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First order self-adjoint multipoint quasi-differential operators

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Abstract: In this paper, using the Calkin–Gorbachuk method, the general form of all self-adjoint operators generated by first order linear singular multipoint quasi-differential expressions in the direct sum of weighted Hilbert spaces of vector functions has been found. Later on, the geometry of the spectrum set of these type extensions was researched.

Key words: Selfadjoint operators, multipoint quasi-differential expression, deficiency indices, spectrum

1. Introduction

The general theory of self-adjoint extensions of linear densely-defined closed symmetric operators in any Hilbert space was mentioned for the first time in the mathematical literature in famous works of Neumann [12] and Stone [14]. Application to scalar linear even order symmetric differential operators and description of all self-adjoint extensions in terms of boundary values was done by Glazman in his seminal work [5] and by Naimark [11] in his book. It is noteworthy to mention that Glazman–Krein–Naimark (or Everitt–Krein–Glazman–Naimark) theorem is very important in the mathematical literature. The Calkin–Gorbachuk method is also another important method in this area (see [6,13]).

Our major motivation originates from some interesting researches [2–4,15] on scalar cases.

In the present study, the representation of all self-adjoint extensions of the multipoint symmetric quasi-differential operators is obtained. These operators are generated by first order symmetric quasi-differential operator expression in the space of the direct sum of weighted Hilbert spaces of vector functions defined on the semiinfinite intervals. In Section 3, we study them in the sense of abstract boundary values. In Section 4, we also examine the spectrum of these self-adjoint extensions.

For the differential operators in Hilbert space three questions are important:
(1) Is this operator symmetric?
(2) What are the boundary conditions by which it is generated?
(3) What is the spectrum of this operator? (see [15])

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2. Statement of the problem

Let H be a separable Hilbert space and $a_1,\ a_2 \in \mathbb{R}$. In the Hilbert space

$$\mathcal{H} = L^2_{\omega_1}(H,(-\infty,a_1)) \oplus L^2_{\omega_2}(H,(a_2,\infty))$$

of vector functions on $(-\infty,a_1)\cup(a_2,\infty)$, consider the following linear multipoint differential operator expression for first order in the form

$$l(u) = (l_1(u_1), l_2(u_2)),$$

where $u = (u_1, u_2)$,

$$l_1(u_1) = i\frac{\alpha_1}{\omega_1}(\alpha_1 u_1)' + A_1 u_1,$$

$$l_2(u_2) = i\frac{\alpha_2}{\omega_2}(\alpha_2 u_2)' + A_2 u_2,$$

where:

(1) $\alpha_1, \omega_1 : (-\infty, a_1) \to (0,\infty)$, $\alpha_2, \omega_2 : (a_2,\infty) \to (0,\infty)$;

(2) $\alpha_1, \omega_1 \in C(-\infty, a_1)$, $\alpha_2, \omega_2 \in C(a_2,\infty)$;

(3) $\int_{-\infty}^{a_1} \frac{\omega_1(t)}{\alpha_1^2(t)}dt = \infty$, $\int_{a_2}^{\infty} \frac{\omega_2(t)}{\alpha_2^2(t)}dt = \infty$;

(4) $A_1 : D(A_1) \subset H \to H$ and $A_2 : D(A_2) \subset H \to H$ are linear self-adjoint operators.

The minimal L_{10} (L_{20}) and maximal L_1 (L_2) operators associated with differential expression l_1 (l_2) in $L^2_{\omega_1}(H,(-\infty,a_1))$ $(L^2_{\omega_2}(H,(a_2,\infty)))$ can be constructed by using the same technique in [7].

The operators $L_0 = L_{10} \oplus L_{20}$ and $L = L_1 \oplus L_2$ in the Hilbert space $\mathcal{H}$ are called minimal and maximal operators associated with differential expression $l(\cdot)$, respectively. It is clear that the operator L_0 is symmetric and $L_0^* = L$ in $\mathcal{H}$. One can easily see that the operator L_0 is not maximal. Furthermore, differential expression $l(\cdot)$ with boundary condition $(\alpha_2 u_2)(a_2) = (\alpha_1 u_1)(a_1)$ generates a self-adjoint extension of L_0.

Our aim in this paper is to obtain all self-adjoint extensions of the minimal operator L_0 in $\mathcal{H}$ in terms of boundary values and examine the spectrum of them.

3. Description of all self-adjoint extensions

In this section, we will study the abstract representation of all self-adjoint extensions of the minimal operator L_0 in terms of boundary values using the method of Calkin and Gorbachuk.

Let us prove the following auxiliary result we will need:

Lemma 1 *The deficiency indices of the operators* L_{10} *and* L_{20} *are in form*

$$(m(L_{10}), n(L_{10})) = (dim H, 0),$$

$$(m(L_{20}), n(L_{20})) = (0, dim H).$$

Proof The general solutions of differential equations can be given as follows:

$$i\frac{\alpha_1(t)}{\omega_1(t)}(\alpha_1 u_1^{\pm})'(t) \pm i u_1^{\pm}(t) = 0, \quad t < a_1,$$

$$i\frac{\alpha_2(t)}{\omega_2(t)}(\alpha_2 u_2^{\pm})'(t) \pm i u_2^{\pm}(t) = 0, \quad t > a_2,$$

where

$$u_1^{\pm}(t) = \frac{1}{\alpha_1(t)} exp\left(\pm \int_t^{a_1} \frac{\omega_1(s)}{\alpha_1^2(s)} ds\right) f_1, \quad t < a_1, \quad f_1 \in H,$$

$$u_2^{\pm}(t) = \frac{1}{\alpha_2(t)} exp\left(\mp \int_{a_2}^{t} \frac{\omega_2(s)}{\alpha_2^2(s)} ds\right) f_2, \quad t > a_2, \quad f_2 \in H,$$

respectively.

Then we obtain that

$$\begin{aligned}
\|u_2^+\|^2_{L^2_{\omega_2}(H,(a_2,\infty))} &= \int_{a_2}^{\infty} \|u_2^+(t)\|_H^2 \omega_2(t) dt \\
&= \int_{a_2}^{\infty} \|\frac{1}{\alpha_2(t)} exp\left(-\int_{a_2}^{t} \frac{\omega_2(s)}{\alpha_2^2(s)} ds\right) f_2\|_H^2 \omega_2(t) dt \\
&= \int_{a_2}^{\infty} \frac{\omega_2(t)}{\alpha_2^2(t)} exp\left(-2\int_{a_2}^{t} \frac{\omega_2(s)}{\alpha_2^2(s)} ds\right) dt \|f_2\|_H^2 \\
&= \int_{a_2}^{\infty} exp\left(-2\int_{a_2}^{t} \frac{\omega_2(s)}{\alpha_2^2(s)} ds\right) d\left(\int_{a_2}^{t} \frac{\omega_2(s)}{\alpha_2^2(s)} ds\right) \|f_2\|_H^2 \\
&= \frac{1}{2}\left(1 - exp\left(-2\int_{a_2}^{\infty} \frac{\omega_2(s)}{\alpha_2^2(s)} ds\right)\right) \|f_2\|_H^2 = \frac{1}{2}\|f_2\|_H^2 < \infty.
\end{aligned}$$

By simple calculations, we also have that

$$u_2^-(t) = \frac{1}{\alpha_2(t)} exp\left(\int_{a_2}^{t} \frac{\omega_2(s)}{\alpha_2^2(s)} ds\right) f_2 \notin L^2_{\omega_2}(H, (a_2, \infty)).$$

Consequently, the deficiency indices of the operator L_{20} can be expressed in the following form:

$$(m(L_{20}), n(L_{20})) = (0, dimH).$$

By using the same technique, one can also show that

$$(m(L_{10}), n(L_{10})) = (dimH, 0),$$

which completes the proof. □

From the last assertion, it is obvious that

$$m(L_0) = m(L_{10}) + m(L_{20}) = dimH$$

and

$$n(L_0) = n(L_{10}) + n(L_{20}) = dimH.$$

Consequently, the symmetric minimal operator L_0 has a self-adjoint extension (see [6]).

In order to describe all self-adjoint extensions of the minimal operator L_0, it is necessary to construct a space of boundary values for it.

Definition 1 *[6] Let $\mathbb{H}$ be any Hilbert space and $S : D(S) \subset \mathbb{H} \to \mathbb{H}$ be a closed densely defined symmetric operator on the Hilbert space having equal finite or infinite deficiency indices. A triplet $(\mathfrak{H}, \gamma_1, \gamma_2)$, where $\mathfrak{H}$ is a Hilbert space and γ_1 and γ_2 are linear mappings from $D(S^*)$ into $\mathfrak{H}$, is called a space of boundary values for the operator S, if for any $f, g \in D(S^*)$*

$$(S^* f, g)_{\mathbb{H}} - (f, S^* g)_{\mathbb{H}} = (\gamma_1(f), \gamma_2(g))_{\mathfrak{H}} - (\gamma_2(f), \gamma_1(g))_{\mathfrak{H}}$$

while for any $F, G \in \mathfrak{H}$, there exists an element $f \in D(S^*)$ such that $\gamma_1(f) = F$ and $\gamma_2(f) = G$.

It is known that for any symmetric operator with equal deficiency indices, we have at least one space of boundary values (see [6]).

Theorem 1 *The triplet (H, γ_1, γ_2), where*

$$\gamma_1 : D(L) \subset H \to H, \quad \gamma_1(u) = \frac{1}{\sqrt{2}}((\alpha_1 u_1)(a_1) - (\alpha_2 u_2)(a_2)), \quad u = (u_1, u_2) \in D(L),$$

$$\gamma_2 : D(L) \subset H \to H, \quad \gamma_2(u) = \frac{1}{i\sqrt{2}}((\alpha_1 u_1)(a_1) + (\alpha_2 u_2)(a_2)), \quad u = (u_1, u_2) \in D(L)$$

is a space of boundary values of the minimal operator L_0 in $\mathcal{H}$.

Proof In this case, for any $u = (u_1, u_2)$ and $v = (v_1, v_2)$ from $D(L)$ one can easily check that

$$\begin{aligned}
(Lu, v)_{\mathcal{H}} - (u, Lv)_{\mathcal{H}} &= (i\frac{\alpha_1}{\omega_1}(\alpha_1 u_1)' + A_1 u_1, v_1)_{L^2_{\omega_1}(H,(-\infty,a_1))} + (i\frac{\alpha_2}{\omega_2}(\alpha_2 u_2)' + A_2 u_2, v_2)_{L^2_{\omega_2}(H,(a_2,\infty))} \\
&- (u_1, i\frac{\alpha_1}{\omega_1}(\alpha_1 v_1)' + A_1 v_1)_{L^2_{\omega_1}(H,(-\infty,a_1))} - (u_2, i\frac{\alpha_2}{\omega_2}(\alpha_2 v_2)' + A_2 v_2)_{L^2_{\omega_2}(H,(a_2,\infty))} \\
&= (i\frac{\alpha_1}{\omega_1}(\alpha_1 u_1)', v_1)_{L^2_{\omega_1}(H,(-\infty,a_1))} + (A_1 u_1, v_1)_{L^2_{\omega_1}(H,(-\infty,a_1))} \\
&+ (i\frac{\alpha_2}{\omega_2}(\alpha_2 u_2)', v_2)_{L^2_{\omega_2}(H,(a_2,\infty))} + (A_2 u_2, v_2)_{L^2_{\omega_2}(H,(a_2,\infty))} \\
&- (u_1, i\frac{\alpha_1}{\omega_1}(\alpha_1 v_1)')_{L^2_{\omega_1}(H,(-\infty,a_1))} - (u_1, A_1 v_1)_{L^2_{\omega_1}(H,(-\infty,a_1))} \\
&- (u_2, i\frac{\alpha_2}{\omega_2}(\alpha_2 v_2)')_{L^2_{\omega_2}(H,(a_2,\infty))} - (u_2, A_2 v_2)_{L^2_{\omega_2}(H,(a_2,\infty))} \\
&= i\left[(\frac{\alpha_1}{\omega_1}(\alpha_1 u_1)', v_1)_{L^2_{\omega_1}(H,(-\infty,a_1))} + (u_1, \frac{\alpha_1}{\omega_1}(\alpha_1 v_1)')_{L^2_{\omega_1}(H,(-\infty,a_1))}\right] \\
&+ i\left[(\frac{\alpha_2}{\omega_2}(\alpha_2 u_2)', v_2)_{L^2_{\omega_2}(H,(a_2,\infty))} + (u_2, \frac{\alpha_2}{\omega_2}(\alpha_2 v_2)')_{L^2_{\omega_2}(H,(a_2,\infty))}\right] \\
&= i\left[\int_{-\infty}^{a_1} (\frac{\alpha_1}{\omega_1}(\alpha_1 u_1)', v_1)_H \omega_1(t) dt + \int_{-\infty}^{a_1} (u_1, \frac{\alpha_1}{\omega_1}(\alpha_1 v_1)')_H \omega_1(t) dt\right] \\
&+ i\left[\int_{a_2}^{\infty} (\frac{\alpha_2}{\omega_2}(\alpha_2 u_2)', v_2)_H \omega_2(t) dt + \int_{a_2}^{\infty} (u_2, \frac{\alpha_2}{\omega_2}(\alpha_2 v_2)')_H \omega_2(t) dt\right]
\end{aligned}$$

$$
\begin{aligned}
&= i\left[\int_{-\infty}^{a_1}((\alpha_1u_1)', (\alpha_1v_1))_H dt + \int_{-\infty}^{a_1}((\alpha_1u_1), (\alpha_1v_1)')_H dt\right] \\
&+ i\left[\int_{a_2}^{\infty}((\alpha_2u_2)', (\alpha_2v_2))_H dt + \int_{a_2}^{\infty}((\alpha_2u_2), (\alpha_2v_2)')_H dt\right] \\
&= i\left(\int_{-\infty}^{a_1}((\alpha_1u_1), (\alpha_1v_1))_H' dt + \int_{a_2}^{\infty}((\alpha_2u_2), (\alpha_2v_2))_H' dt\right) \\
&= i\left[((\alpha_1u_1)(a_1), (\alpha_1v_1)(a_1))_H - ((\alpha_2u_2)(a_2), (\alpha_2v_2)(a_2))_H\right] \\
&= (\gamma_1(u), \gamma_2(v))_H - (\gamma_2(u), \gamma_1(v))_H.
\end{aligned}
$$

Now let $f_1, f_2 \in H$. Let us find the function $u = (u_1, u_2) \in D(L)$ such that

$$\gamma_1(u) = \frac{1}{\sqrt{2}}((\alpha_1u_1)(a_1) - (\alpha_2u_2)(a_2)) = f_1$$

and

$$\gamma_2(u) = \frac{1}{i\sqrt{2}}((\alpha_1u_1)(a_1) + (\alpha_2u_2)(a_2)) = f_2.$$

From this we can obtain that

$$(\alpha_1u_1)(a_1) = \frac{(if_2 + f_1)}{\sqrt{2}}, \quad (\alpha_2u_2)(a_2) = \frac{(if_2 - f_1)}{\sqrt{2}}$$

If we choose the functions $u_1(\cdot)$ and $u_2(\cdot)$ as

$$u_1(t) = \frac{1}{\alpha_1(t)} exp\left(-\int_t^{a_1} \frac{\omega_1(s)}{\alpha_1^2(s)} ds\right) \frac{(if_2 + f_1)}{\sqrt{2}}, \quad t < a_1,$$

$$u_2(t) = \frac{1}{\alpha_2(t)} exp\left(-\int_{a_2}^{t} \frac{\omega_2(s)}{\alpha_2^2(s)} ds\right) \frac{(if_2 - f_1)}{\sqrt{2}}, \quad t > a_2$$

then it is obvious that $(u_1, u_2) \in D(L)$ and $\gamma_1(u_1) = f_1$, $\gamma_2(u_2) = f_2$. □

With the use of the Calkin–Gorbachuk method [6], we obtain the following:

Theorem 2 *If $\tilde{L}$ is a self-adjoint extension of the minimal operator L_0 in $\mathcal{H}$, then it is generated by the differential operator expression $l = (l_1, l_2)$ and the boundary condition*

$$(\alpha_2u_2)(a_2) = W(\alpha_1u_1)(a_1),$$

where $W : H \to H$ is a unitary operator. Moreover, the unitary operator W in H is determined uniquely by the extension $\tilde{L}$, i.e. $\tilde{L} = L_W$, and vice versa.

Proof It is known that all self-adjoint extensions of the minimal operator $\tilde{L_0}$ are described by the differential-operator expression $l = (l_1, l_2)$ with boundary condition

$$(V - E)\gamma_1(u) + i(V + E)\gamma_2(u) = 0, \quad u = (u_1, u_2) \in D(L),$$

where $V : H \to H$ is a unitary operator. Therefore, from Lemma 3.3, we obtain

$$(V-E)\frac{1}{\sqrt{2}}((\alpha_1 u_1)(a_1)-(\alpha_2 u_2)(a_2))+i(V+E)\frac{1}{i\sqrt{2}}((\alpha_1 u_1)(a_1)+(\alpha_2 u_2)(a_2))=0.$$

Hence it is obtained that

$$(\alpha_2 u_2)(a_2)=-V(\alpha_1 u_1)(a_1).$$

Choosing $W=-V$ in the last boundary condition we have

$$(\alpha_2 u_2)(a_2)=W(\alpha_1 u_1)(a_1).$$

□

4. Spectrum of self-adjoint extensions

In this section we will investigate the structure of the spectrum of the self-adjoint extension L_W of the minimal operator L_0 in $\mathcal{H}$.

Theorem 3 *The point spectrum $\sigma_p(L_W)$ of the self-adjoint extension L_W is empty.*

Proof Let us consider the following eigenvalue problem defined by

$$l(u)=\lambda u, \quad u=(u_1,u_2)\in\mathcal{H}, \quad \lambda\in\mathbb{R},$$

with boundary condition

$$(\alpha_2 u_2)(a_2)=W(\alpha_1 u_1)(a_1).$$

Then we have

$$\begin{gathered} i\tfrac{\alpha_1(t)}{\omega_1(t)}(\alpha_1 u_1)'(t)+A_1 u_1(t)=\lambda u_1(t), \quad t<a_1, \\ i\tfrac{\alpha_2(t)}{\omega_2(t)}(\alpha_2 u_2)'(t)+A_2 u_2(t)=\lambda u_2(t), \quad t>a_2, \\ (\alpha_2 u_2)(a_2)=W(\alpha_1 u_1)(a_1). \end{gathered}$$

The general solutions of these differential equations are as follows:

$$\begin{gathered} u_1(t;\lambda)=\tfrac{1}{\alpha_1(t)}exp\left(-i(A_1-\lambda E)\int_t^{a_1}\tfrac{\omega_1(s)}{\alpha_1^2(s)}ds\right)f_\lambda^{(1)}, \quad f_\lambda^{(1)}\in H, \quad t<a_1, \\ u_2(t;\lambda)=\tfrac{1}{\alpha_2(t)}exp\left(i(A_2-\lambda E)\int_{a_2}^{t}\tfrac{\omega_2(s)}{\alpha_2^2(s)}ds\right)f_\lambda^{(2)}, \quad f_\lambda^{(2)}\in H, \quad t>a_2. \end{gathered}$$

In this case

$$\begin{aligned} \|u_1(t;\lambda)\|^2_{L^2_{\omega_1}(H,(-\infty,a_1))} &= \|\frac{1}{\alpha_1(t)}exp\left(-i(A_1-\lambda E)\int_t^{a_1}\frac{\omega_1(s)}{\alpha_1^2(s)}ds\right)f_\lambda^{(1)}\|^2_{L^2_{\omega_1}(H,(-\infty,a_1))} \\ &= \int_{-\infty}^{a_1}\frac{\omega_1(t)}{\alpha_1^2(t)}dt\|f_\lambda^{(1)}\|_H^2=\infty \end{aligned}$$

and

$$\begin{aligned} \|u_2(t;\lambda)\|^2_{L^2_{\omega_2}(H,(a_2,\infty))} &= \|\frac{1}{\alpha_2(t)}exp\left(i(A_2-\lambda E)\int_{a_2}^{t}\frac{\omega_2(s)}{\alpha_2^2(s)}ds\right)f_\lambda^{(2)}\|^2_{L^2_{\omega_2}(H,(a_2,\infty))} \\ &= \int_{a_2}^{\infty}\frac{\omega_2(t)}{\alpha_2^2(t)}dt\|f_\lambda^{(2)}\|_H^2=\infty. \end{aligned}$$

Then one can notice that $u_1(\cdot,\lambda) \notin L^2_{\omega_1}(H,(-\infty,a_1))$ and $u_2(\cdot,\lambda) \notin L^2_{\omega_2}(H,(a_2,\infty))$.

Consequently, we obtain that $\sigma_p(L_W) = \emptyset$ for very unitary operator W in H. □

Notice that the residual spectrum of any self-adjoint operator in any Hilbert space is empty. Therefore, it is enough to study the continuous spectrum of the self-adjoint extensions L_W of the minimal operator L_0 in $\mathcal{H}$. It is well known that

$$\sigma(L_W) \subset \mathbb{R}$$

in the theory of linear self-adjoint operators in Hilbert spaces.

One can immediately obtain the following:

Theorem 4 *The continuous spectrum $\sigma_c(L_W)$ of the self-adjoint extension L_W in $\mathcal{H}$ coincides with $\mathbb{R}$, i.e. $\sigma_c(L_W) = \mathbb{R}$.*

Proof For $\lambda \in \mathbb{C},\ \lambda_i = Im\lambda > 0$ and $f = (f_1, f_2) \in \mathcal{H}$ one can see that

$$\begin{aligned}
\|R_\lambda(L_W))f(t)\|^2_{\mathcal{H}} &= \|\frac{1}{\alpha_1(t)} exp\left(i(A_1-\lambda E)\int_{a_1}^{t}\frac{\omega_1(s)}{\alpha_1^2(s)}ds\right)f_\lambda \\
&+\frac{i}{\alpha_1(t)}\int_{t}^{a_1}\exp\left(i(A_1-\lambda E)\int_{s}^{t}\frac{\omega_1(\tau)}{\alpha_1^2(\tau)}d\tau\right)\frac{\omega_1(s)}{\alpha_1(s)}f_1(s)ds\|^2_{L^2_{\omega_1}(H,(-\infty,a_1))} \\
&+\|\frac{i}{\alpha_2(t)}\int_{t}^{\infty}\exp\left(i(A_2-\lambda E)\int_{s}^{t}\frac{\omega_2(\tau)}{\alpha_2^2(\tau)}d\tau\right)\frac{\omega_2(s)}{\alpha_2(s)}f_2(s)ds\|^2_{L^2_{\omega_2}(H,(a_2,\infty))} \\
&\geq \|\frac{1}{\alpha_2(t)}\int_{t}^{\infty}\exp\left(i(A_2-\lambda E)\int_{s}^{t}\frac{\omega_2(\tau)}{\alpha_2^2(\tau)}d\tau\right)\frac{\omega_2(s)}{\alpha_2(s)}f_2(s)ds\|^2_{L^2_{\omega_2}(H,(a_2,\infty))}.
\end{aligned}$$

The vector functions $f^*(t;\lambda)$ have the form $f^*(t;\lambda) = \left(0, \frac{1}{\alpha_2(t)} exp\left(i(A_2-\overline{\lambda})\int_{a_2}^{t}\frac{\omega_2(s)}{\alpha_2^2(s)}ds\right)f\right)$, $\lambda \in \mathbb{C}$, $\lambda_i = Im\lambda > 0,\ f \in H$ belong to $\mathcal{H}$. Indeed,

$$\begin{aligned}
\|f^*(t;\lambda)\|^2_{\mathcal{H}} &= \int_{a_2}^{\infty}\frac{1}{\alpha_2^2(t)}\|exp\left(i(A_2-\overline{\lambda})\int_{a_2}^{t}\frac{\omega_2(s)}{\alpha_2^2(s)}ds\right)f\|^2_H\omega_2(t)dt \\
&= \int_{a_2}^{\infty}\frac{1}{\alpha_2^2(t)}exp\left(-2\lambda_i\int_{a_2}^{t}\frac{\omega_2(s)}{\alpha_2^2(s)}ds\right)\omega_2(t)dt\|f\|^2_H \\
&= \frac{1}{2\lambda_i}\|f\|^2_H < \infty.
\end{aligned}$$

For such functions $f^*(\lambda;\ \cdot\ )$, we have

$$\begin{aligned}
\|R_\lambda(L_W)f^*(\lambda;\ \cdot\ )\|_{\mathcal{H}}^2 &\geq \|\frac{i}{\alpha_2(t)}\int_t^\infty \frac{1}{\alpha_2(s)} exp\left(i(A_2-\lambda E)\int_s^t \frac{\omega_2(\tau)}{\alpha_2^2(\tau)}d\tau + i(A_2-\overline{\lambda}E)\int_{a_2}^t \frac{\omega_2(s)}{\alpha_2^2(s)}ds\right) \\
&\quad \frac{\omega_2(s)}{\alpha_2(s)} f ds\|_{L^2_{\omega_2}(H,(a_2,\infty))}^2 \\
&= \|\frac{1}{\alpha_2(t)} exp\left(-i\lambda\int_{a_2}^t \frac{\omega_2(\tau)}{\alpha_2^2(\tau)}d\tau + iA_2\int_{a_2}^t \frac{\omega_2(\tau)}{\alpha_2^2(\tau)}d\tau\right) \\
&\quad \int_t^\infty \frac{1}{\alpha_2(s)} exp\left(-2\lambda_i\int_{a_2}^s \frac{\omega_2(\tau)}{\alpha_2^2(\tau)}d\tau\right)\frac{\omega_2(s)}{\alpha_2(s)} f(s)ds\|_{L^2_{\omega_2}(H,(a_2,\infty))}^2 \\
&= \|\frac{1}{\alpha_2(t)} exp\left(\lambda_i\int_{a_2}^t \frac{\omega_2(\tau)}{\alpha_2^2(\tau)}d\tau\right)\int_t^\infty \frac{\omega_2(s)}{\alpha_2^2(s)} exp\left(-2\lambda_i\int_{a_2}^s \frac{\omega_2(\tau)}{\alpha_2^2(\tau)}d\tau\right) ds\|_{L^2_{\omega_2}(H,(a_2,\infty))}^2\|f\|_H^2 \\
&= \|\frac{1}{2\lambda_i\alpha_2(t)} exp\left(-\lambda_i\int_{a_2}^t \frac{\omega_2(s)}{\alpha_2^2(\tau)}d\tau\right)\|_{L^2_{\omega_2}(H,(a_2,\infty))}^2\|f\|_H^2 \\
&= \frac{1}{4\lambda_i^2}\int_{a_2}^\infty \frac{1}{\alpha_2^2(t)} exp\left(-2\lambda_i\int_{a_2}^t \frac{\omega_2(\tau)}{\alpha_2^2(\tau)}d\tau\right) dt\|f\|_H^2 \\
&= \frac{1}{8\lambda_i^3}\|f\|_H^2.
\end{aligned}$$

Using the above inequality we get

$$\|R_\lambda(L_W)f^*(\lambda;\ \cdot\ )\|_{\mathcal{H}} \geq \frac{\|f\|_H^2}{2\sqrt{2}\lambda_i\sqrt{\lambda_i}} = \frac{1}{2\lambda_i}\|f^*(\lambda;t)\|_{\mathcal{H}},$$

i.e. for $\lambda_i = Im\lambda > 0$ and $f \neq 0$ we can write

$$\frac{\|R_\lambda(L_W)f^*(\lambda;\ \cdot\ )\|_{\mathcal{H}}}{\|f^*(\lambda;t)\|_{\mathcal{H}}} \geq \frac{1}{2\lambda_i}$$

and it is also obvious that

$$\|R_\lambda(L_W)\| \geq \frac{\|R_\lambda(L_W)f^*(\lambda;\ \cdot\ )\|_{\mathcal{H}}}{\|f^*(\lambda;t)\|_{\mathcal{H}}},\ f \neq 0.$$

As a consequence, we get

$$\|R_\lambda(L_W)\| \geq \frac{1}{2\lambda_i} \text{ for } \lambda \in \mathbb{C},\ \lambda_i = Im\lambda > 0,$$

which shows that every $\lambda_r \in \mathbb{R}$ belongs to the continuous spectrum of the extension L_W. This completes the proof. □

Note: Some interesting models related to the theory of singular multipoint ordinary self-adjoint operators have been investigated in [8-10].

Note: When $\alpha_1 = \alpha_2 = 1$, similar results were obtained in [1].

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