Received: 30 March 2023 | Revised: 5 October 2023 | Accepted: 16 October 2023

DOI: 10.1002/rnc.7061

RESEARCH ARTICLE

WILEY

Cluster consensus for inherently nonlinear cooperative networks

Ümit Develer[1] | **Onur Cihan**[2] | **Mehmet Akar**[1]

[1]Department of Electrical and Electronics Engineering, Bogazici University, Istanbul, Turkey

[2]Department of Electrical and Electronics Engineering, Marmara University, Istanbul, Turkey

Correspondence
Onur Cihan, Department of Electrical and Electronics Engineering, Marmara University, Istanbul, 34854 Turkey.
Email: onur.cihan@marmara.edu.tr

Funding information
Scientific and Technical Research Council of Turkey (TUBITAK), Grant/Award Number: 118E218

Summary

This paper investigates the cluster consensus problem for inherently nonlinear cooperative networks with first and second-order system dynamics. For a first order multi-agent network evolving over a directed graph, not necessarily containing a spanning tree, a sufficient condition based on Lyapunov theory is derived on the single controller parameter so that the system achieves cluster consensus. These stability results are extended to second-order nonlinear systems that utilize two parameters in the distributed control law. Furthermore, the total number of clusters and members of each cluster are explicitly computed from the primary and secondary layer subgraphs of the underlying directed graph. The results are the first in the literature to address the cluster consensus problem for inherently nonlinear cooperative networks evolving over directed graphs that do not contain a spanning tree.

KEYWORDS

cluster consensus, inherent dynamics, multi-agent systems

1 | INTRODUCTION

In a distributed multi-agent system, cluster consensus refers to reaching more than one agreement value or decision among agents.[1-4] In real world, size of the systems, the complexity of the decision-making process and the level of communication between the agents lead to multi-agent systems that converge to different clusters where each cluster reaches its own consensus within the group. Cluster consensus algorithms in multi-agent systems have numerous applications, including in robotics, swarm intelligence, and distributed control systems.

In Reference 5, the single consensus problem has been investigated for first and second-order cooperative systems with inherent nonlinear dynamics that satisfy a Lipschitz condition. Sufficient conditions have been derived such that networks evolving over digraphs with a spanning tree achieve consensus in Reference 6. However, the case where the directed graph does not contain a spanning tree has not been considered in References 5,6, which has been studied in References 7-11 and 12 as the *containment control problem*. In containment control, the goal is to canalize followers to the dynamic convex hull spanned by multiple leaders. While the literature consists of studies that consider the containment control problem, no research investigates on grouping of followers and the number of groups. In Reference 7, an algorithm based on high-frequency feedback robust control for second-order nonlinear networks has been proposed to ensure that all followers stay in the dynamic convex hull spanned by the leaders. The distributed containment control problem has also been investigated for second-order network with inherent nonlinear dynamics in Reference 8.

Two control methods have been derived to guarantee that the followers converge to the dynamic convex hull space. However, the problem of how many clusters the followers converge to has not been investigated. In Reference 9, the authors have established a distributed event-triggered control law for nonlinearly networked cooperative systems to guarantee that the containment objective is achieved in fixed time without dependence of initial states of agents. In Reference 10, a method on selection of control gain has been developed to realize containment control of the followers utilizing inherent nonlinear dynamics. While this method ensures that the followers stay in the area spanned by the leaders, it does not give any further information about the behaviours of the followers in that area. The authors in Reference 11 have designed a centralized method to achieve the containment control objective and extended their results to the decentralized counterpart for second-order nonlinear systems. In Reference 12, the same problem has been investigated for high-order networks with inherent nonlinear dynamics. The authors have proposed an adaptive method based on the relative state information to keep the followers in the dynamic convex hull spanned by the leaders. In Reference 13, a distributed algorithm has been proposed to ensure that the followers converge to the region covered by multiple leaders. A control protocol has been presented to achieve containment control for networks with nonlinear dynamics in Reference 14 and the finite convergence time has been analysed. In Reference 15, the authors have investigated the second-order consensus problem for systems having inherent nonlinear dynamics. For strongly connected networks, a new generalized algebraic connectivity has been characterized to derive some sufficient conditions for attaining consensus.

As discussed above, research on multi-agent systems[5,7-15] with inherent nonlinear dynamics is limited to ensure that the followers converge to the dynamic convex hull. However, the problems of determining the numbers and members of clusters formed by the followers and obtaining the stability conditions for cluster consensus with nonlinear dynamics evolving over arbitrary directed graphs are unsolved by state-of-the-art methods. In this paper, we contribute to the growing area of cluster consensus research by discussing the containment control problem as a cluster consensus problem and investigating first and second-order cooperative networks with inherent nonlinear dynamics. We develop methods that compute the number of clusters to which systems converge and the members of each cluster. The networks considered in this paper have no structural restrictions such as the existence of a spanning tree, being strongly connected etc. Due to these nonrestrictive characteristics of this study, the results of this paper can be applied to a wider range of real-world applications consisting of multi agent systems.

The contributions of this paper can be summarized as follows:

1. In first-order networks with nonlinear inherent dynamics, a sufficient condition is obtained so that the systems can achieve cluster consensus.
2. Sufficient conditions on controller parameters are derived to attain cluster consensus in second-order inherently nonlinear systems.
3. For a given first or second-order inherently nonlinear network, the number of clusters and their members are explicitly computed.

The rest of this paper is organized as follows. The main results for first-order multi-agent systems with inherent nonlinear dynamics are given in Section 2. These results are extended to second-order networks in Section 3. In Section 4, two numerical examples are provided to verify the obtained results. Finally, there are some concluding remarks in Section 5.

2 | FIRST-ORDER NONLINEAR NETWORKS

Given a cooperative network consisting of n agents, the connections between agents are modelled via a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{v_1, \dots, v_n\}$ is the set of agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of directed edges. A directed edge, (v_j, v_i), indicates that there exists an information flow from agent v_j to agent v_i. The adjacency matrix, $\mathcal{A}$, holds the weights of the directed edges and its elements, a_{ij}, satisfy $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$, otherwise $a_{ij} = 0$. The neighbor set of agent v_i is the set of vertices that are adjacent to v_i and defined by $\mathcal{N}_i = \{v_j : (v_j, v_i) \in \mathcal{E}\}$. L is the Laplacian matrix associated with the adjacency matrix, $\mathcal{A}$, and its elements, l_{ij}, are defined by

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} a_{ik}, & \text{if } i = j \\ -a_{ij}, & \text{if } i \neq j. \end{cases} \tag{1}$$

In this section, the goal is to derive conditions leading first-order multi-agent systems to reach cluster consensus under inherent nonlinear dynamics. Each agent, v_i, has the following first-order nonlinear dynamics

$$\dot{x}_i(t) = f(x_i, t) + u_i(t), \quad i = 1, \dots, n \tag{2}$$

where $x_i(t) \in \mathbb{R}$ and $f(x_i, t)$ are the state and the inherent nonlinear dynamics of agent v_i, respectively; and $u_i(t)$ is the control input of agent v_i given by

$$u_i(t) = \gamma_1 \sum_{v_j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)) \tag{3}$$

where $\gamma_1 > 0$ is control parameter. By using (1), the control input can be rewritten in the following form:

$$u_i(t) = -\gamma_1 \sum_{v_j \in \mathcal{N}_i} l_{ij} x_j(t). \tag{4}$$

When (2) and (4) are combined, the dynamics of agents are represented as

$$\dot{x}_i(t) = f(x_i, t) - \gamma_1 \sum_{v_j \in \mathcal{N}_i} l_{ij} x_j(t), \quad i = 1, \dots, n, \tag{5}$$

or equivalently in the matrix form as

$$\dot{x}(t) = f(x, t) - \gamma_1 L x(t) \tag{6}$$

where $x(t) = [x_1(t), \dots, x_n(t)]^T$ and $f(x, t) = [f(x_1, t), \dots, f(x_n, t)]^T$.

Remark 1. Note that since each agent utilizes local and neighboring agents' information in its update law, the algorithm is fully distributed. In this section, we will determine the stability conditions for which the network utilizing distributed algorithm (5) achieves cluster consensus.

Before presenting the main results, we state the definition of the first-order cluster consensus.

Definition 1. FIRST-ORDER CLUSTER CONSENSUS: The first-order system (2) is said to achieve cluster consensus with $\mathcal{K}$ non-empty distinct sets C_m, $m = 1, \dots, \mathcal{K}$ such that for each cluster, if we have

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0 \quad \forall v_i, v_j \in C_m.$$

In the analysis of the cluster consensus problem, primary and secondary layer subgraph concepts are used to determine the number of clusters. The definitions are given below:

Definition 2 (16). PRIMARY AND SECONDARY LAYER SUBGRAPHS (PLS AND SLS): A digraph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, can be uniquely parted into l_p primary and l_s secondary layer subgraphs with the following properties:

- a PLS ($\mathcal{G}_{p,i}$, $i = 1, \dots, l_p$)
  - (i) has a vertex set, $\mathcal{V}_{p,i}$, containing a spanning tree and consisting of the maximum possible number of vertices that do not receive information from $\mathcal{V} \setminus \mathcal{V}_{p,i}$.
- a SLS ($\mathcal{G}_{s,j}$, $j = 1, \dots, l_s$)
  - (i) has a vertex set, $\mathcal{V}_{s,j}$, containing a spanning tree and a root vertex that receives information from vertices of at least two different subgraphs in $\mathcal{V} \setminus \mathcal{V}_{s,j}$ and
  - (ii) except for its root vertex, does not receive information from $\mathcal{V} \setminus \mathcal{V}_{s,j}$.

The PLS and SLS concepts are functional tools in analysis of the cluster consensus problem. The following remark gives detailed information on PLS and SLS notions:

Remark 2. The PLSs and SLSs in Definition 2 can be applied to an arbitrary digraph without any structural limitations to determine clusters by using the detection algorithms in Reference 16. In order to determine PLSs and SLSs, the following procedure can be followed:

1. Firstly, PLS detection algorithm in Reference 16 is utilized to determine PLS vertex sets $\mathcal{V}_{p,i}$, $i = 1, \dots, l_p$. Each PLS should contain a spanning tree and cannot receive information from out of the group.
2. Secondly, SLS vertex sets $\mathcal{V}_{s,j}$, $j = 1, \dots, l_s$ is obtained from the remaining vertex sets $\mathcal{V} \setminus \bigcup_{i=1,\dots,l_p} \mathcal{V}_{p,i}$ by using SLS detection algorithm in Reference 16. Each SLS should contain a spanning tree, its root vertex should receive information from at least two different groups and other vertices cannot receive information from out of the group.

In order to illustrate determination of PLS and SLS, an example is given below.

Example 1. Consider the network with 10 agents in Figure 1A. By following the procedure in Remark 2, two PLS vertex sets ($\mathcal{V}_{p,1}$ and $\mathcal{V}_{p,2}$) are determined in accordance with the properties of PLS in Definition 2. The corresponding subgraphs ($\mathcal{G}_{p,1}$ and $\mathcal{G}_{p,2}$) are shown with dotted line in Figure 1B. In light of Definition 2 note that two PLSs ($\mathcal{G}_{p,1}$ and $\mathcal{G}_{p,2}$), contain a spanning tree and do not have a link coming from other groups. Subsequently, three SLS vertex sets ($\mathcal{V}_{s,1}$, $\mathcal{V}_{s,2}$ and $\mathcal{V}_{s,3}$) are obtained from the remaining vertex set. The corresponding subgraphs ($\mathcal{G}_{s,1}$, $\mathcal{G}_{s,2}$ and $\mathcal{G}_{s,3}$) are shown with dashed line in Figure 1C. Each SLS ($\mathcal{G}_{s,1}$, $\mathcal{G}_{s,2}$ and $\mathcal{G}_{s,3}$) consists of a spanning tree whose root vertex receives information from at least two different vertices in other clusters.

By using PLS and SLS detection algorithms in Reference 16, PLSs and SLSs and their members can be obtained as $\mathcal{V}_{p,1} = \{v_1, v_2, v_3\}$, $\mathcal{V}_{p,2} = \{v_4, v_5\}$ and $\mathcal{V}_{s,1} = \{v_6\}$, $\mathcal{V}_{s,2} = \{v_7, v_8, v_9\}$, $\mathcal{V}_{s,3} = \{v_{10}\}$, respectively. PLS and SLS decompositions can be applied to any given digraph without any topological structure constraints. PLSs and SLSs can be uniquely determined by the algorithms proposed in Reference 16.

A given multi-agent system dynamics can be rewritten in terms of the PLS and SLS concepts. Let $n_{p,i}$ $(i = 1, \dots, l_p)$ and $n_{s,j}$ $(j = 1, \dots, l_s)$ be the number of agents in the $i-$th PLS and $j-$th SLS, respectively. $n_p = \sum_{i=1}^{l_p} n_{p,i}$ and $n_s = \sum_{j=1}^{l_s} n_{s,j}$ denote the total number of agents in PLSs and SLSs, respectively. In this context, we can rewrite (2) as

$$\begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_s(t) \end{bmatrix} = \begin{bmatrix} f(x_p, t) \\ f(x_s, t) \end{bmatrix} - \gamma_1 \begin{bmatrix} L_p & 0_{n_p \times n_s} \\ L_{sp} & L_s \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_s(t) \end{bmatrix} \tag{7}$$

where $x_p(t) \in \mathbb{R}^{n_p}$ and $x_s(t) \in \mathbb{R}^{n_s}$ are the state vectors for PLSs and SLSs, respectively; $0_{n_p \times n_s}$ is the $n_p \times n_s$ zero matrix and the system matrices are given as

$$L_p = \text{blkdiag}\{L_{1,1}, \dots, L_{l_p,l_p}\}, \quad L_{sp} = \begin{bmatrix} L_{l_p+1,1} & \cdots & L_{l_p+1,l_p} \\ \vdots & \ddots & \vdots \\ L_{l_p+l_s,1} & \cdots & L_{l_p+l_s,l_p} \end{bmatrix}, \quad L_s = \begin{bmatrix} L_{l_p+1,l_p+1} & \cdots & L_{l_p+1,l_p+l_s} \\ \vdots & \ddots & \vdots \\ L_{l_p+l_s,l_p+1} & \cdots & L_{l_p+l_s,l_p+l_s} \end{bmatrix}$$

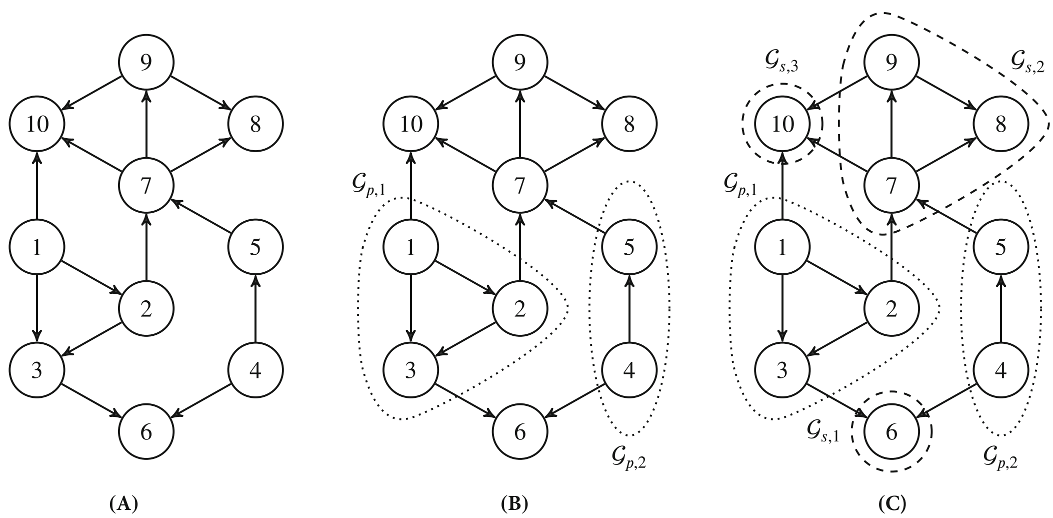

FIGURE 1 A network consisting of 10 nodes and 14 edges.

where blkdiag$\{L_{1,1}, \dots, L_{l_p,l_p}\}$ denotes the block diagonal matrix with $L_{1,1}, \dots, L_{l_p,l_p}$ on the diagonal. Note that each row of $-L_s^{-1}L_{sp}$ adds up to 1 and all members of $-L_s^{-1}L_{sp}$ are non-negative.[17,18] In the light of this fact, the following assumption is utilized for analysis of cluster consensus.

Assumption 1 (5). Given the function $f(x,t)$ with nonnegative constants $\eta_i \geq 0$ satisfying $\sum_{i=1}^{n_p} \eta_i = 1$, there exist a constant l such that the inequality

$$\left| f(y,t) - \sum_{i=1}^{n_p} \eta_i f(x_i,t) \right| \leq l \left| y - \sum_{i=1}^{n_p} \eta_i x_i \right|, \forall y, x_i \in \mathbb{R} \tag{8}$$

holds.

Remark 3. For $n_p = 1$, the system using (2) has only one PLS and the condition (8) reduces to the Lipschitz condition as follows:

$$|f(y,t) - f(x,t)| \leq l|y - x|, \forall y, x \in \mathbb{R}.$$

The condition in Assumption 1 ensures that followers stay in a region spanned by leaders.[5,7-12] All linear functions and some nonlinear functions ($x\sin(t)$, $-x\cos(t^2)$ etc.) are suitable to use the condition (8). Note that, if the condition (8) holds, then the Lipschitz condition is satisfied, but not vice-versa. Assumption 1 ensures that the SLSs always converge to the convex hull spanned by PLSs. Therefore, the convergence points of SLSs do not depend on the initial values of SLSs, but on the initial values of PLSs. By using this assumption, the containment control property for system with inherently nonlinear dynamics can be achieved. In Example 4, this phenomenon is simulated in multi-robot systems to illustrate the rationale behind using Assumption 1.

Under Assumption 1, consider the following transformation:

$$\begin{bmatrix} y_p(t) \\ y_s(t) \end{bmatrix} = \begin{bmatrix} S_p & 0_{n_p \times n_s} \\ 0_{n_s \times n_p} & I_{n_s} \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_s(t) \end{bmatrix} \tag{9}$$

where I_{n_s} is the $n_s \times n_s$ identity matrix and $S_p = \text{blkdiag}\{S_1, \dots, S_{l_p}\}$ with

$$S_i = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}_{n_{p,i} \times n_{p,i}}$$

for $i = 1, \dots, l_p$. Note that $S_i = S_i^{-1}$ which also implies $S_p = S_p^{-1}$. Let $y_{p,i} = [y_{p,i,1}, e_{p,i}^T]^T$ be the transformed PLS states where

$$\begin{aligned} y_{p,i,1} &= x_{p,i,1} \\ e_{p,i} &= [y_{p,i,2}, y_{p,i,3}, \dots, y_{p,i,n_{p,i}}]^T \\ &= [x_{p,i,1} - x_{p,i,2}, x_{p,i,1} - x_{p,i,3}, \dots, x_{p,i,1} - x_{p,i,n_{p,i}}]^T \end{aligned}$$

for $i = 1, \dots, l_p$. Rewriting (7) with respect to y_p and y_s, we have

$$\begin{bmatrix} \dot{y}_p(t) \\ \dot{y}_s(t) \end{bmatrix} = \begin{bmatrix} S_p f(y_p,t) \\ f(y_s,t) \end{bmatrix} - \gamma_1 \begin{bmatrix} S_p L_p S_p^{-1} & 0_{n_p \times n_s} \\ L_{sp} S_p^{-1} & L_s \end{bmatrix} \begin{bmatrix} y_p(t) \\ y_s(t) \end{bmatrix}. \tag{10}$$

From the above equation, the PLS and SLS dynamics can be rewritten in the following form:

$$\dot{y}_p(t) = S_p f(y_p, t) - \gamma_1 S_p L_p S_p^{-1} y_p(t) \tag{11a}$$
$$\dot{y}_s(t) = f(y_s, t) - \gamma_1 L_s y_s(t) - \gamma_1 L_{sp} S_p^{-1} y_p(t) \tag{11b}$$

Since each PLS dynamics is independent of the other PLS dynamics, the overall system dynamics can be written as

$$\begin{aligned}\dot{y}_{p,1}(t) &= S_1 f(y_{p,1}, t) - \gamma_1 S_1 L_{1,1} S_1^{-1} y_{p,1}(t)\\ &\vdots\\ \dot{y}_{p,l_p}(t) &= S_{l_p} f(y_{p,l_p}, t) - \gamma_1 S_{l_p} L_{l_p,l_p} S_{l_p}^{-1} y_{p,l_p}(t)\\ \dot{y}_s(t) &= f(y_s, t) - \gamma_1 L_s y_s(t) - \gamma_1 L_{sp} S_p^{-1} y_p(t)\end{aligned}$$

Define $E_i = \begin{bmatrix}\mathbf{1}_{n_{p,i}-1} & -I_{n_{p,i}-1}\end{bmatrix}$ and $F_i = \begin{bmatrix}0_{n_{p,i}-1} & -I_{n_{p,i}-1}\end{bmatrix}^T$ for $i = 1, \dots, l_p$ where $\mathbf{1}_{n_{p,i}-1} = \begin{bmatrix}1 & \dots & 1\end{bmatrix}^T \in \mathbb{R}^{n_{p,i}-1}$ and $0_{n_{p,i}-1} = \begin{bmatrix}0 & \dots & 0\end{bmatrix}^T \in \mathbb{R}^{n_{p,i}-1}$. In this context, each PLS dynamics can be written as the following two subsystems

$$\dot{y}_{p,i,1}(t) = f(y_{p,i,1}, t) - \gamma_1 l_{i,1} F_i e_{p,i}(t) \tag{12a}$$
$$e_{p,i}(t) = f_e(y_{p,i}, t) - \gamma_1 E_i L_{i,i} F_i e_{p,i}(t) \tag{12b}$$

where $l_{i,1}$ is the first row of $L_{i,i}$ and $f_e(y_{p,i}, t) = [f(y_{p,i,1}) - f(y_{p,i,1} - y_{p,i,2}), \dots, f(y_{p,i,1}) - f(y_{p,i,1} - y_{p,i,n_{p,i}})]^T$.

The error dynamics of all PLSs can be written in the following form:

$$e_p(t) = \overline{f}_e(y_p, t) - \gamma_1 E L_p F e_p(t)$$

where $e_p(t) = [e_{p,1}^T(t), \dots, e_{p,l_p}^T(t)]^T$, $E = \text{blkdiag}\{E_1, \dots, E_{l_p}\}$, $F = \text{blkdiag}\{F_1, \dots, F_{l_p}\}$ and $\overline{f}_e(y_p, t) = [f_e^T(y_{p,1}, t), \dots, f_e^T(y_{p,l_p}, t)]^T$. Before presenting the main result, we need the following lemmas:

Lemma 1 (19). *For any two real vectors a and b with the same dimension, the inequality*

$$2a^T b \le a^T \Phi a + b^T \Phi^{-1} b$$

holds where Φ is any positive definite matrix with an appropriate dimension.

Lemma 2 (20). *For a real, symmetric matrix P, the inequality*

$$\lambda_m(P) x^T x \le x^T P x \le \lambda_M(P) x^T x$$

holds where $\lambda_m(P)$ and $\lambda_M(P)$ denote the minimum and maximum eigenvalues of P, respectively.

The following theorem states the conditions on the control parameter γ_1 that guarantee cluster consensus of a first-order multi-agent system with inherent nonlinear dynamics.

Theorem 1. *Suppose that system (2) evolves over a digraph with l_p PLSs and l_s SLSs and there exists a positive coefficient l that satisfies Assumption 1. The system in (2) converges to $l_p + l_s$ clusters if there exists a scalar control parameter γ_1 that satisfies*

$$-\gamma_1[(EL_pF)^T P_p + P_p(EL_pF)] + l^2 \lambda_M(P_p) I_{n_p - l_p} + P_p < 0 \tag{13a}$$
$$-\gamma_1[L_s^T P_s + P_s L_s] + l^2 \lambda_M(P_s) I_{n_s} + P_s < 0 \tag{13b}$$

for some symmetric positive definite matrices $P_p \in \mathbb{R}^{n_p - l_p \times n_p - l_p}$ and $P_s \in \mathbb{R}^{n_s \times n_s}$ that yield $(EL_pF)^T P_p + P_p(EL_pF)$ and $L_s^T P_s + P_s L_s$ positive definite, respectively.

Proof. We investigate the PLS and SLS dynamics consecutively.

Step 1 (Stability and the number of clusters for the PLS dynamics): Due to the structure of L_p, each PLS dynamics can be individually investigated as

$$\dot{y}_{p,i,1}(t) = f(y_{p,i,1}, t) - \gamma_1 l_{i,1} F_i e_{p,i}(t)$$
$$e_{p,i}(t) = f_e(y_{p,i}, t) - \gamma_1 E_i L_{i,i} F_i e_{p,i}(t)$$

To prove the stability of all PLSs at once, the following candidate Lyapunov function is considered:

$$V_1(t) = e_p(t)^T P_p e_p(t)$$

where $P_p = \text{blkdiag}\{P_1, \dots, P_{l_p}\}$ with each P_i being also a symmetric positive definite matrix that satisfy $(E_i L_{i,i} F_i)^T P_i + P_i(E_i L_{i,i} F_i) < 0$. Under Assumption 1, differentiating $V_1(t)$ yields

$$\dot{V}_1(t) = -\gamma_1 e_p(t)^T[(EL_pF)^T P_p + P_p(EL_pF)]e_p(t) + 2e_p(t)^T P_p \overline{f}_e(y_{p,i}, t)$$

Using Lemmas 1 and 2, one can obtain the following:

$$\dot{V}_1(t) \le -\gamma_1 e_p(t)^T[(EL_pF)^T P_p + P_p(EL_pF)]e_p(t) + e_p(t)^T P_p e_p(t) + \overline{f}_e(y_p, t)^T P_p \overline{f}_e(y_p, t)$$
$$\dot{V}_1(t) \le -e_p(t)^T[\gamma_1\left((EL_pF)^T P_p + P_p(EL_pF)\right) - l^2 \lambda_{\max}(P_p) I_{n_p - l_p} - P_p]e_p(t)^T$$

We have $\dot{V}_1(t) < 0$ for $e_p(t) \neq 0$, which implies that each PLS dynamics is asymptotically stable. The agents in a particular PLS act together, that is, $\lim_{t\to\infty} \|x_{p,i,j_1} - x_{p,i,j_2}\| = 0$ for $i = 1, \dots, l_p$ and $j_1, j_2 = 1, \dots, n_{p,i}$. This implies that one can find a total of l_p clusters in l_p PLSs.

Step 2 (Stability and the number of clusters for the SLS dynamics): In this step, we will prove that the following holds

$$\lim_{t\to\infty} \|y_s(t) + L_s^{-1} L_{sp} S_p^{-1} y_p(t)\| = 0. \tag{14}$$

Let $e_s(t) = y_s(t) + L_s^{-1} L_{sp} S_p^{-1} y_p(t)$. Taking derivatives gives

$$\begin{aligned}
\dot{e}_s(t) &= \dot{y}_s(t) + L_s^{-1} L_{sp} S_p^{-1} \dot{y}_p(t) \\
&= f(y_s, t) - \gamma_1 L_s y_s(t) - \gamma_1 L_{sp} S_p^{-1} y_p(t) + L_s^{-1} L_{sp} f(y_p, t) - \gamma_1 L_s^{-1} L_{sp} L_p S_p^{-1} y_p(t) \\
&= f(y_s, t) + L_s^{-1} L_{sp} f(y_p, t) - \gamma_1 L_s(y_s(t) + L_s^{-1} L_{sp} S_p^{-1} y_p(t)) - \gamma_1 L_s^{-1} L_{sp} L_p S_p^{-1} y_p(t) \\
&= f(y_s, t) + L_s^{-1} L_{sp} f(y_p, t) - \gamma_1 L_s e_s(t) - \gamma_1 L_s^{-1} L_{sp} L_p S_p^{-1} y_p(t)
\end{aligned}$$

Let $\omega(t) = L_p S_p^{-1} y_p(t) = L_p x_p(t)$. The above dynamics become as follows:

$$\dot{e}_s(t) = f(y_s, t) + L_s^{-1} L_{sp} f(y_p, t) - \gamma_1 L_s e_s(t) - \gamma_1 L_s^{-1} L_{sp} \omega(t)$$

Note that $\omega(t)$ converges to zero as $t \to \infty$ from Step 1 since $L_p x_p(t) \to 0$. In order to analyze the stability of the above system dynamics, it is required to investigate the stability of the unforced system, that is, $\omega(t) = 0$ for all t. If the origin of unforced system is asymptotically stable, then the states of forced system converge to zero with $\omega(t) = 0$ as $t \to \infty$ (p. 107 in Reference 21). For this reason, choose the Lyapunov function candidate

$$V_2(t) = e_s(t)^T P_s e_s(t)$$

for the unforced SLS dynamics, that is, $\dot{e}_s(t) = f(y_s, t) + L_s^{-1} L_{sp} f(y_p, t) - \gamma_1 L_s e_s(t)$. Taking derivative of $V_2(t)$ yields the following:

$$\dot{V}_2(t) = -\gamma_1 e_s(t)^T[L_s^T P_s + P_s L_s]e_s(t) + 2e_s(t)^T P_s[f(y_s, t) + L_s^{-1} L_{sp} f(y_p, t)]$$

Using Lemmas 1 and 2, one can obtain the following:

$$\dot{V}_2(t) \leq -\gamma_1 e_s(t)^T[L_s^T P_s + P_s L_s]e_s(t) + e_s(t)^T P_s e_s(t) + [f(y_s,t) + L_s^{-1}L_{sp}f(y_p,t)]^T P_s[f(y_s,t) + L_s^{-1}L_{sp}f(y_p,t)]$$
$$\dot{V}_2(t) \leq -e_s(t)^T[\gamma_1(L_s^T P_s + P_s L_s) - l^2\lambda_M(P_s)I_{n_s} - P_s]e_s(t)$$

By (13b), we have $\dot{V}_2(t) < 0$ for $e_s(t) \neq 0$, which implies that the origin of $e_s(t)$ is asymptotically stable. This implies that the following expression holds

$$y_s(t) = -L_s^{-1}L_{sp}S_p^{-1}y_p(t), \quad t \to \infty$$

Note that original states $x_p(t)$ and $x_s(t)$ can be obtained from $x_p(t) = S_p^{-1}y_p(t)$ and $x_s(t) = y_s(t)$. Therefore, as $t \to \infty$, we have

$$x_s(t) = -L_s^{-1}L_{sp}x_p(t), \quad t \to \infty$$

Let $x_p^*(t) = \lim_{t\to\infty} x_p(t)$ and $x_s^*(t) = \lim_{t\to\infty} x_s(t)$. As $t \to \infty$, $x_p(t)$ converges to trajectory $x_p^*(t)$ containing l_p distinct routes from Step 1. From PLS and SLS dynamics in Reference 17, it is known that l_p distinct points at any t gives l_s clusters in $x_s^*(t)$ which is a convergent trajectory for $x_s(t)$.

Step 3 (Total number of clusters): We have l_p distinct PLSs from Step 1 and l_s distinct SLSs from Step 2. Hence, one can conclude that the total number of cluster is computed as $\mathcal{K} = l_p + l_s$. ■

Remark 4. By using Theorem 1, the selection method of control parameter γ_1 can be established as follows:

1. Find a symmetric positive definite matrix P_p such that $(EL_pF)^T P_p + P_p(EL_pF)$ is positive definite (e.g., solve for P_p from $(EL_pF)^T P_p + P_p(EL_pF) = I_{n_p - l_p}$).
2. Calculate the maximum eigenvalue of P_p.
3. Find a symmetric positive definite matrix P_s such that $L_s^T P_s + P_s L_s$ is positive definite (e.g., solve for P_s from $L_s^T P_s + P_s L_s = I_{n_s}$).
4. Calculate the maximum eigenvalue of P_s.
5. Choose a control parameter γ_1 that satisfies (13a) and (13b) by using data from the previous steps. (Note (13a) and (13b) can be simultaneously satisfied by choosing a high enough γ_1).

Recall that the protocol (5) used as update rule is fully distributed from Remark 1. However, L_p and L_s global matrices should be known only for the stability analysis and the selection of control parameter γ_1.

Remark 5. Any inherently nonlinear network satisfying Assumption 1 for first-order dynamics can be investigated in the context of the cluster consensus problem. The control parameter (γ_1) can be selected by using the condition in Theorem 1 and the number of clusters can be calculated by applying PLS and SLS detection algorithms. In the existing literature, the total number of clusters is not explicitly computed for a multi-agent system. In this context, this is the first study that states the stability conditions and number of clusters in a first-order network with inherent nonlinear dynamics under an arbitrary digraph. The main challenges of our study are that determining the groups which will be formed in a network with an arbitrary directed graph requires graph theoretic techniques; and while the dynamics of the agents in some of these groups are to be examined as a standard consensus system, some must be considered as a leader-follower system. The results of this section show that the members of these two different types of groups can be systematically determined by using detection algorithms while the conditions stated in Theorem 1 guarantee that first-order cluster consensus dynamics are stable.

3 | SECOND-ORDER NONLINEAR NETWORKS

The system is modeled by a second-order dynamics as follows:

$$\dot{x}_i(t) = v_i(t)$$
$$\dot{v}_i(t) = f(x_i, v_i, t) + u_i(t) \tag{15}$$

where $x_i(t)$ and $v_i(t)$ are the states of vertex v_i, $f(x_i, v_i, t)$ is a second-order inherent nonlinear function and $u_i(t)$ is the control input described as follows:

$$u_i(t) = -\gamma_1 \sum_{v_j \in \mathcal{N}_i} l_{ij} x_j(t) - \gamma_2 \sum_{v_j \in \mathcal{N}_i} l_{ij} v_j(t) \tag{16}$$

where γ_1 and γ_2 are coupling strengths. The system dynamics with control input can be rewritten in matrix form as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0_n \\ f(x, v, t) \end{bmatrix} + \begin{bmatrix} 0_{n\times n} & I_n \\ -\gamma_1 L & -\gamma_2 L \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \tag{17}$$

where $x(t) = [x_1(t), \dots, x_n(t)]^T$, $v(t) = [v_1(t), \dots, v_n(t)]^T$ and $f(x, v, t) = [f(x_1, v_1, t), \dots, f(x_n, v_n, t)]^T$. PLS and SLS dynamics can be written as follows:

$$\begin{bmatrix} \dot{x}_p(t) \\ \dot{v}_p(t) \end{bmatrix} = \begin{bmatrix} 0_{n_p} \\ f(x_p, v_p, t) \end{bmatrix} + \begin{bmatrix} 0_{n_p\times n_p} & I_{n_p} \\ -\gamma_1 L_p & -\gamma_2 L_p \end{bmatrix} \begin{bmatrix} x_p(t) \\ v_p(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_s(t) \\ \dot{v}_s(t) \end{bmatrix} = \begin{bmatrix} 0_{n_s} \\ f(x_s, v_s, t) \end{bmatrix} + \begin{bmatrix} 0_{n_s\times n_s} & I_{n_s} \\ -\gamma_1 L_s & -\gamma_2 L_s \end{bmatrix} \begin{bmatrix} x_s(t) \\ v_s(t) \end{bmatrix} + \begin{bmatrix} 0_{n_s\times n_p} & 0_{n_s\times n_p} \\ -\gamma_1 L_{sp} & -\gamma_2 L_{sp} \end{bmatrix} \begin{bmatrix} x_p(t) \\ v_p(t) \end{bmatrix} \tag{18}$$

Remark 6. Note that the second-order consensus protocol (17) is a fully distributed algorithm since each agent exploits its own and neighbors' information. In this section, the stability conditions are derived for the network operating distributed algorithm (17) to achieve cluster consensus.

Before presenting the main results on second-order networks with inherent nonlinear dynamics, the definition of the second-order cluster consensus is given below.

Definition 3. SECOND-ORDER CLUSTER CONSENSUS: The second-order system (15) is said to achieve cluster consensus with $\mathcal{K}$ non-empty distinct sets C_m, $m = 1, \dots, \mathcal{K}$ such that for each cluster, if we have

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0 \quad \text{and} \quad \lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0 \quad \forall v_i, v_j \in C_m$$

The following assumption is utilized for analysis of cluster consensus in second-order systems:

Assumption 2 (8). Given the function $f(x, v, t)$ with nonnegative constants $\eta_i \geq 0$ satisfying $\sum_{i=1}^{n_p} \eta_i = 1$, there exist constants l_1, l_2 such that the inequality

$$\left| f(y, z, t) - \sum_{i=1}^{n_p} \eta_i f(x_i, v_i, t) \right| \leq l_1 \left| y - \sum_{i=1}^{n_p} \eta_i x_i \right| + l_2 \left| z - \sum_{i=1}^{n_p} \eta_i v_i \right|, \quad \forall y, z, x_i, v_i \in \mathbb{R} \tag{19}$$

holds.

Remark 7. For $n_p = 1$, the system using (15) has only one PLS and the condition (19) reduces to the Lipschitz condition as follows:

$$|f(y, z, t) - f(x_i, v_i, t)| \leq l_1 |y - x_i| + l_2 |z - v_i|, \quad \forall y, z, x_i, v_i \in \mathbb{R}.$$

Similar assumptions are utilized in Reference 8. Under Assumption 2, consider $y = Sx$ and $z = Sv$ transformations as follows:

$$\begin{bmatrix} y_p(t) \\ y_s(t) \end{bmatrix} = \begin{bmatrix} S_p & 0_{n_p\times n_s} \\ 0_{n_s\times n_p} & I_{n_s} \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_s(t) \end{bmatrix}, \quad \begin{bmatrix} z_p(t) \\ z_s(t) \end{bmatrix} = \begin{bmatrix} S_p & 0_{n_p\times n_s} \\ 0_{n_s\times n_p} & I_{n_s} \end{bmatrix} \begin{bmatrix} v_p(t) \\ v_s(t) \end{bmatrix}.$$

Let $y_{p,i} = [y_{p,i,1}, e^T_{p_1,i}]^T$ and $z_{p,i} = [z_{p,i,1}, e^T_{p_2,i}]^T$ be the transformed second-order PLS dynamics where

$$\begin{aligned}
y_{p,i,1} &= x_{p,i,1} \\
e_{p_1,i} &= [y_{p,i,2}, y_{p,i,3}, \dots, y_{p,i,n_{p,i}}]^T \\
&= [x_{p,i,1} - x_{p,i,2}, x_{p,i,1} - x_{p,i,3}, \dots, x_{p,i,1} - x_{p,i,n_{p,i}}]^T \\
z_{p,i,1} &= v_{p,i,1} \\
e_{p_2,i} &= [z_{p,i,2}, z_{p,i,3}, \dots, z_{p,i,n_{p,i}}]^T \\
&= [v_{p,i,1} - v_{p,i,2}, v_{p,i,1} - v_{p,i,3}, \dots, v_{p,i,1} - v_{p,i,n_{p,i}}]^T
\end{aligned}$$

Rewriting (18) with respect to y_p, y_s, z_p and z_s, we have

$$\begin{aligned}
\begin{bmatrix} \dot{y}_p(t) \\ \dot{y}_s(t) \end{bmatrix} &= \begin{bmatrix} z_p(t) \\ z_s(t) \end{bmatrix} \\
\begin{bmatrix} \dot{z}_p(t) \\ \dot{z}_s(t) \end{bmatrix} &= \begin{bmatrix} S_p f(y_p, z_p, t) - S_p L_p S_p^{-1} z_p(t) \\ f(y_s, z_s, t) - L_s z_s(t) - L_{sp} S_p^{-1} z_p(t) \end{bmatrix}
\end{aligned} \tag{20}$$

From the above equations, the PLS and SLS dynamics can be rewritten in the following form:

$$\begin{aligned}
\begin{bmatrix} \dot{y}_p(t) \\ \dot{z}_p(t) \end{bmatrix} &= \begin{bmatrix} 0_{n_p \times n_p} & I_{n_p} \\ -\gamma_1 S_p L_p S_p^{-1} & -\gamma_2 S_p L_p S_p^{-1} \end{bmatrix} \begin{bmatrix} y_p(t) \\ z_p(t) \end{bmatrix} + \begin{bmatrix} 0_{n_p} \\ S_p f(y_p, z_p, t) \end{bmatrix} \\
\begin{bmatrix} \dot{y}_s(t) \\ \dot{z}_s(t) \end{bmatrix} &= \begin{bmatrix} 0_{n_s \times n_s} & I_{n_s} \\ -\gamma_1 L_s & -\gamma_2 L_s \end{bmatrix} \begin{bmatrix} y_s(t) \\ z_s(t) \end{bmatrix} + \begin{bmatrix} 0_{n_s \times n_p} & 0_{n_s \times n_p} \\ -\gamma_1 L_{sp} S_p^{-1} & -\gamma_2 L_{sp} S_p^{-1} \end{bmatrix} \begin{bmatrix} y_p(t) \\ z_p(t) \end{bmatrix} + \begin{bmatrix} 0_{n_s} \\ f(y_s, z_s, t) \end{bmatrix}
\end{aligned} \tag{21}$$

Since each PLS dynamics is independent of the other PLS dynamics, the overall system dynamics can be written as

$$\begin{aligned}
\begin{bmatrix} \dot{y}_{p,1}(t) \\ \dot{z}_{p,1}(t) \end{bmatrix} &= \begin{bmatrix} 0_{n_{p,1} \times n_{p,1}} & I_{n_{p,1}} \\ -\gamma_1 S_1 L_{1,1} S_1^{-1} & -\gamma_2 S_1 L_{1,1} S_1^{-1} \end{bmatrix} \begin{bmatrix} y_{p,1}(t) \\ z_{p,1}(t) \end{bmatrix} + \begin{bmatrix} 0_{n_{p,1}} \\ S_1 f(y_{p,1}, z_{p,1}, t) \end{bmatrix} \\
&\vdots \\
\begin{bmatrix} \dot{y}_{p,l_p}(t) \\ \dot{z}_{p,l_p}(t) \end{bmatrix} &= \begin{bmatrix} 0_{n_{p,l_p} \times n_{p,l_p}} & I_{n_{p,l_p}} \\ -\gamma_1 S_{l_p} L_{l_p,l_p} S_{l_p}^{-1} & -\gamma_2 S_{l_p} L_{l_p,l_p} S_{l_p}^{-1} \end{bmatrix} \begin{bmatrix} y_{p,l_p}(t) \\ z_{p,l_p}(t) \end{bmatrix} + \begin{bmatrix} 0_{n_{p,l_p}} \\ S_{l_p} f(y_{p,l_p}, z_{p,l_p}, t) \end{bmatrix} \\
\begin{bmatrix} \dot{y}_s(t) \\ \dot{z}_s(t) \end{bmatrix} &= \begin{bmatrix} 0_{n_s \times n_s} & I_{n_s} \\ -\gamma_1 L_s & -\gamma_2 L_s \end{bmatrix} \begin{bmatrix} y_s(t) \\ z_s(t) \end{bmatrix} + \begin{bmatrix} 0_{n_s \times n_p} & 0_{n_s \times n_p} \\ -\gamma_1 L_{sp} S_p^{-1} & -\gamma_2 L_{sp} S_p^{-1} \end{bmatrix} \begin{bmatrix} y_p(t) \\ z_p(t) \end{bmatrix} + \begin{bmatrix} 0_{n_s} \\ f(y_s, z_s, t) \end{bmatrix}
\end{aligned}$$

Define $E_i = \begin{bmatrix} 1_{n_{p,i}-1} & -I_{n_{p,i}-1} \end{bmatrix}$ and $F_i = \begin{bmatrix} 0_{n_{p,i}-1} & -I_{n_{p,i}-1} \end{bmatrix}^T$ for $i = 1, \dots, l_p$. In this context, each PLS dynamics can be written as the following two subsystems

$$\begin{aligned}
\begin{bmatrix} \dot{y}_{p,i,1}(t) \\ \dot{z}_{p,i,1}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_{p,i,1}(t) \\ z_{p,i,1}(t) \end{bmatrix} + \begin{bmatrix} 0^T_{n_{p,i}-1} & 0^T_{n_{p,i}-1} \\ -\gamma_1 l_{i,1} F_i & -\gamma_1 l_{i,1} F_i \end{bmatrix} \begin{bmatrix} e_{p_1,i}(t) \\ e_{p_2,i}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f(y_{p,i,1}, z_{p,i,1}, t) \end{bmatrix} \\
\begin{bmatrix} \dot{e}_{p_1,i}(t) \\ \dot{e}_{p_2,i}(t) \end{bmatrix} &= \begin{bmatrix} 0_{n_{p,i}-1 \times n_{p,i}-1} & I_{n_{p,i}-1} \\ -\gamma_1 E_i L_{i,i} F_i & -\gamma_2 E_i L_{i,i} F_i \end{bmatrix} \begin{bmatrix} e_{p_1,i}(t) \\ e_{p_2,i}(t) \end{bmatrix} + \begin{bmatrix} 0_{n_{p,i}-1} \\ F_e(y_{p,i}, z_{p,i}, t) \end{bmatrix}
\end{aligned}$$

where $F_e(y_{p,i}, z_{p,i}, t) = \begin{bmatrix} f(y_{p,i,1}, z_{p,i,1}, t) - f(y_{p,i,1} - y_{p,i,2}, z_{p,i,1} - z_{p,i,2}, t) & \dots & f(y_{p,i,1}, z_{p,i,1}, t) - f(y_{p,i,1} - y_{p,i,n_{p,i}}, z_{p,i,1} - z_{p,i,n_{p,i}}, t) \end{bmatrix}$.

The error dynamics of all PLSs can be written in the following form:

$$\begin{bmatrix}\dot{e}_{p_1}(t)\\ \dot{e}_{p_2}(t)\end{bmatrix} = \begin{bmatrix}0_{n_p-l_p\times n_p-l_p} & I_{n_p-l_p}\\ -\gamma_1 EL_pF & -\gamma_2 EL_pF\end{bmatrix}\begin{bmatrix}e_{p_1}(t)\\ e_{p_2}(t)\end{bmatrix} + \begin{bmatrix}0_{n_p-l_p}\\ \overline{F}_e(y_p, z_p, t)\end{bmatrix}$$

where $\overline{F}_e(y_p, z_p, t) = \begin{bmatrix}F_e^T(y_{p,1}, z_{p,1}, t) & \dots & F_e^T(y_{p,l_p}, z_{p,l_p}, t)\end{bmatrix}^T$. Before presenting the main result on second-order networks, we need the following lemma:

Lemma 3 (20). *Let M be a symmetric matrix of the form*

$$M = \begin{bmatrix}M_1 & M_2\\ M_2^T & M_3\end{bmatrix}$$

where M_1 is invertible. Then $M > 0$ if and only if $M_1 > 0$ and $M_3 - M_2^T M_1^{-1} M_2 > 0$.

The following theorem states the conditions on control parameters that guarantee cluster consensus of a second-order multi-agent system with inherent nonlinear dynamics.

Theorem 2. *Suppose that system (15) evolves over a digraph with l_p PLSs and l_s SLSs and there exists positive coefficients l_1 and l_2 that satisfy Assumption 2. The system in (15) converges to $l_p + l_s$ clusters if there exists a pair of control parameters (γ_1, γ_2) that satisfies $\gamma_1 \geq l_1$ $\gamma_2 \geq l_2$ and*

$$\frac{\gamma_1}{\gamma_2^2} < \min\left\{\frac{1}{2\lambda_M(P_p)}, \frac{1}{2\lambda_M(P_s)}\right\} - 2 \tag{22}$$

for the symmetric positive definite matrices $P_p \in \mathbb{R}^{n_p-l_p\times n_p-l_p}$, $P_s \in \mathbb{R}^{n_s\times n_s}$ that are obtained from

$$(EL_pF)^T P_p + P_p(EL_pF) = I_{n_p-l_p} \tag{23a}$$

$$L_s^T P_s + P_s L_s = I_{n_s} \tag{23b}$$

Proof. Step 1 (Stability and the number of clusters for the PLS dynamics): Each PLS dynamics can be individually investigated as

$$\begin{bmatrix}\dot{y}_{p,i,1}(t)\\ \dot{z}_{p,i,1}(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}y_{p,i,1}(t)\\ z_{p,i,1}(t)\end{bmatrix} + \begin{bmatrix}0^T_{n_{p,i}-1} & 0^T_{n_{p,i}-1}\\ -\gamma_1 l_{i,1}F_i & -\gamma_1 l_{i,1}F_i\end{bmatrix}\begin{bmatrix}e_{p_1,i}(t)\\ e_{p_2,i}(t)\end{bmatrix} + \begin{bmatrix}0\\ f(y_{p,i,1}, z_{p,i,1}, t)\end{bmatrix}$$

$$\begin{bmatrix}\dot{e}_{p_1,i}(t)\\ \dot{e}_{p_2,i}(t)\end{bmatrix} = \begin{bmatrix}0_{n_{p,i}-1\times n_{p,i}-1} & I_{n_{p,i}-1}\\ -\gamma_1 E_i L_{i,i} F_i & -\gamma_2 E_i L_{i,i} F_i\end{bmatrix}\begin{bmatrix}e_{p_1,i}(t)\\ e_{p_2,i}(t)\end{bmatrix} + \begin{bmatrix}0_{n_{p,i}-1}\\ F_e(y_{p,i}, z_{p,i}, t)\end{bmatrix}$$

To prove the stability of all PLSs at once, we consider the Lyapunov function candidate

$$V_1(t) = e_p^T(t) H_p e_p(t)$$

where $e_p(t) = \begin{bmatrix}e_{p_1}^T(t) & e_{p_2}^T(t)\end{bmatrix}^T$ with $e_{p_1}(t) = \begin{bmatrix}e^T_{p_1,1}, \dots, e^T_{p_1,l_p}\end{bmatrix}^T$, $e_{p_2}(t) = \begin{bmatrix}e^T_{p_2,1}, \dots, e^T_{p_2,l_p}\end{bmatrix}^T$ and

$$H_p = \begin{bmatrix}\gamma_1\gamma_2 I_{n_{p-l_p}} & \gamma_1 P_p\\ \gamma_1 P_p & \gamma_2 P_p\end{bmatrix}$$

with $P_p = \text{blkdiag}\{P_1, \dots, P_{l_p}\}$. From Lemma 3, $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{\lambda_M(P_p)}$ ensures that $H_p > 0$. Therefore, when $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{2\lambda_M(P_p)} - 2$ holds, $V(t) > 0$ if $e_p(t) \neq 0$ and $V(t) = 0$ if and only if $e_p(t) = 0$. $\dot{V}_1(t)$ can be computed as

$$\dot{V}_1(t) = 2\gamma_1\gamma_2 e_{p_1}^T(t)e_{p_2}(t) + 2\gamma_1 e_{p_2}^T(t)P_p e_{p_2}(t) - \gamma_1^2 e_{p_1}^T(t)[(EL_pF)^T P_p + P_p(EL_pF)]e_{p_1}(t)$$
$$- 2\gamma_1\gamma_2 e_{p_1}^T(t)[(EL_pF)^T P_p + P_p(EL_pF)]e_{p_2}(t) - \gamma_2^2 e_{p_2}^T(t)[(EL_pF)^T P_p + P_p(EL_pF)]e_{p_2}(t)$$
$$+ 2(\gamma_1 e_{p_1}^T(t) + \gamma_2 e_{p_2}^T(t))P_p\overline{F}_e(t) \tag{24}$$

Note that the inequality

$$\|\overline{F}_e(t)\| \leq l_1\|e_{p_1}(t)\| + l_2\|e_{p_2}(t)\|$$

holds and by choosing $\gamma_1 \geq l_1$ and $\gamma_2 \geq l_2$, we have

$$(\gamma_1 e_{p_1}^T(t) + \gamma_2 e_{p_2}^T(t))P_p\overline{F}_e(t) \leq \lambda_M(P_p)(\|\gamma_1 e_{p_1}(t)\| + \|\gamma_2 e_{p_2}(t)\|) \tag{25}$$

Together with (24) and (25), we can obtain the following inequality:

$$\dot{V}_1(t) \leq -[\lambda_m(Q_p) - 4\lambda_M(P_p)](\|\gamma_1 e_{p_1}(t)\|^2 + \|\gamma_2 e_{p_2}(t)\|^2)$$

where $Q_p = \begin{bmatrix} I_{n_p-l_p} & 0_{n_p-l_p} \\ 0_{n_p-l_p} & I_{n_p-l_p} - \frac{2\gamma_1}{\gamma_2^2}P_p \end{bmatrix}$. Note that $\lambda_m(Q_p) - 4\lambda_M(P_p) > 0$ if and only if $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{2\lambda_M(P_p)} - 2$ holds. By $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{2\lambda_M(P_p)} - 2$, we have $\dot{V}(t) < 0$ for $e_p(t) \neq 0$, which implies that each PLS dynamics is asymptotically stable. The agents in a particular PLS act together, that is, $\lim_{t\to\infty} \|x_{p,i,j_1} - x_{p,i,j_2}\| = 0$ and $\lim_{t\to\infty} \|v_{p,i,j_1} - v_{p,i,j_2}\| = 0$ for $i = 1, \dots, l_p$ and $j_1, j_2 = 1, \dots, n_{p,i}$. This implies that one can find a total of l_p clusters in l_p PLSs.

Step 2 (Stability and the number of clusters for the SLS dynamics): In this step, we will prove that the followings hold

$$\lim_{t\to\infty}\|y_s(t) + L_s^{-1}L_{sp}S_p^{-1}y_p(t)\| = 0,$$
$$\lim_{t\to\infty}\|z_s(t) + L_s^{-1}L_{sp}S_p^{-1}z_p(t)\| = 0 \tag{26}$$

as $t \to \infty$. Let $e_{s_1}(t) = y_s(t) + L_s^{-1}L_{sp}S_p^{-1}y_p(t)$ and $e_{s_2}(t) = z_s(t) + L_s^{-1}L_{sp}S_p^{-1}z_p(t)$. We have

$$e(t) = \begin{bmatrix} e_{s_1}(t) \\ e_{s_2}(t) \end{bmatrix} = \begin{bmatrix} y_s(t) + L_s^{-1}L_{sp}S_p^{-1}y_p(t) \\ z_s(t) + L_s^{-1}L_{sp}S_p^{-1}z_p(t) \end{bmatrix} \tag{27}$$

Taking derivatives yields

$$\dot{e}_{s_1}(t) = \dot{y}_s(t) + L_s^{-1}L_{sp}S_p^{-1}\dot{y}_p(t)$$
$$\dot{e}_{s_2}(t) = \dot{z}_s(t) + L_s^{-1}L_{sp}S_p^{-1}\dot{z}_p(t)$$
$$\dot{e}_{s_1}(t) = e_{s_2}(t)$$
$$\dot{e}_{s_2}(t) = \gamma_1 L_s e_{s_1}(t) - \gamma_2 L_s e_{s_2}(t) + f(y_s, z_s, t) + L_s^{-1}L_{sp}f(y_p, z_p, t)$$
$$= -\gamma_1 L_s^{-1}L_{sp}L_pS_p^{-1}y_p(t) - \gamma_2 L_s^{-1}L_{sp}L_pS_p^{-1}z_p(t)$$

Let $\omega(t) = -\gamma_1 L_s^{-1}L_{sp}L_pS_p^{-1}y_p(t) - \gamma_2 L_s^{-1}L_{sp}L_pS_p^{-1}z_p(t) = -\gamma_1 L_s^{-1}L_{sp}L_px_p(t) - \gamma_2 L_s^{-1}L_{sp}L_pv_p(t)$. The above dynamics can be expressed as follows:

$$\dot{e}_{s_1}(t) = e_{s_2}(t)$$
$$\dot{e}_{s_2}(t) = \gamma_1 L_s e_{s_1}(t) - \gamma_2 L_s e_{s_2}(t) + f(y_s, z_s, t) + L_s^{-1}L_{sp}f(y_p, z_p, t) + \omega(t)$$

Note that $\omega(t)$ converges to zero as $t \to \infty$ from Step 1. In order to analyze the stability of the above system dynamics, it is required to investigate the stability of the unforced system, that is, $\omega(t) = 0$ for all t (p. 107 in

Reference 21). One can obtain the derivative of the unforced system's error dynamics as

$$\begin{bmatrix} \dot{e}_{s_1}(t) \\ \dot{e}_{s_2}(t) \end{bmatrix} = \begin{bmatrix} 0_{n_s} \\ f(y_s, z_s, t) + L_s^{-1} L_{sp} f(y_p, z_p, t) \end{bmatrix} + \begin{bmatrix} 0_{n_s \times n_s} & I_{n_s} \\ -\gamma_1 L_s & -\gamma_2 L_s \end{bmatrix} \begin{bmatrix} e_{s_1}(t) \\ e_{s_2}(t) \end{bmatrix}.$$

In order to prove asymptotic stability of (27) choose the Lyapunov function candidate

$$V_2(t) = e_s(t)^T H_s e_s(t)$$

for the unforced SLS dynamics where $e_p(t) = \begin{bmatrix} e_{s_1}^T(t) & e_{s_2}^T(t) \end{bmatrix}^T$ and

$$H_s = \begin{bmatrix} \gamma_1 \gamma_2 I_{n_s} & \gamma_1 P_s \\ \gamma_1 P_s & \gamma_2 P_s \end{bmatrix}.$$

From Lemma 3, $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{\lambda_M(P_s)}$ ensures that $H_s > 0$. Therefore, when $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{2\lambda_M(P_s)} - 2$ holds, $V_2(t) > 0$ if $e_s(t) \neq 0$ and $V_2(t) = 0$ if and only if $e_s(t) = 0$. $\dot{V}_2(t)$ can be computed as

$$\begin{aligned} \dot{V}_2(t) = {} & 2\gamma_1\gamma_2 e_{s_1}^T(t) e_{s_2}(t) + 2\gamma_1 e_{s_2}^T(t) P_s e_{s_2}(t) - \gamma_1^2 e_{s_1}^T(t)[L_s^T P_s + P_s L_s] e_{s_1}(t) - 2\gamma_1\gamma_2 e_{s_1}^T(t)[L_s^T P_s + P_s L_s] e_{s_2}(t) \\ & - \gamma_2^2 e_{s_2}^T(t)[L_s^T P_s + P_s L_s] e_{s_2}(t) + 2(\gamma_1 e_{s_1}^T(t) + \gamma_2 e_{s_2}^T(t)) P_s [f(y_s, z_s, t) + L_s^{-1} L_{sp} f(y_p, z_p, t)] \end{aligned}$$

Following a similar procedure as in Step 1, the following inequality can be obtained:

$$\dot{V}_2(t) \leq -[\lambda_m(Q_s) - 4\lambda_M(P_s)](\|\gamma_1 e_{s_1}\|^2 + \|\gamma_2 e_{s_2}\|^2)$$

where $Q_s = \begin{bmatrix} I_{n_s} & 0_{n_s \times n_s} \\ 0_{n_s \times n_s} & I_{n_s} - \frac{2\gamma_1}{\gamma_2^2} P_s \end{bmatrix}$. Note that $\lambda_m(Q_s) - 4\lambda_M(P_s) > 0$ if and only if $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{2\lambda_M(P_s)} - 2$ holds, $V_2(t) > 0$ holds. By $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{2\lambda_M(P_s)} - 2$ holds, $V_2(t) > 0$, we have $\dot{V}_2(t) < 0$ for $e_s(t) \neq 0$, which implies that the origin of (27) is asymptotically stable. This implies that the following expression holds

$$\begin{aligned} y_s(t) &= -L_s^{-1} L_{sp} S_p^{-1} y_p(t), \\ z_s(t) &= -L_s^{-1} L_{sp} S_p^{-1} z_p(t), \quad t \to \infty \end{aligned}$$

Note that original states $x_p(t) = S_p^{-1} y_p(t)$, $x_s(t) = y_s(t)$, $v_p(t) = S_p^{-1} z_p(t)$ and $v_s(t) = z_s(t)$. One can obtain the following

$$\begin{aligned} x_s(t) &= -L_s^{-1} L_{sp} x_p(t) \\ v_s(t) &= -L_s^{-1} L_{sp} v_p(t) \end{aligned}$$

Let $x_p^*(t) = \lim_{t\to\infty} x_p(t)$, $x_s^*(t) = \lim_{t\to\infty} x_s(t)$, $v_p^*(t) = \lim_{t\to\infty} v_p(t)$ and $v_s^*(t) = \lim_{t\to\infty} v_s(t)$. As $t \to \infty$, $x_p(t)$ converges to trajectory $x_p^*(t)$ and $v_p(t)$ converges to trajectory $v_p^*(t)$ containing l_p distinct routes from Step 1. From PLS and SLS dynamics in Reference 17, it is known that l_p points at any t gives l_s clusters in $x_s^*(t)$ and $v_s^*(t)$ which are convergent trajectories for $x_s(t)$ and $v_s(t)$, respectively.

Step 3 (Total number of clusters): We have l_p distinct PLSs from Step 1 and l_s distinct SLSs from Step 2. Hence, one can conclude that the total number of cluster is computed as $\mathcal{K} = l_p + l_s$ since (22) is satisfied. ■

Remark 8. By applying Theorem 2, the selection method of control parameters γ_1 and γ_2 can be established as follows:

1. Calculate the maximum eigenvalue of P_p which is a symmetric positive definite matrix that satisfies (23a).
2. Calculate the maximum eigenvalue of P_s which is a symmetric positive definite matrix that satisfies (23b).

3. Obtain a pair of control parameters γ_1 and γ_2 that satisfies (22) by using the maximum eigenvalues of P_p and P_s.

Note that the protocol (17), which is used as update rule, is fully distributed from Remark 6. However, L_p and L_s global matrices should be known only for the stability analysis and the selection of control parameters γ_1 and γ_2.

Remark 9. Given a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, Theorems 1 and 2 state the cluster consensus results on the total number of clusters for first and second-order systems with inherently nonlinear dynamics, respectively. According to these theorems, the total number of clusters for both systems can be calculated as $l_p + l_s$ with the same cluster members. Although the clusters are the same, the stability conditions and the methods of the proofs are different from each other. First of all, the Lyapunov functions that we utilized in the proofs of Theorems 1 and 2 are not the same. Furthermore, Lemma 1 is used to prove the result of Theorem 1 while Lemma 3 is used in the proof of Theorem 2. Therefore, the second-order cluster consensus problem is not exactly an extension of its first-order counterpart.

4 | SIMULATION RESULTS

In this section, numerical simulations are provided to verify the results obtained in Sections 2 and 3. Reconsider the network with 10 agents in Figure 1A. The Laplacian matrix is selected as follows:

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & -6 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 & 7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -8 & 0 & 11 & 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & -6 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 & 14 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 4 & 0 \\ -8 & 0 & 0 & 0 & 0 & 0 & -9 & 0 & -6 & 23 \end{bmatrix} \tag{28}$$

Example 2. (First-order nonlinear dynamics) Consider the first-order consensus dynamics with inherent nonlinear functions in (2). The nonlinear function is given as $f(x, t) = x \sin(t)$. Note that Assumption 1 is satisfied with $l = 1$ and for the choice of

$$P_p = \begin{bmatrix} 0.1919 & 0.0126 & 0 \\ 0.0126 & 0.0357 & 0 \\ 0 & 0 & 0.1429 \end{bmatrix}, \quad P_s = \begin{bmatrix} 0.0909 & 0 & 0 & 0 & 0 \\ 0 & 0.0972 & 0.0167 & 0.0931 & 0.0110 \\ 0 & 0.0167 & 0.0714 & 0.0357 & 0 \\ 0 & 0.0931 & 0.0357 & 0.3448 & 0.0097 \\ 0 & 0.0110 & 0 & 0.0097 & 0.0435 \end{bmatrix},$$

the conditions (13a) and (13b) hold. By Theorem 1, if $\gamma_1 > 0.3858$ then the first-order dynamics of the given system reach cluster consensus. When $\gamma_1 = 1$, the simulation result is depicted in Figure 2 and the network acts as five distinct clusters. These are the same clusters, as in Example 1, which are $\mathcal{V}_{p,1} = \{v_1, v_2, v_3\}$, $\mathcal{V}_{p,2} = \{v_4, v_5\}$ and $\mathcal{V}_{s,1} = \{v_6\}$, $\mathcal{V}_{s,2} = \{v_7, v_8, v_9\}$, $\mathcal{V}_{s,3} = \{v_{10}\}$.

Example 3. (Second-order nonlinear dynamics) Consider the second-order consensus dynamics with inherent nonlinear functions in (15). The nonlinear function is given as $f(x, v, t) = -x - v \sin(t)$. Note that

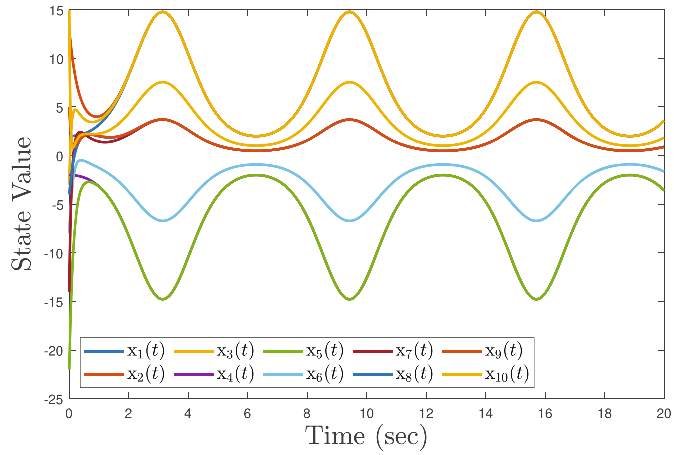

FIGURE 2 State evolution of first-order nonlinear dynamics for network in Figure 1A.

Assumption 2 is satisfied with $l_1 = 1$ and $l_2 = 1$. By Theorem 2, P_p and P_s matrices can be obtained as

$$P_p = \begin{bmatrix} 0.1919 & 0.0126 & 0 \\ 0.0126 & 0.0357 & 0 \\ 0 & 0 & 0.0714 \end{bmatrix}, \quad P_s = \begin{bmatrix} 0.0455 & 0 & 0 & 0 & 0 \\ 0 & 0.0486 & 0.0083 & 0.0465 & 0.0055 \\ 0 & 0.0083 & 0.0357 & 0.0179 & 0 \\ 0 & 0.0465 & 0.0179 & 0.1724 & 0.0048 \\ 0 & 0.0055 & 0 & 0.0048 & 0.0217 \end{bmatrix}.$$

The conditions on γ_1 and γ_2 for PLSs and SLSs can be obtained as

$$\frac{\gamma_1}{\gamma_2^2} < \min\left\{\frac{1}{2\lambda_M(P_p)}, \frac{1}{2\lambda_M(P_s)}\right\} - 2 = \min\{5.1844, 5.2447\} - 2 = 3.1844$$

When $\gamma_1 = \gamma_2 = 1$, the simulation result is depicted in Figure 3 and the network acts as five distinct clusters. Note that clusters and their members are the same as in Example 2.

Example 4. (Multi-robot system) In this example, we consider the rendezvous problem as a potential application in multi-robot systems. Consider the communication topology in Figure 4. Note that the network does not contain a spanning tree. By using PLS and SLS detection algorithms in Reference 16, PLSs and SLSs and their members can be obtained as $\mathcal{V}_{p,1} = \{v_1, v_2\}$, $\mathcal{V}_{p,2} = \{v_3\}$, $\mathcal{V}_{p,3} = \{v_4\}$ and $\mathcal{V}_{s,1} = \{v_5, v_6\}$, $\mathcal{V}_{s,2} = \{v_7\}$, $\mathcal{V}_{s,3} = \{v_8\}$, respectively. The following Laplacian matrix is used as system matrix:

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & -2 & 0 & 10 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -7 & 7 & 0 & 0 \\ 0 & 0 & 0 & -3 & -6 & 0 & 9 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & -2 & 6 \end{bmatrix} \tag{29}$$

Each robot is randomly placed in 2-D coordinate system. Inherent nonlinear functions are chosen as $f(x, t) = 0.5\sin(x)$ and $f(y, t) = 0.5\sin(y)$ for x and y-coordinates, respectively. The goal is to have robots in the same

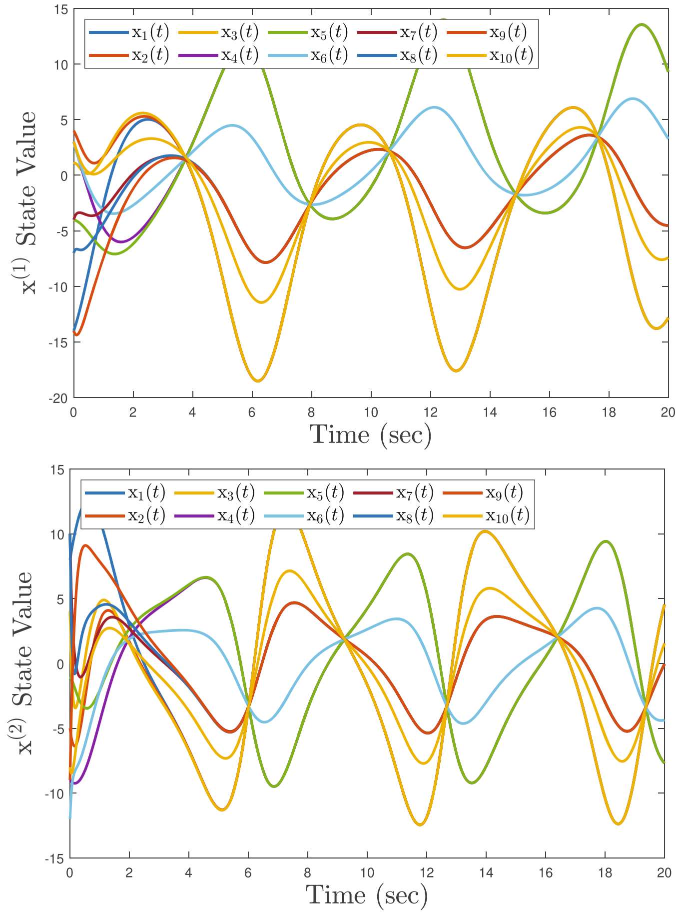

FIGURE 3 State evolution of second-order nonlinear dynamics for network in Figure 1A.

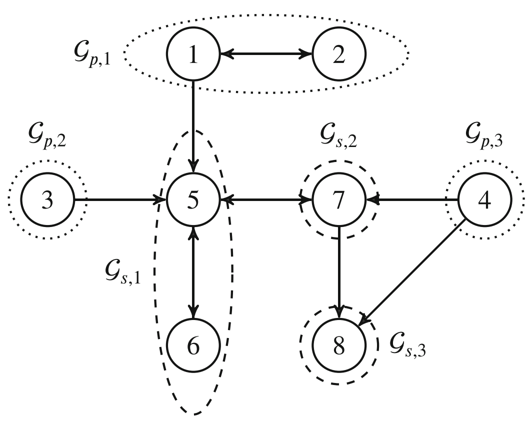

FIGURE 4 A network consisting of eight nodes and 11 edges.

cluster converge to a common coordinate. Note that Assumption 1 is satisfied with $l = 1$ and for the choice of

$$P_p = 0.1667, \quad P_s = \begin{bmatrix} 0.2554 & 0.1678 & 0.0632 & 0.0099 \\ 0.1678 & 0.2388 & 0.0267 & 0.0031 \\ 0.0632 & 0.0267 & 0.1232 & 0.0229 \\ 0.0099 & 0.0031 & 0.0229 & 0.1667 \end{bmatrix},$$

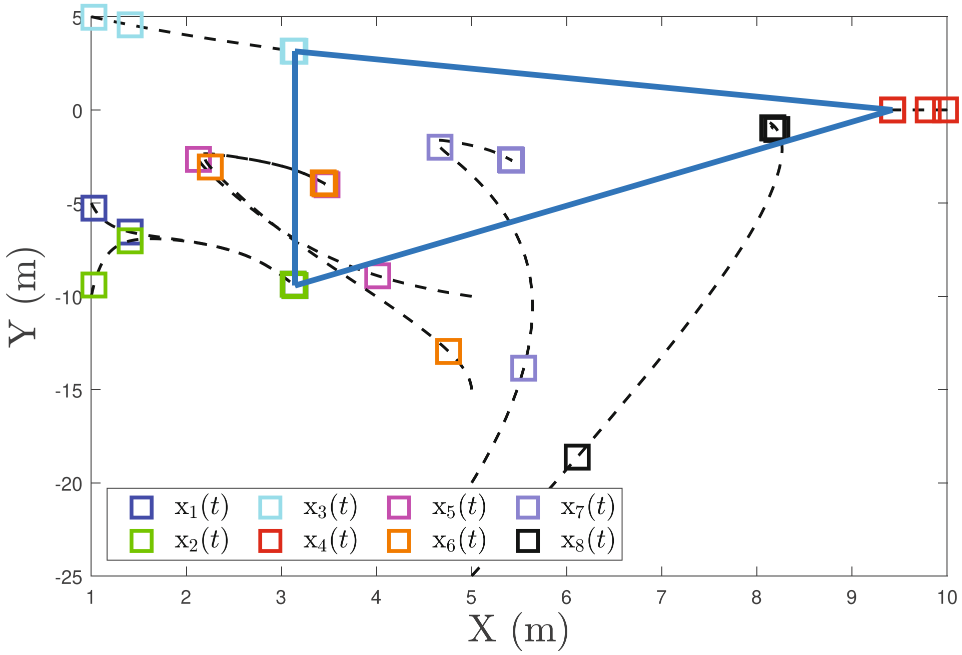

FIGURE 5 Position of the robots are marked by squares for the time instants $t = 0.07$ s, $t = 0.88$ s, $t = 10.43$ s and $t = 20$ s.

the conditions (13a) and (13b) hold. By Theorem 1, if $\gamma_1 > 0.8588$ then the multi-robot system reaches cluster consensus. As can be seen from the simulation results in Figure 5 for $\gamma_1 = 1$, SLSs converge to convex hull spanned by PLSs.

5 | CONCLUSION

This paper has discussed the cluster consensus problem for first and second-order networks with nonlinear inherent dynamics. We have analyzed convergence properties of the given nonlinear systems. The number of clusters and the agents in each cluster have been calculated by using PLS and SLS concepts. Two different control parameter selection methods are provided for first and second-order systems. The obtained results in this paper have been justified by two numerical examples. The extension of the results for nonlinear cooperative-competitive networks expressed by signed digraphs is currently under investigation.

ACKNOWLEDGMENTS

This research was supported by the Scientific and Technical Research Council of Turkey (TUBITAK) under TUBITAK grant 118E218.

CONFLICT OF INTEREST STATEMENT

The authors declare no conflict of interest.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID

Ümit Develer https://orcid.org/0000-0002-6838-8676
Onur Cihan https://orcid.org/0000-0002-5729-2417
Mehmet Akar https://orcid.org/0000-0002-0166-9932

REFERENCES

1. Chen Y, Lü J, Han F, Yu X. On the cluster consensus of discrete-time multi-agent systems. *Syst Control Lett*. 2011;60(7):517-523.
2. Qin J, Yu C. Cluster consensus control of generic linear multi-agent systems under directed topology with acyclic partition. *Automatica*. 2013;49(9):2898-2905.
3. Han Y, Lu W, Chen T. Achieving cluster consensus in continuous-time networks of multi-agents with inter-cluster non-identical inputs. *IEEE Trans Automat Contr*. 2014;60(3):793-798.
4. Gambuzza LV, Frasca M. A criterion for stability of cluster synchronization in networks with external equitable partitions. *Automatica.* 2019;100:212-218.
5. Liu K, Xie G, Ren W, Wang L. Consensus for multi-agent systems with inherent nonlinear dynamics under directed topologies. *Syst Control Lett*. 2013;62(2):152-162.
6. Defoort M, Di Gennaro S, Djemai M. Self-triggered control for multi-agent systems with unknown non-linear inherent dynamics. *IET Control Theory Appl*. 2014;8(18):2266-2275.
7. Liu Z, Jin Q, Chen Z. Distributed containment control for bounded unknown second-order nonlinear multi-agent systems with dynamic leaders. *Neurocomputing*. 2015;168:1138-1143.
8. Wang P, Jia Y. Distributed containment control of second-order multi-agent systems with inherent non-linear dynamics. *IET Control Theory Appl*. 2014;8(4):277-287.
9. Xu T, Lv G, Duan Z, Sun Z, Yu J. Distributed fixed-time triggering-based containment control for networked nonlinear agents under directed graphs. *IEEE Trans Circuits Syst I: Regul Pap*. 2020;67(10):3541-3552.
10. Mei J, Ren W, Ma G. Distributed containment control for multiple nonlinear systems with identical dynamics. Proceedings of the 30th Chinese Control Conference IEEE. 2011 6544–6549.
11. Zou W, Xiang Z. Event-triggered containment control of second-order nonlinear multi-agent systems. *J Franklin Inst*. 2019;356(17):10421-10438.
12. Wang Q, Fu J, Wang J. Fully distributed containment control of high-order multi-agent systems with nonlinear dynamics. *Syst Control Lett*. 2017;99:33-39.
13. Ning B, Zuo Z, Jin J, Zheng J. Distributed fixed-time coordinated tracking for nonlinear multi-agent systems under directed graphs. *Asian J Control*. 2018;20(2):646-658.
14. Zhang Y, Yang Y, Zhao Y, Wen G. Distributed finite-time tracking control for nonlinear multi-agent systems subject to external disturbances. *Int J Control*. 2013;86(1):29-40.
15. Yu W, Chen G, Cao M, Kurths J. Second-order consensus for multiagent systems with directed topologies and nonlinear dynamics. *IEEE Trans Syst Man Cybern B Cybern*. 2009;40(3):881-891.
16. Erkan ÖF, Cihan O, Akar M. Analysis of distributed consensus protocols with multi-equilibria under time-delays. *J Franklin Inst*. 2018;355(1):332-360.
17. Develer Ü, Akar M. Cluster consensus in first and second-order continuous-time networks with input and communication delays. *Int J Control*. 2021;94(4):961-976.
18. Develer Ü, Akar M. Higher-order cluster consensus of a multi-agent network with continuous-time dynamics. *Trans Inst Meas Control*. 2021;43(8):1734-1744.
19. Wu M, He Y, She JH. *Stability Analysis and Robust Control of Time-Delay Systems*. Vol 22. Springer; 2010.
20. Horn RA, Johnson CR. *Matrix Analysis*. Cambridge University Press; 2012.
21. Khalil HK. *Nonlinear Control*. Vol 406. Pearson New York; 2015.

How to cite this article: Develer Ü, Cihan O, Akar M. Cluster consensus for inherently nonlinear cooperative networks. *Int J Robust Nonlinear Control*. 2023;1-18. doi: 10.1002/rnc.7061