain within the two ellipsoids, i.e., $x_i(t) \in \mathcal{E}(P, x_T(t)) \cap \mathcal{E}\left(P/c^2, x_T(t)\right)$. ∎

Corollary 1: If the conditions of the theorem hold for all the n agents, the TCPUD is solved since all agents can reach and remain in the ellipsoidal ring.

Remark 3: Clearly there is no guarantee that an edge between two agents cannot be lost when their mutual distance is greater than r and it can also be the case that one or more agents become singletons, i.e., they are not connected with any other agent. However, also in these cases, agents tend to enter the ellipsoidal ring and remain therein indefinitely. Note that the results of the paper are clearly valid also in the case of fixed (connected) graph where the neighborhood of each agent is constant.

Remark 4: According to the Theorem 1, in order to properly choose the control parameters each agent must know the mass of all the other ones. In the robotic context this is not usually a problem since the vehicles of a team are all known. However, if some uncertainty affects these data, it is always possible to use an upper bound value of the mass, $\overline{m}$, in place of the single mass in (16) and consequently $\underline{c}(b_v,b_a)=\max\{1, \overline{m}\|\sqrt{P}\|(\overline{m}b_a+\gamma b_v)/\gamma^2\}$. As expected, this larger mass makes the admissible containment regions wider.

This theorem does not provide any guidance on how to choose the γ control parameter. Its effect, however, is decisive for two characteristics of the swarm: i) the relative speed of the agent with respect to the target, $w_i(t)$, is inversely proportional to it (see (13)); ii) if γ exceeds a certain value, it allows to arbitrarily choose the containment region even in the case of uncertain data. According to (16), this is possible if $m_i\|\sqrt{P}\|(m_ib_a+\gamma b_v)/\gamma^2<1\ \forall i=1,2,\dots,n$ that is satisfied if

$$\gamma>\max_{i=1,2,\dots,n}\left\{\frac{1}{2}m_i\left\|\sqrt{P}\right\|b_v\left(1+\sqrt{1+\frac{4b_a}{b_v^2\left\|\sqrt{P}\right\|}}\right)\right\}. \tag{31}$$

Finally, we provide an outline of the steps necessary to solve the TCPUD:

1) According to the TCPUD definition, set the distancing region P, the estimation bounds b_v,b_a and the agents mass m_i (or alternatively an upper bound value $\overline{m}$).

2) Choose γ giving priority to the agents' maximum relative velocity (see (13)) or the smallest possible containment region (see (31)).

3) Choose the parameter c (containment region) according to (16).

4) Each agent, given the above parameters, chooses the control parameters α_i,β_i according to (17) and (19), respectively.

Following these indications, which respect the formal conditions required by Theorem 1, it is always possible to find a set of parameters that solve the TCPUD.

C. Stationary Behavior

This section shows some characteristics of the swarm model in the case of known target. It proves that asymptotically the individuals will converge to a static position w.r.t. the target and therefore to a constant relative configuration. In particular it is shown (see Theorem 3) that, thanks to the interaction term, when the agents are in the ellipsoidal ring and, at steady-state at least two agents are connected then, they organize so that the target is a convex combination of the agents' positions. With that objective, let us consider the case in which:

i) *No uncertainty* affects the data about the target, i.e.,

$$b_v=b_a=0. \tag{32}$$

ii) All the agents have the *same mass*, i.e.,

$$m_i=m,\ \forall i=1,2,\dots,n. \tag{33}$$

Note that assuming the same mass for all agents does not imply an additional restriction since the effect of the mass can easily be compensated by appropriately scaling the agents' control input. In terms of the perfect knowledge of the velocity and acceleration, the analysis here can also represent the case in which the estimation errors on the velocity and acceleration of the target vanish as time progresses. For example, it could be the case that a tool such as a high gain observer [29] is utilized to estimate the velocity and acceleration of the target from its position data guaranteeing asymptotic convergence of the estimation errors to zero.

In the described case, we still can apply the results of Lemma 2 and Theorem 1 to solve the TCPUD. The resulting control law is

$$\begin{aligned}u_i(t)=&-\alpha Pz_i(t)+\beta h_i(t)+P\frac{z_i(t)}{(z_i^T(t)Pz_i(t)-1)}\\&-\gamma w_i(t)+ma_T(t),\quad i=1,2,\dots,n\end{aligned} \tag{34}$$

with $c>1$, $\beta<c\gamma^2/\left(mnl\left\|\sqrt{P}\right\|\right)$, and $\alpha>\max\{W_0(1/e),\overline{\alpha}\}$ where

$$\overline{\alpha}=\frac{\left(c+(c^2-1)\beta nl\left\|\sqrt{P^{-1}}\right\|\right)\left(c\gamma^2+m\beta nl\left\|\sqrt{P}\right\|\right)}{c(c^2-1)\left(c\gamma^2-m\beta nl\left\|\sqrt{P}\right\|\right)}. \tag{35}$$

The previous control parameters α_i,β_i, $i=1,2,\dots,n$ of the control law (8), become now the common values α,β thanks to the equal mass. This homogeneity, moreover, transforms the result of Lemma 2 in

$$\|w(t)\|\le\overline{w}=b_v+\frac{\beta nl+mb_a}{\gamma}. \tag{36}$$

In the following theorem it is shown how, at steady-state, the swarm reaches a static configuration around the moving target.

Theorem 2: Consider the swarm described by the model (6) under the conditions (32), (33) and the control protocol (34). If there exists a function $J:\mathbb{R}^+\to\mathbb{R}$ such that $\nabla_{q(t)}J(\|q(t)\|)=g(\|q(t)\|)q(t)$, and $\alpha>W_0(1/e)$ in (34) then $\lim_{t\to\infty}w_i(t)=0_d\quad\forall i=1,2,\dots,n$, or, in other words, all agents reach a velocity consensus.

Proof: Let $U(t)=\sum_{i=1}^{n}U_i(t)$ be the candidate Lyapunov function where

$$\begin{aligned}U_i(t)=&\frac{\alpha}{2m}z_i^T(t)Pz_i(t)-\frac{1}{2m}\ln\left(z_i^T(t)Pz_i(t)-1\right)\\&-\frac{\beta}{2m}\left(\sum_{j\in\mathcal{N}_i(t)}J(\|z_i(t)-z_j(t)\|)-nl_2\right)\\&+\frac{1}{2}w_i^T(t)w_i(t)\end{aligned} \tag{37}$$

with $z_i(t)\in\mathcal{D}\triangleq\left\{z\in\mathbb{R}^d\mid z^TPz>1\right\}$ and

$$
\begin{aligned}
l_2 &= \max_{z_i(t),z_j(t)\in\mathcal{E}(P/c^2,0)\setminus\mathcal{E}(P,0)} J(\|z_i(t)-z_j(t)\|) \\
&= J(2\max\{\mathcal{S}(P/c^2)\}).
\end{aligned} \tag{38}
$$

Note that, according to Lemma 1, $U_i(t)$ is positive definite in $\mathcal{E}(P/c^2,0)\setminus\mathcal{E}(P,0)$ if $\alpha > W_0(1/e)$.

For any $u>0$, let $\Omega_u = \{z(t)\in\mathcal{D}\,|\,U(t)\le u\}$ denote the level sets of $U(t)$ and observe that given i and $j\neq i$,

$$
\begin{aligned}
\nabla_{z_i} U_i &= \frac{1}{m}\left(\alpha P z_i - \frac{P z_i}{z_i^T(t)Pz_i(t)-1} - \frac{\beta}{2}h_i\right) \\
&= -\dot{w}_i - \frac{\gamma}{m}w_i + \frac{\beta}{2m}h_i
\end{aligned} \tag{39}
$$

$$
\nabla_{z_j} U_i = \frac{\beta}{2m} g(\|z_i-z_j\|)(z_i-z_j) \tag{40}
$$

$$
\nabla_{w_j} U_i = \begin{cases} w_i, & \text{if } i=j \\ 0, & \text{otherwise} \end{cases} \tag{41}
$$

and then $\dot{U}(t) = \sum_{i=1}^n \dot{U}_i(t)$ results in

$$
\begin{aligned}
\dot{U}(t) = &\sum_{i=1}^n \Big(\nabla_{z_i(t)} U_i^T(t)\dot{z}_i(t) + \nabla_{w_i(t)} U_i^T(t)\dot{w}_i(t) \\
&+ \sum_{j\in\mathcal{N}_i(t)} \left(\nabla_{z_j(t)} U_i^T(t)\dot{z}_j(t) + \nabla_{w_j(t)} U_i^T(t)\dot{w}_j(t)\right)\Big) \\
= &-\frac{\gamma}{m}\sum_{i=1}^n \|w_i(t)\|^2 \\
&+ \frac{\beta}{2m}\sum_{i=1}^n \sum_{j\in\mathcal{N}_i(t)} g(\|z_i(t)-z_j(t)\|)(z_i(t)-z_j(t))^T w_i(t) \\
&+ \frac{\beta}{2m}\sum_{i=1}^n \sum_{j\in\mathcal{N}_i(t)} g(\|z_i(t)-z_j(t)\|)(z_i(t)-z_j(t))^T w_j(t).
\end{aligned} \tag{42}
$$

Since connections are bidirectional and the summations in the last two terms include all the possible couples of neighbour agents, such terms have zero sum and therefore,

$$
\dot{U}(t) = -\frac{\gamma}{m}\sum_{i=1}^n \|w_i(t)\|^2 \le 0. \tag{43}
$$

Clearly, $\dot{U}(t)$ is always nonpositive, hence the level sets Ω_u are positively invariant. Then, by continuity of $U(t)\in\mathcal{D}$, it follows that $z_i^T(t)Pz_i(t) > 1\,\forall i$ and thus Ω_u are compact [30]. By the LaSalle's Invariance Principle [31], every solution starting in Ω_u asymptotically converges to the largest invariant set in $\{z(t)\in\mathcal{D}\,|\,\dot{U}(t)=0\}$. This means that the agents will reach a static configuration w.r.t. the reference $x_T(t)$. ■

The benefits of the agents' interaction term on the swarm steady-state configuration will now be highlighted showing that the $g(\cdot)$ term properties drive the swarm to a configuration where the target position results in a convex combination of the agents' positions.

Lemma 3: Under the conditions of Theorem 2, at steady-state $z_i^T P z_i \ge \frac{\alpha+1}{\alpha}, \forall i$ is satisfied.

Proof: The proof straightforwardly follows, yielding to $z_i^T P z_i = \frac{\alpha+1}{\alpha}$, for agents without neighbours. If $\mathcal{N}_i$ is not empty then, starting from (34) it follows that at steady-state:

$$
\begin{aligned}
&\frac{1}{m}\sum_{i=1}^n\left(\alpha - \frac{1}{z_i^T P z_i - 1}\right) z_i^T P z_i \\
&\quad = \frac{\beta}{m}\sum_{i=1}^n \sum_{j\in\mathcal{N}_i} g(\|z_i-z_j\|) z_i^T(z_i-z_j) > 0
\end{aligned} \tag{44}
$$

from which

$$
\alpha > \sum_{i=1}^n \frac{z_i^T P z_i}{\sum_{j=1}^n z_j^T P z_j}\frac{1}{z_i^T P z_i - 1} \tag{45}
$$

that represents a convex combination of $\frac{1}{z_i^T P z_i - 1}, i=1,\ldots,n$. As a consequence $\alpha > \frac{1}{z_i^T P z_i - 1}$, $\forall i$ follows from (45) and the same arguments in [32]. ■

Theorem 3: Under the conditions of Theorem 2 and assuming that at steady-state there exist at least two agents i, j such that $j\in\mathcal{N}_i$, then,

$$
x_T = \sum_{i=1}^n \xi_i x_i, \sum_{i=1}^n \xi_i = 1, \xi_i > 0 \;\; \forall i \tag{46}
$$

i.e., the agents reach a configuration where the target is in the convex hull of their positions.

Proof: In a steady-state condition it follows that:

$$
-\alpha P z_i + \beta h_i + \frac{P z_i}{z_i^T P z_i - 1} = 0. \tag{47}
$$

By summing on i and considering that $z_i = x_i - x_T$ the proof follows with $\xi_i = \tilde{\xi}_i / \sum_{k=1}^n \tilde{\xi}_k$ with $\tilde{\xi}_i = \alpha - \frac{1}{(x_i-x_T)^T P(x_i-x_T)-1}$ and $\tilde{\xi}_i \ge 0$ by Lemma 3. Note that, at least one $\tilde{\xi}_i$ is not zero by the assumptions of the theorem. ■

V. Simulations

In this section we will use three simulations in order to better show the behavior of the swarm described in the previous results. The first two are numerical simulations of the proposed protocol while the last is based on the SIL methodology.

A. Example 1

Let us consider a group of six agents in the 2D-space with mass $m_i = i\times 0.1\,\text{kg}, i=1,2,\ldots,6$ and the common detection radius is $r=10\,\text{m}$. The actual target trajectory is a hypotrochoid described by the equation

$$
x_T(t) = \begin{bmatrix} 20 \\ 10 \end{bmatrix} + 1.5\begin{bmatrix} 2\cos(0.8t) + 5\cos(0.53t) \\ 2\sin(0.8t) - 5\sin(0.53t) \end{bmatrix}. \tag{48}
$$

The estimates of its velocity and acceleration made by each agent are

$$
\hat{v}_T^i(t) = v_T(t) + [0,\; b_v^i\cos(it)]^T \tag{49}
$$

$$
\hat{a}_T^i(t) = a_T(t) + [0,\; b_a^i\sin(it)]^T \tag{50}
$$

where b_v^i and b_a^i, $i=1,2,\ldots,6$, have been randomly chosen with uniform distribution in $(0,b_v)$ and $(0,b_a)$, respectively. For this simulation $b_v = 0.5\,\text{m/s}$ and $b_a = 1\,\text{m/s}^2$ have been chosen. We set a distancing region defined by the matrix $P = R^T(\pi/6)\text{diag}(1/4,1)R(\pi/6)$ where $R(\pi/6)\in SO(2)$ represents a clockwise rotation of $\pi/6$ rad. To solve the TCPUD, we use the control protocol (8) where the damping coefficient is $\gamma = 10$ and the other control parameters have been chosen according to Theorem 1. In particular, $c = 2, \beta_i = 1, i = 1, 2, \ldots, 6$ and $[\alpha_1, \alpha_2, \ldots, \alpha_6] = [11.57, 11.80, 12.04, 12.29, 12.54, 12.80]$. Finally, to properly define $h_i(t)$ (see (9)), we choose

$$g(\|q(t)\|) = \begin{cases} \dfrac{1}{\|q(t)\| + \varepsilon} - \dfrac{1}{r + \varepsilon}, & \text{if } 0 \le \|q(t)\| \le r \\ 0, & \text{if } \|q(t)\| > r \end{cases} \tag{51}$$

with $\varepsilon = 0.01$. Moreover, according to (10), $g(\|q(t)\|)\|q(t)\| \le l = 1 \ \forall t$. With these choices and random agents' initial positions outside the containment region as required by Theorem 1 (see Table I), a simulation with $t \in [0, 30]$ s leads to the paths shown in Fig. 2 . As expected by the theoretical results, the agents converge to an ellipsoidal ring around the moving target defined by the distancing and containment regions.

TABLE I
INITIAL POSITIONS OF THE AGENTS IN THE FIRST TWO EXAMPLES

Agent	1	2	3	4	5	6
1st axis (m)	50.08	52.74	55.34	48.56	50.90	47.50
2nd axis (m)	39.51	42.69	40.82	32.02	43.47	40.69

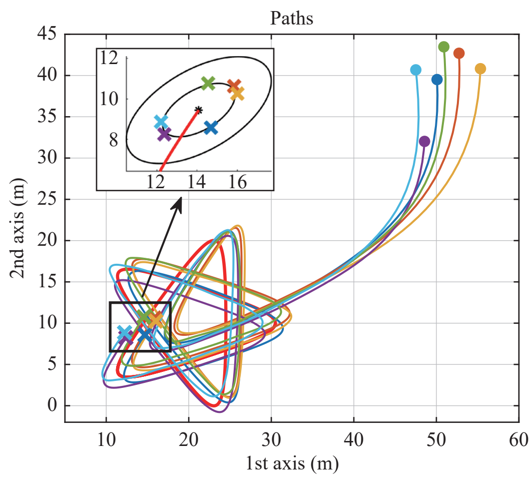


Fig. 2. *Example 1*. Agents' paths in coloured lines along the target trajectory in red. Circles are the initial positions while crosses are the final ones. In the magnified view, the final agents' positions inside the ellipsoidal ring are shown.

B. Example 2

In this simulation we want to highlight the ability of the swarm to reach a static configuration around the target when the conditions of Theorem 2 are met. To this aim we consider the same scenario of the previous simulation with some important differences: all agents have the same mass $m_i = 0.5$ kg, $i = 1, 2, \ldots, 6$; the target position, velocity and acceleration, are perfectly known by all the agents thanks to $b_v = b_a = 0$; the detection radius is $r = 3$ m and is such that agents are not guaranteed to know all the other ones once in the ellipsoidal ring ($\max\{\mathcal{S}(P)\} = 2$ m); $\beta = 1$ and $\alpha = 36$ according to Theorem 1; a new interaction term $h_i(t)$ is used thanks to

$$g(\|q(t)\|) = \begin{cases} \dfrac{1}{\|q(t)\|^2 + \varepsilon} - \dfrac{1}{r^2 + \varepsilon}, & \text{if } 0 \le \|q(t)\| \le r \\ 0, & \text{if } \|q(t)\| > r \end{cases} \tag{52}$$

with $\varepsilon = 0.01$. According to (10), $g(\|q(t)\|)\|q(t)\| \le l = \frac{1}{2\sqrt{\varepsilon}} \ \forall t$, holds and the function $J(\cdot)$ that satisfies Theorem 2 is

$$J(\|q(t)\|) = \frac{1}{2}\left(\ln\left(\|q(t)\|^2 + \varepsilon\right) - \frac{\|q(t)\|^2}{r^2 + \varepsilon}\right). \tag{53}$$

Starting from the same positions of the previous example, a simulation with $t \in [0, 60]$ s has been run. Fig. 3 shows how the velocities of the agents relative to the target tend to zero, and consequently a static configuration around the moving target is achieved. Such a configuration is shown in Fig. 4(a) that also highlights another important feature of the proposed control law: the convex hull formed by the agents' positions includes the target ensuring an effective enclosing (see Theorem 3). It is worth pointing out that such a goal is achieved even if not all the agents know each other as evident from Fig. 4(b).

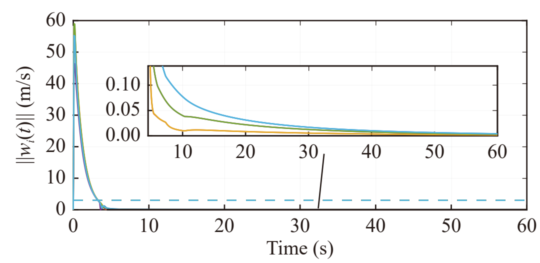
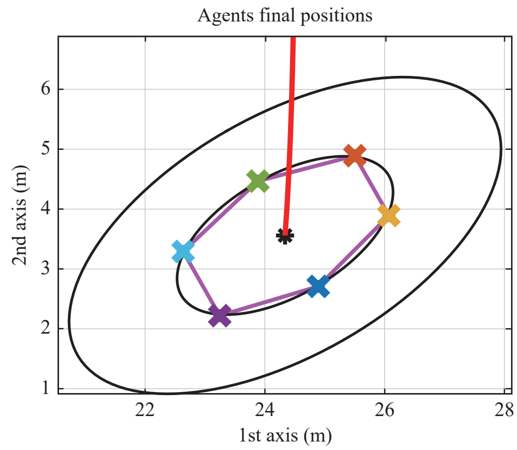


Fig. 3. *Example 2*. The evolution of velocities of the agents w.r.t. the target, i.e., $w_i(t)$. The magnified view helps showing how the relative velocities tend to zero according to Theorem 2. After about 60 s, the agents reach a static configuration around the target (see Fig. 4). The dashed line, finally, is the common upper bound $\overline{w} = 3$ m/s provided by Lemma 2 (see (36)).

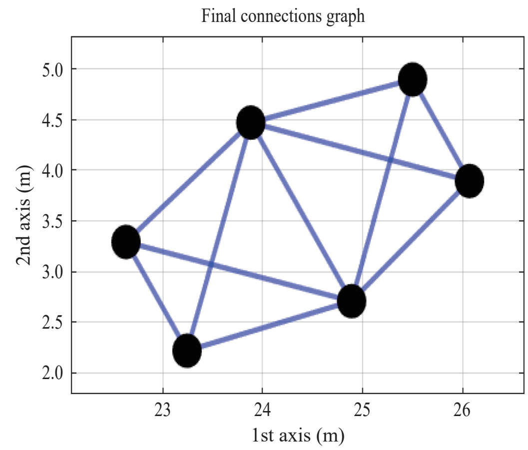
C. Software in the Loop Simulation

This last simulation aims to show how the proposed control protocol can be used in a realistic scenario where the TCPUD is assigned to a team of differential drive UGVs.

For this purpose, we have chosen the Simulink-Gazebo co-simulation environment [33]. The advantage of such a tool is to quickly implement the control logic in Simulink and, at the same time, to exploit the high fidelity ODE physics engine of Gazebo for the dynamics simulation. This relationship can be appreciated in Fig. 5 , where a scheme with the main components of the simulation is shown. Here the proposed protocol is not used directly as control input for the single robot. The actual controller, in fact, is the classic non linear tracker for differential drive robots reported in [12] followed by a proportional controller that computes the torque commands for the wheels. Since this algorithm is based on the differential flat property of the unicycle model, a reference trajectory and its derivative up to the second order must be



(a) The convex hull formed by the agents' positions (magenta polytope) includes the target (black asterisk) as stated in Theorem 3. The red line represents the target trajectory in the last time instants.



(b) The connections graph: the positions of the nodes are equal to the real positions of the agents. It is evident how the detection radius does not allow a complete connectivity.

Fig. 4. *Example 2*. Agents' configuration at the final instant $t = 60\,\text{s}$.

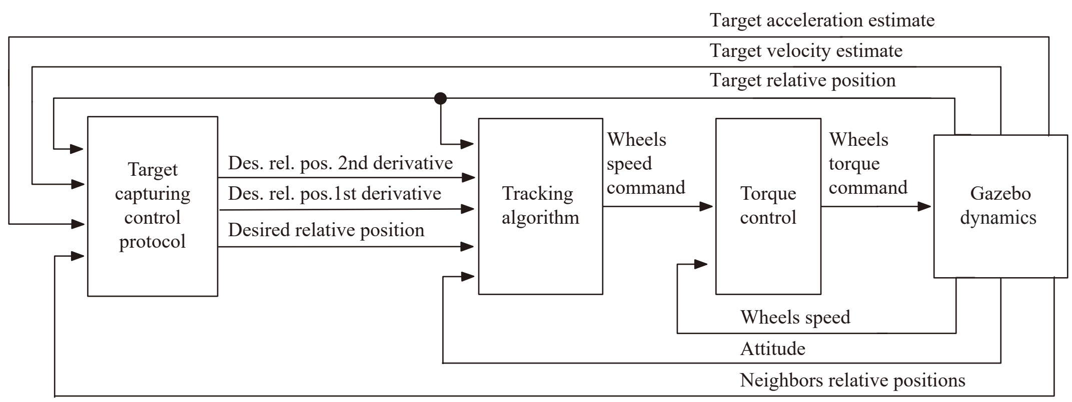


Fig. 5. *SIL simulation*. The input/output relationships among the main components of the simulation.

provided. The evolution of the error (6) and the control input (8) are sufficient to generate these required data if we assume that the coordinate bases of the different agents are directionally aligned (this is commonly achieved by using compasses or landmark data). To better show the behavior of the team due to the control protocol (8) we also assume that each robot receives from Gazebo the relative positions of the target and all the other robots inside its detection radius. In this way we avoided the influence of all the issues related to the measurements and localization processes. The exchange of data between Simulink and Gazebo occurs at each sampling time $T_s = 0.01\,\text{s}$, that is also used to compute an Euler-forward discretized version of the proposed protocol (8) and the tracking algorithm [12].

With this setup, we have simulated six Pioneer 2DX [34] mobile robots whose mass is $m_i = 5.67\,\text{kg}$, $i = 1,2,\ldots,6$ and equipped with a detection radius $r = 10\,\text{m}$. The target robot is of the same kind and tracks the following figure-eight trajectory:

$$p(t) = \begin{bmatrix} 22 \\ -1 \end{bmatrix} + 8 \begin{bmatrix} \cos(0.2\,t) \\ \sin(0.4\,t)/2 \end{bmatrix}. \tag{54}$$

While all robots know the target relative position ($z_i(t) = x_i(t) - x_T(t)$), the estimates of its velocity and acceleration are affected by the same kind of error reported in the first simulation. The chosen distancing region is an ellipse defined by the matrix $P^* = \operatorname{diag}(1/4, 9/16)$. By tracking the references given by the protocol, however, there is no guarantee for a real robot not to enter the distancing region. This is due to two factors: the real size of the robot that is not taken into account in the protocol and the performance of the tracking algorithm. Such a drawback can be easily overcome by using a larger region in the protocol than the one actually desired. In this simulation, for example, we have enlarged both the axes of a factor 1.5 obtaining $P = \operatorname{diag}(1/9, 1/4)$. The velocities matching coefficient is $\gamma = 25$ and the other control parameters have been chosen according to Theorem 1. In particular, $c = 1.5$, $\beta_i = 2$, and $\alpha_i = 73.42$, $i = 1,2,\ldots,6$. To define the $h_i(t)$ term the same $g(\|q(t)\|)$ function of the first simulation has been chosen (see (51)). The simulation run with these parameters gives satisfactory results as confirmed by the tracking errors reported in Fig. 6 and by the convergence in the containment region that occurs at $t = 11.58$ s. A snapshot of the simulation at the final instant $t = 60$ s, reported in Fig. 7, shows that all the agents are in fact inside the ellipsoidal ring. A video of the entire experiment is also available at the following link: https://www.youtube.com/watch?v=nS3vMuO8j3g.

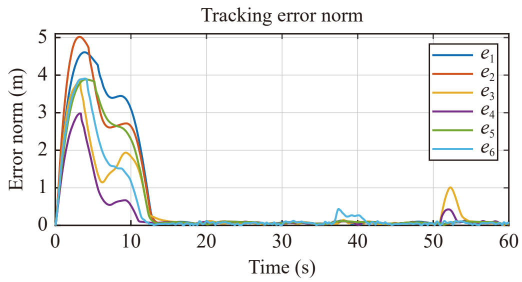


Fig. 6. *SIL simulation.* The norm of the tracking error for the relative position reference: e_i, $i = 1,\ldots,6$, represents the norm of the tracking error between the position of the robot ith and its reference trajectory.

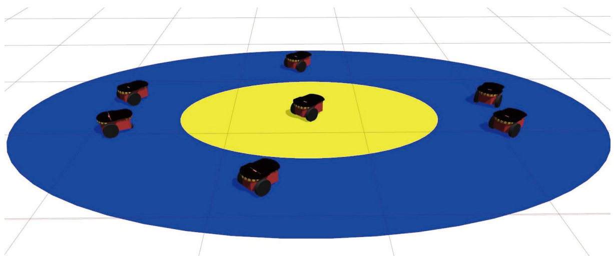


Fig. 7. *SIL simulation.* A snapshot from the simulation video focusing on the formation around the target robot. The yellow ellipse is the distancing region and the blue is the containment one.

VI. Conclusions

In this paper a strategy for the uncertain target capturing problem has been proposed. It has been shown that the swarm enters an ellipsoidal ring around the target and remains therein, indefinitely. Moreover, steady-state properties of the swarm have been investigated particularly stressing the capability of the agents to organize such that the target is a convex combination of the agents' positions. Future developments will be devoted to consider more general agent's dynamics and removing hypotheses on the communication facilities, e.g., send/receive operations amongst the agents may be subject to latency phenomena.

References

[1] W. Ren and Y. Cao, *Distributed Coordination of Multi-Agent Networks: Emergent Problems, Models, and Issues*. Springer Science & Business Media, 2010.

[2] V. Gazi and K. M. Passino, *Swarm Stability and Optimization*. Springer Science & Business Media, 2011.

[3] C. Robin and S. Lacroix, "Multi-robot target detection and tracking: Taxonomy and survey," *Autonomous Robots*, vol. 40, no. 4, pp. 729–760, 2016.

[4] T.-H. Kim and T. Sugie, "Cooperative control for target-capturing task based on a cyclic pursuit strategy," *Automatica*, vol. 43, no. 8, pp. 1426–1431, 2007.

[5] J. Li and X. Chen, "Multi-AUV circular formation sliding mode control based on cyclic pursuit," in *Proc. IEEE Int. Conf. Mechatronics and Automation*, 2020, pp. 1365–1370.

[6] G. Fedele, L. D'Alfonso, F. Chiaravalloti, and G. D'Aquila, "Obstacles avoidance based on switching potential functions," *Journal of Intelligent & Robotic Systems*, vol. 90, no. 3–4, pp. 387–405, 2018.

[7] L. Blázovics, T. Lukovszki, and B. Forstner, "Target surrounding solution for swarm robots," in *Information and Communication Technologies*, R. Szabó and A. Vidács, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2012, pp. 251–262.

[8] Y. Kobayashi, K. Otsubo, and S. Hosoe, "Design of decentralized capturing behavior by multiple mobile robots," in *Proc. IEEE Workshop Distributed Intelligent Systems: Collective Intelligence and Its Applications*, 2006, pp. 13–18.

[9] J. Yao, R. Ordonez, and V. Gazi, "Swarm tracking using artificial potentials and sliding mode control," *Journal of Dynamic Systems, Measurement and Control*, vol. 129, no. 5, pp. 749–754, Sept. 2007.

[10] H. Kawakami and T. Namerikawa, "Cooperative target-capturing strategy for multi-vehicle systems with dynamic network topology," in *Proc. American Control Conf.*, 2009, pp. 635–640.

[11] M. Kothari, R. Sharma, I. Postlethwaite, R. W. Beard, and D. Pack, "Cooperative target-capturing with incomplete target information," *Journal of Intelligent & Robotic Systems*, vol. 72, no. 3–4, pp. 373–384, 2013.

[12] B. Siciliano, L. Sciavicco, L. Villani, and G. Oriolo, *Robotics: Modelling, Planning and Control*. Springer Publishing Company, Incorporated, 2010.

[13] M. Lv, W. Yu, J. Cao, and S. Baldi, "Consensus in high-power multiagent systems with mixed unknown control directions via hybrid nussbaum-based control," *IEEE Trans. Cybernetics*, pp. 1–13, 2020. DOI: 10.1109/TCYB.2020.3028171

[14] M. Lv, W. Yu, J. Cao, and S. Baldi, "A separation-based methodology to consensus tracking of switched high-order nonlinear multiagent systems," *IEEE Trans. Neural Networks and Learning Systems*, pp. 1–13, 2021. DOI: 10.1109/TNNLS.2021.3070824

[15] O. Y. Nieto and L. Colombo, "A geometric path planning strategy based on variational calculus for the shape control of multi-agent Lagrangian systems," in *Proc. 29th Mediterranean Conf. Control and Automation*, 2021, pp. 1092–1099.

[16] W. Wang, H. Liang, Y. Pan, and T. Li, "Prescribed performance adaptive fuzzy containment control for nonlinear multi-agent systems using disturbance observer," *IEEE Trans. Cybernetics*, vol. 50, no. 9, pp. 3879–3891, 2020.

[17] A. Bono, G. Fedele, and G. Franzè, "A swarm-based distributed model predictive control scheme for autonomous vehicle formations in uncertain environments," *IEEE Trans. Cybernetics*, 2021. DOI: 10.1109/TCYB.2021.3070461.

[18] Q. Shi, X. Cui, S. Zhao, and M. Lu, "Sequential TOA-based moving target localization in multi-agent networks," *IEEE Communications Letters*, vol. 24, no. 8, pp. 1719–1723, 2020.

[19] M. Deghat, I. Shames, B. D. Anderson, and C. Yu, "Localization and circumnavigation of a slowly moving target using bearing measurements," *IEEE Trans. Automatic Control*, vol. 59, no. 8, pp. 2182–2188, 2014.

[20] S. Chun and Y.-P. Tian, "Multi-targets localization and elliptical circumnavigation by multi-agents using bearing-only measurements in two-dimensional space," *Int. Journal of Robust and Nonlinear Control*, vol. 30, no. 8, pp. 3250–3268, 2020.

[21] S. Chun, "Bearing-only-based formation circumnavigation guided by multiple unknown targets," *IEEE Access*, vol. 8, pp. 228377–228391, 2020.

[22] Y. Yu, Z. Li, X. Wang, and L. Shen, "Bearing-only circumnavigation control of the multi-agent system around a moving target," *IET Control Theory & Applications*, vol. 13, no. 17, pp. 2747–2757, 2019.

[23] G. Fedele and L. D'Alfonso, "A kinematic model for swarm finite-time trajectory tracking," *IEEE Trans. Cybernetics*, vol. 49, no. 10, pp. 3806–3815, 2018.

[24] G. Fedele, L. D'Alfonso, and A. Bono, "Vortex formation in a swarm of agents with a coordinates mixing matrix-based model," *IEEE Control Systems Letters*, vol. 4, pp. 420–425, 2020.

[25] G. Fedele and L. D'Alfonso, "A coordinates mixing matrix-based model for swarm formation," *Int. Journal of Control*, DOI: 10.1080/00207179.2019.1613561.

[26] G. Fedele, L. D'Alfonso, and A. Bono, "A discrete-time model for swarm formation with coordinates coupling matrix," *IEEE Control*

Systems Letters, DOI: 10.1109/LCSYS.2020.2998669.

[27] A. Filippov, *Differential Equations With Discontinuous Righthand Sides*, A. F.M., Ed. Springer, Dordrecht, 1988.

[28] R. M. Corless, G. H. Gonnet, D. E. Hare, D. J. Jeffrey, and D. E. Knuth, "On the LambertW function," *Advances in Computational Mathematics*, vol. 5, no. 1, pp. 329–359, 1996.

[29] H. K. Khalil, *High-Gain Observers In Nonlinear Feedback Control*. SIAM, 2017.

[30] M. M. Zavlanos, A. Jadbabaie, and G. J. Pappas, " Flocking while preserving network connectivity," in *Proc. 46th IEEE Conf. Decision and Control*, 2007, pp. 2919–2924.

[31] H. K. Khalil, *Nonlinear Systems, 3rd*. Prentice Hall, 2002.

[32] L. Sabattini, C. Secchi, and C. Fantuzzi, "Arbitrarily shaped formations of mobile robots: Artificial potential fields and coordinate transformation," *Autonomous Robots*, vol. 30, no. 4, pp. 385–397, 2011.

[33] Perform co-simulation between simulink and gazebo. [Online]. Available: https://www.mathworks.com/help/robotics/ug/perform-co-simulation-between-simulink-and-gazebo.html

[34] Pioneer 2 operations manual. [Online]. Available: http://www.iri.upc.edu/groups/lrobots/private/Pioneer2/AT_DISK1/DOCUMENTS/p2opman9.pdf

Antonio Bono (Graduate Student Member, IEEE) received the master degree (cum laude) in control engineering from University of Calabria, Italy, in 2018. He is currently a Ph.D. student at the ICT School of the Department of Informatics, Modeling, Electronics and Systems Engineering at University of Calabria, Italy. His research interests are mainly focused on networked control of multi-agent systems and distributed model predictive control for robotics applications. From September 2019 to March 2020, he was a visiting Research Scholar with the School of Electrical and Computer Engineering (ECE) at the Georgia Institute of Technology, USA.

Luigi D'Alfonso received the Ph.D. degree in computer science and system engineering from the University of Calabria, Italy, in 2014. Since 2011 to 2014 he was with the Department of Informatics, Modeling, Electronics and Systems Engineering of the University of Calabria as a Ph.D. student. In 2012, as a Visiting Ph.D. Scholar, he was with the Service d'Automatique et d'Analyse des Systems at the Université Libre de Bruxelles, Belgium. Since 2015 to 2021 he was the Principal Researcher at GiPStech s.r.l.. Since 2022 he is a Research Fellow with the Department of Informatics, Modeling, Electronics and Systems Engineering, University of Calabria, Italy. His current research interests include mobile robots control, localization/mapping/SLAM for single-agent and multi-agents systems, perspective-n-point problem using cameras and inertial measurements units, study and control of swarms of agents.

Giuseppe Fedele (Member, IEEE) received the laurea (M.Sc) degree (cum laude) in computer science engineering and the Ph.D. degree in computer science and system engineering from the University of Calabria, Italy, in 1999 and 2005, respectively. Since 2006, he has been an Assistant Professor in control engineering with the Department of Informatics, Modeling, Electronics and Systems Engineering, University of Calabria. He is a Founding Member of GiPStech s.r.l., Rende, a startup, and spin-off of the University of Calabria, which develops novel solutions for the indoor localization and navigation problems. His current research interests include power systems, identification and filtering methods, adaptive control, adaptive algorithms for active noise and vibration control, signal pro-cessing for localization, navigation, and tracking, multiagent systems. Dr. Fedele is a Guest Editor of the Special Issue "Recent Advances in Adaptive Methods for Frequency Estimation with Applications," International Journal of Adaptive Control and Signal Processing, 2016. He currently serves as an Academic Editor for the *Mathematical Problems in Engineering*.

Veysel Gazi (Senior Member, IEEE) received the B.S. degree in electrical and electronics engineering from the Middle East Technical University, Ankara Turkey in 1996, and M.S. and Ph.D. degrees in electrical engineering from the Ohio State University in 1998 and 2002, respectively. From September 1996 to April 1997 he was a Scholar of the Scientific and Technological Research Council of Turkey (TUBITAK). He is currently with the Department of Control and Automation Engineering, Faculty of Electrical and Electronics Engineering, Yildiz Technical University, Turkey. His current research interests focus on modeling, analysis, and decentralized coordination and control of multi-agent dynamical systems. He has co-authored many scientific papers and is a Recipient (shared with Kevin M. Passino) of the 2005 Systems, Man, and Cybernetics Society Andy P. Sage Best Transactions Paper Award. He is also co-author of the books "The RCS Handbook: Tools for Real-Time Control Systems Software Development" published by John Wiley and Sons (Interscience) in 2001 and "Swarm Stability and Optimization" published by Springer in 2011. He is serving as a technical publication Reviewer for many journals and conferences. He has also served in the technical program committees and in the organizing committees of many respected national or international conferences and as a Guest Editor for the Turkish Journal of Electrical Engineering and Computer Sciences Special Issue on "Swarm Robotics" published in July 2007.