

## $r$ -Submodules and $sr$ -Submodules

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**Abstract:** In this article, we introduce new classes of submodules called  $r$ -submodule and special  $r$ -submodule, which are two different generalizations of  $r$ -ideals. Let  $M$  be an  $R$ -module, where  $R$  is a commutative ring. We call a proper submodule  $N$  of  $M$  an  $r$ -submodule (resp., special  $r$ -submodule) if the condition  $am \in N$  with  $ann_M(a) = 0_M$  (resp.,  $ann_R(m) = 0$ ) implies that  $m \in N$  (resp.,  $a \in (N :_R M)$ ) for each  $a \in R$  and  $m \in M$ . We also give various results and examples concerning  $r$ -submodules and special  $r$ -submodules.

**Key words:**  $r$ -Ideal, prime ideal,  $r$ -submodule, special  $r$ -submodule, prime submodule

### 1. Introduction

Throughout, all rings will be commutative with  $1 \neq 0$  and all modules will be unitary. In particular,  $R$  will always denote such a ring. The concept of  $r$ -ideals was introduced and studied by Mohamadian in [9]. Recall from [9] that a proper ideal  $I$  of  $R$  is an  $r$ -ideal if  $ab \in I$  and  $ann(a) = \{r \in R : ra = 0\} = 0$ , and then  $b \in I$  for each  $a, b \in R$ . In this article, we give two different generalizations of this concept to modules by  $r$ -submodules and special  $r$ -submodules.

Let us give some definitions and notations we will need throughout this study. Let  $M$  be an  $R$ -module. Then a submodule  $N$  of  $M$  is proper whenever  $N \neq M$ . If  $N$  is a submodule of  $M$  and  $K$  is a nonempty subset of  $M$ , then the ideal  $\{r \in R : rK \subseteq N\}$  is denoted by  $(N :_R K)$ . In particular, we use  $Ann_R(M)$  instead of  $(0_M :_R M)$ . Furthermore, for each element  $m$  of  $M$ , we denote  $(0_M :_R \{m\})$  by  $ann_R(m)$ . Suppose that  $N$  is a submodule of  $M$  and  $S$  is a nonempty subset of  $R$ . Denote by  $(N :_M S)$  the set of all  $m \in M$  with  $Sm \subseteq N$ . In particular, we use  $ann_M(a)$  instead of  $(0_M :_M \{a\})$  for each  $a \in R$ . Also, the sets  $\{a \in R : ann_M(a) \neq 0_M\}$  and  $\{m \in M : ann_R(m) \neq 0\}$  will be designated by  $Z(M)$  and  $T(M)$ , respectively.

The prime submodule, which is an important subject of module theory, has been widely studied by various authors. See, for example, [2, 4, 8] and [3, 5, 7]. Recall that a prime submodule is a proper submodule  $N$  of  $M$  with the property that  $am \in N$  implies that  $a \in (N :_R M)$  or  $m \in N$  for each  $a \in R, m \in M$ . In that case,  $(N :_R M)$  is a prime ideal of  $R$ . In Section 2, we extend the concept of  $r$ -ideals to modules by  $r$ -submodules, and we investigate some properties of  $r$ -submodules with similar prime submodules. We define a proper submodule  $N$  of  $M$  as an  $r$ -submodule if whenever  $am \in N$  with  $ann_M(a) = 0_M$ , then  $m \in N$  for each  $a \in R$  and  $m \in M$ . Since there is no proper submodule of zero module, from now on we assume that  $R$ -module  $M$  is nonzero. Among many results in this paper, it is shown in Proposition 4 that

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a proper submodule  $N$  of  $M$  is an  $r$ -submodule if and only if  $N = (N :_M a)$  for every  $a \in R - Z(M)$ . In Theorem 1 we show that a proper submodule  $N$  of  $M$  is an  $r$ -submodule of  $M$  if and only if whenever  $I$  is an ideal of  $R$  such that  $I \cap (R - Z(M)) \neq \emptyset$  and  $L$  is a submodule of  $M$  with  $IL \subseteq N$ , then  $L \subseteq N$ . Also, it is proved in Proposition 7 that if  $N$  is a maximal  $r$ -submodule of  $M$ , then  $N$  is prime submodule. Finally, in Theorem 8, we characterize the  $r$ -submodules of Cartesian products of modules.

In Section 3, we introduce the special  $r$ -submodule, which is another generalization of  $r$ -ideals. We call a proper submodule  $N$  of  $M$  a special  $r$ -submodule (briefly  $sr$ -submodule) if for each  $a \in R$  and  $m \in M$ ,  $am \in N$  with  $ann_R(m) = 0$ , and then  $a \in (N :_R M)$ . In Example 11, it is shown that  $r$ -submodules and  $sr$ -submodules are different concepts, i.e. neither implies the other. In Theorem 13, we show that an  $R$ -module  $M$  is torsion-free if and only if  $M$  is faithful and the zero submodule is the only  $sr$ -submodule of  $M$ . We characterize, in Theorem 14, all  $R$ -modules in which every proper submodule is an  $sr$ -submodule. Finally we characterize, in Theorem 15, the  $sr$ -submodules of Cartesian products of modules.

## 2. $r$ -Submodules

**Definition 1** Let  $M$  be an  $R$ -module. A proper submodule  $N$  of  $M$  is said to be an  $r$ -submodule if  $am \in N$  with  $ann_M(a) = 0_M$  implies that  $m \in N$  for each  $a \in R, m \in M$ .

Note that a proper submodule  $N$  of  $M$  being an  $r$ -submodule means simply that  $Z(M/N) \subseteq Z(M)$  and also the  $r$ -submodules of  $R$ -module  $R$  are precisely the  $r$ -ideals of  $R$ . Now we give some examples of  $r$ -submodules.

**Example 1** Consider the  $\mathbb{Z}$ -module  $\mathbb{Z}_n$  for  $n \geq 2$ . Let  $\langle \bar{x} \rangle$  be a proper submodule of  $\mathbb{Z}_n$ . Then  $\gcd(x, n) = d > 1$ . This implies that  $\langle \bar{x} \rangle = \langle \bar{d} \rangle$ , and also note that  $\mathbb{Z}_n / \langle \bar{x} \rangle$  is isomorphic to  $\mathbb{Z}$ -module  $\mathbb{Z}_d$ . Since  $Z(\mathbb{Z}_d) \subseteq Z(\mathbb{Z}_n)$ , it follows that  $\langle \bar{x} \rangle$  is an  $r$ -submodule of  $\mathbb{Z}_n$ .

**Example 2** Consider  $\mathbb{Z}$ -module  $\mathbb{Q}/\mathbb{Z}$ . We know that  $E(p) = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = \frac{r}{p^t} + \mathbb{Z} \text{ for } t \in \mathbb{N} \cup \{0\} \text{ and } r \in \mathbb{Z}\}$  is a submodule of  $\mathbb{Q}/\mathbb{Z}$ , where  $p$  is a prime number. Then any proper submodule of  $E(p)$  is of the form  $G_{t_0} = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = \frac{r}{p^{t_0}} + \mathbb{Z} \text{ for some } r \in \mathbb{Z}\}$  for some  $t_0 \in \mathbb{N} \cup \{0\}$  [12].  $E(p)$  does not have any prime submodule. However, we show that every proper submodule of  $E(p)$  is an  $r$ -submodule. First, note that  $ann_{E(p)}(m) = 0_{E(p)}$  if and only if  $\gcd(p, m) = 1$  for  $m \in \mathbb{Z}$ . Let  $m \in \mathbb{Z}$ ,  $\frac{r}{p^t} + \mathbb{Z} \in E(p)$  such that  $m \left( \frac{r}{p^t} + \mathbb{Z} \right) = \frac{mr}{p^t} + \mathbb{Z} \in G_{t_0}$  and  $\gcd(p, m) = 1$ . If  $t \leq t_0$ , then we have  $\frac{r}{p^t} + \mathbb{Z} \in G_{t_0}$ . Now, assume that  $t > t_0$ . Since  $\frac{mr}{p^t} + \mathbb{Z} \in G_{t_0}$ , we have  $\frac{mr}{p^t} + \mathbb{Z} = \frac{k}{p^{t_0}} + \mathbb{Z}$  for some  $k \in \mathbb{Z}$ , and so  $\frac{mr}{p^t} - \frac{k}{p^{t_0}} \in \mathbb{Z}$ . Then we have  $mr \equiv kp^{t-t_0} \pmod{p^t}$ . Since  $\gcd(m, p^t) = 1$ , we get  $r \equiv k'kp^{t-t_0} \pmod{p^t}$  for some  $k' \in \mathbb{Z}$ , and so  $\frac{r}{p^t} + \mathbb{Z} = \frac{k'k}{p^{t_0}} + \mathbb{Z} \in G_{t_0}$ . Hence,  $G_{t_0}$  is an  $r$ -submodule of  $E(p)$ .

**Lemma 1** If  $N$  is an  $r$ -submodule of  $M$ , then  $(N :_R M) \subseteq Z(M)$ .

**Proof** It follows from the fact that  $(N :_R M) = Ann(M/N) \subseteq Z(M/N) \subseteq Z(M)$ . □

The converse of Lemma 1 is not always valid, i.e. if  $N$  is a submodule of  $M$  with  $(N :_R M) \subseteq Z(M)$ , then  $N$  need not be an  $r$ -submodule of  $M$ . We give a counter example in the following.

**Example 3** Consider the  $\mathbb{Z}$ -module  $\mathbb{Z} \times \mathbb{Z}$  and the submodule  $N = 2\mathbb{Z} \times 0$  of  $M = \mathbb{Z} \times \mathbb{Z}$ . Note that  $(N :_{\mathbb{Z}} M) = \langle 0 \rangle \subseteq Z(M)$  and also  $M/N$  is isomorphic to  $\mathbb{Z}$ -module  $\mathbb{Z}_2 \times \mathbb{Z}$ . Since  $2 \in Z(\mathbb{Z}_2 \times \mathbb{Z}) - Z(M)$ , we have  $Z(\mathbb{Z}_2 \times \mathbb{Z}) \not\subseteq Z(M)$  and thus  $N$  is not an  $r$ -submodule of  $M$ .

The following examples show that the concepts of prime submodule and  $r$ -submodule are different.

**Example 4** (i) Consider the  $\mathbb{Z}$ -module  $\mathbb{Z}$ . Of course,  $3\mathbb{Z}$  is a prime submodule of  $\mathbb{Z}$ , since  $(3\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z}) = 3\mathbb{Z} \not\subseteq Z(\mathbb{Z})$ , it follows that  $3\mathbb{Z}$  is not an  $r$ -submodule of  $\mathbb{Z}$ .

(ii) Consider the  $\mathbb{Z}$ -module  $\mathbb{Z}_{18}$ . By Example 1, we know that  $\langle \bar{9} \rangle$  is an  $r$ -submodule of  $\mathbb{Z}_{18}$  but it is not a prime submodule. Since  $3\bar{3} = \bar{9} \in \langle \bar{9} \rangle$  but  $3 \notin (\langle \bar{9} \rangle :_{\mathbb{Z}_{18}}) = 9\mathbb{Z}$  and  $\bar{3} \notin \langle \bar{9} \rangle$ .

Note that in a vector space, any proper subspace is a prime submodule. In the following proposition, we show it is true for  $r$ -submodules and so in a vector space the prime submodule coincides with the  $r$ -submodule.

**Proposition 1** Let  $V$  be a vector space over a field  $F$ . Then every proper subspace  $W$  of  $V$  is an  $r$ -submodule.

**Proof** Follows from  $Z(V/W) = 0$ . □

**Proposition 2** For a prime submodule  $N$  of  $M$ ,  $N$  is an  $r$ -submodule if and only if  $(N :_R M) \subseteq Z(M)$ .

**Proof** If  $N$  is prime submodule, then  $Z(M/N) = (N :_R M)$  so that  $N$  is an  $r$ -submodule iff  $(N :_R M) \subseteq Z(M)$ . □

**Proposition 3** Let  $M$  be an  $R$ -module. Then the following hold:

(i) The zero submodule is an  $r$ -submodule.

(ii) The intersection of an arbitrary nonempty set of  $r$ -submodules is an  $r$ -submodule.

**Proof** (i) It is clear that  $Z(M/0_M) = Z(M)$  and so the zero submodule is an  $r$ -submodule.

(ii) Let  $N_i$  be an  $r$ -submodule of  $M$  for every  $i \in \Delta$ . Suppose that  $am \in \bigcap_{i \in \Delta} N_i$  with  $ann_M(a) = 0_M$  for  $a \in R, m \in M$ . Then we have  $am \in N_i$  for every  $i \in \Delta$ . Since  $N_i$  is an  $r$ -submodule, we conclude that  $m \in N_i$  for every  $i \in \Delta$ , and thus  $m \in \bigcap_{i \in \Delta} N_i$ . Hence,  $\bigcap_{i \in \Delta} N_i$  is an  $r$ -submodule. □

Note that the sum of two  $r$ -submodules need not be an  $r$ -submodule. See the following example.

**Example 5** Consider the  $\mathbb{Z}$ -module  $\mathbb{Z}_{10}$ . Then  $\langle \bar{2} \rangle$  and  $\langle \bar{5} \rangle$  are  $r$ -submodules but  $\langle \bar{2} \rangle + \langle \bar{5} \rangle = \mathbb{Z}_{10}$  is not an  $r$ -submodule of  $\mathbb{Z}_{10}$ .

It is well known if  $N$  is prime submodule of  $M$ , then  $(N :_R M)$  is prime ideal of  $R$ . However, the following example shows that this is not always correct for  $r$ -submodules.

**Example 6** Consider the  $\mathbb{Z}$ -module  $\mathbb{Z}_4$ .  $\langle \bar{2} \rangle$  is an  $r$ -submodule but  $(\langle \bar{2} \rangle :_{\mathbb{Z}} \mathbb{Z}_4) = 2\mathbb{Z}$  is not an  $r$ -ideal of  $\mathbb{Z}$ , since a domain has no nonzero  $r$ -ideals.

Recall that a nonempty subset  $S$  of  $R$  is multiplicatively closed precisely when  $ab \in S$  for all  $a, b \in S$ . For instance,  $S = R - Z(M)$  is a multiplicatively closed subset of  $R$ . Suppose that  $S$  is a multiplicatively closed subset of  $R$  and  $M$  is an  $R$ -module. Then we denote the module of fraction at  $S$  by  $S^{-1}M$ . Note that  $S^{-1}M$  is both an  $S^{-1}R$ -module and an  $R$ -module. Also, for every submodule  $N$  of  $M$ ,  $S^{-1}N$  is an  $S^{-1}R$ -submodule of  $S^{-1}M$ . Let  $M$  be an  $R$ -module. Consider  $S^{-1}M$  as an  $R$ -module. The natural  $R$ -homomorphism is defined as follows:

$$\pi : M \rightarrow S^{-1}M, \text{ for all } m \in M, \pi(m) = \frac{m}{1}.$$

**Proposition 4** Let  $N$  be a proper submodule of  $M$ . Then the following are equivalent:

- (i)  $N$  is an  $r$ -submodule of  $M$ .
- (ii)  $aM \cap N = aN$  for every  $a \in R - Z(M)$ .
- (iii)  $(N :_M a) = N$  for every  $a \in R - Z(M)$ .
- (iv)  $N = \pi^{-1}(L)$ , where  $S = R - Z(M)$  and  $L$  is an  $S^{-1}R$ -submodule of  $S^{-1}M$ .

**Proof** (i)  $\Rightarrow$  (ii): Suppose that  $N$  is an  $r$ -submodule. For every  $a \in R$ , the inclusion  $aN \subseteq aM \cap N$  always holds. Let  $a \in R$  with  $ann_M(a) = 0_M$  and  $x \in aM \cap N$ . Then we get  $x = am \in N$  for some  $m \in M$ . Since  $N$  is an  $r$ -submodule,  $m \in N$  and thus  $x = am \in aN$ . Hence, we get  $aM \cap N = aN$ .

(ii)  $\Rightarrow$  (iii): It is well known that  $N \subseteq (N :_M a)$  for every  $a \in R$ . Let  $a \in R$  such that  $ann_M(a) = 0_M$  and  $m \in (N :_M a)$ . Then we have  $am \in N$ , and so  $am \in aM \cap N = aN$  by (ii). Thus, we have  $am = an$  for some  $n \in N$ . Since  $ann_M(a) = 0_M$ , we conclude that  $m = n \in N$ . Hence, we have  $(N :_M a) \subseteq N$ .

(iii)  $\Rightarrow$  (iv): Since  $N \subseteq \pi^{-1}(S^{-1}N)$ , it is sufficient to show that  $\pi^{-1}(S^{-1}N) \subseteq N$ . Let  $m \in \pi^{-1}(S^{-1}N)$ . Then we have  $\pi(m) = \frac{m}{1} \in S^{-1}N$  and so  $am \in N$  for some  $a \in S$ . Thus, by (iii), we conclude that  $m \in (N :_M a) = N$ .

(iv)  $\Rightarrow$  (i): Suppose that  $N = \pi^{-1}(L)$ , where  $S = R - Z(M)$  and  $L$  is an  $S^{-1}R$ -submodule of  $S^{-1}M$ . Let  $am \in N$  and  $ann_M(a) = 0_M$ . Then we have  $\pi(am) = \frac{am}{1} \in L$ . Since  $a \in S$  and  $L$  is an  $S^{-1}R$ -submodule, we conclude that  $\frac{1}{a} \frac{am}{1} = \frac{m}{1} = \pi(m) \in L$  and so  $m \in \pi^{-1}(L) = N$ , as needed.  $\square$

In [11], Ribenboim defined the pure submodule as a proper submodule  $N$  of  $M$  if  $aM \cap N = aN$  for every  $a \in R$ . By Proposition 4, every pure submodule is also an  $r$ -submodule. However, in the following, we show that the converse is not necessarily correct.

**Example 7** Consider the  $\mathbb{Z}$ -module  $\mathbb{Z}_{16}$  and the submodule  $N = \langle \bar{2} \rangle$ . Then  $N$  is an  $r$ -submodule of  $\mathbb{Z}_{16}$ , but  $N$  is not a pure submodule of  $\mathbb{Z}_{16}$ , because  $2N = \langle \bar{4} \rangle \subsetneq 2\mathbb{Z}_{16} \cap N = \langle \bar{2} \rangle$ .

**Proposition 5** Suppose that  $N$  is an  $r$ -submodule of  $M$  and  $S$  is a nonempty subset of  $R$  with  $S \not\subseteq (N :_R M)$ . Then  $(N :_M S)$  is an  $r$ -submodule of  $M$ . In particular,  $(0_M :_M S)$  is always an  $r$ -submodule if  $S \not\subseteq Ann_R(M)$ .

**Proof** Let  $am \in (N :_M S)$  with  $ann_M(a) = 0_M$  for  $a \in R, m \in M$ . Then we have  $asm \in N$  for every  $s \in S$ . Since  $N$  is an  $r$ -submodule, we get  $sm \in N$  for every  $s \in S$  and this yields  $m \in (N :_M S)$ , as is needed. The rest follows easily.  $\square$

**Corollary 1** *If  $a \notin Ann_R(M)$ , then  $ann_M(a)$  is an  $r$ -submodule of  $M$ .*

**Proposition 6** *For any  $R$ -module  $M$ , the following hold if the zero submodule is the only  $r$ -submodule:*

- (i) *The zero submodule is a prime submodule of  $M$ .*
- (ii)  *$Ann_R(M)$  is a prime ideal of  $R$ .*

**Proof** (i) Let  $am = 0_M$  and  $a \notin Ann_R(M)$ , where  $a \in R, m \in M$ . Then by previous corollary,  $ann_M(a)$  is an  $r$ -submodule and thus  $ann_M(a) = 0_M$ . Hence, we have  $m = 0_M$ , as needed.

(ii) It follows from (i).  $\square$

Remember that a proper submodule  $N$  of  $M$  is prime if and only if for every ideal  $I$  of  $R$  and submodule  $L$  of  $M$  with  $IL \subseteq N$ , then either  $I \subseteq (N :_R M)$  or  $L \subseteq N$ . Now we present a similar result for  $r$ -submodules as follows.

**Theorem 1** *For a proper submodule  $N$  of  $M$ , the following hold:*

- (i)  *$N$  is an  $r$ -submodule of  $M$  if and only if whenever  $I$  is an ideal of  $R$  such that  $I \cap (R - Z(M)) \neq \emptyset$  and  $L$  is a submodule of  $M$  with  $IL \subseteq N$ , then  $L \subseteq N$ .*
- (ii) *If  $(N :_R M) \subseteq Z(M)$  and  $N$  is not an  $r$ -submodule of  $M$ , then there exist an ideal  $I$  of  $R$  and a submodule  $L$  of  $M$  such that  $I \cap (R - Z(M)) \neq \emptyset, N \not\subseteq L, (N :_R M) \not\subseteq I$ , and  $IL \subseteq N$ .*

**Proof** (i) Suppose that  $N$  is an  $r$ -submodule and  $IL \subseteq N$  for some ideal  $I$  of  $R$  with  $I \cap (R - Z(M)) \neq \emptyset$  and submodule  $L$  of  $M$ . Then there exist  $a \in I$  such that  $ann_M(a) = 0_M$ . Since  $al \in N$  for every  $l \in L$  and  $N$  is an  $r$ -submodule, we conclude that  $l \in N$ , and thus  $L \subseteq N$ . For the converse, let  $am \in N$  and  $ann_M(a) = 0_M$  for  $a \in R, m \in M$ . We take  $I = aR$  and  $L = Rm$ . Note that  $I \cap (R - Z(M)) \neq \emptyset$  and  $IL \subseteq N$ . Then by assumption we have  $Rm \subseteq N$ , and so  $m \in N$ . Hence,  $N$  is an  $r$ -submodule.

(ii) Since  $N$  is not an  $r$ -submodule, there exist  $a \in R, m \in M$  such that  $am \in N$  with  $ann_M(a) = 0_M$  and  $m \notin N$ . We take  $I = (N :_R m)$ . Note that  $a \in I$  and  $a \notin (N :_R M)$  since  $ann_M(a) = 0_M$ . Thus,  $(N :_R M) \not\subseteq I$ . Now we take  $L = (N :_M I)$ . Since  $m \notin N$  and  $m \in L, N \not\subseteq L$ . Hence, we get  $N \not\subseteq L, (N :_R M) \not\subseteq I$  and  $IL = I(N :_M I) \subseteq N$ .  $\square$

**Theorem 2** *Suppose that  $K_1, K_2, L$  are submodules of  $M$  and  $I$  is an ideal of  $R$  with  $I \cap (R - Z(M)) \neq \emptyset$ . Then the following hold:*

- (i) *If  $K_1, K_2$  are  $r$ -submodules of  $M$  with  $IK_1 = IK_2$ , then  $K_1 = K_2$ .*
- (ii) *If  $IL$  is an  $r$ -submodule, then  $IL = L$ . In particular,  $L$  is an  $r$ -submodule.*

**Proof** (i) Since  $IK_1 \subseteq K_2$  and  $K_2$  is an  $r$ -submodule, we have  $K_1 \subseteq K_2$  by Theorem 1(i). Similarly, we have  $K_2 \subseteq K_1$ , and so  $K_1 = K_2$ .

(ii) Since  $IL$  is an  $r$ -submodule and  $IL \subseteq IL$ , we have  $L \subseteq IL \subseteq L$  by Theorem 1(i), and so  $IL = L$ .

$\square$

**Theorem 3** Suppose that  $N_1, N_2, \dots, N_n$  are prime submodules of  $M$  such that  $(N_i :_R M)$ s are not comparable.

If  $\bigcap_{i=1}^n N_i$  is an  $r$ -submodule, then  $N_i$  is an  $r$ -submodule for each  $i \in \{1, 2, \dots, n\}$ .

**Proof** Let  $am \in N_k$  with  $ann_M(a) = 0_M$  for  $a \in R, m \in M$ . Since  $(N_i :_R M)$ s are not comparable, we have  $r \in \left( \bigcap_{\substack{i=1 \\ i \neq k}}^n (N_i :_R M) \right) - (N_k :_R M)$  for some  $r \in R$ . Then we have  $ram \in \bigcap_{i=1}^n N_i$ . Since  $\bigcap_{i=1}^n N_i$  is an  $r$ -submodule, we conclude that  $rm \in \bigcap_{i=1}^n N_i \subseteq N_k$ . Thus, we have  $m \in N_k$ , because  $N_k$  is a prime submodule and  $r \notin (N_k :_R M)$ . Hence,  $N_k$  is an  $r$ -submodule.  $\square$

**Proposition 7** If  $N$  is a maximal  $r$ -submodule of  $M$ , then  $N$  is prime submodule.

**Proof** Let  $am \in N$  and  $m \notin N$ ; we show that  $a \in (N :_R M)$ . Assume that  $a \notin (N :_R M)$ . Then  $(N :_M a)$  is an  $r$ -submodule by Proposition 5. Since  $N$  is a maximal  $r$ -submodule, we conclude that  $m \in (N :_M a) = N$ , a contradiction. Thus, we have  $a \in (N :_R M)$ , as needed.  $\square$

Let recall the following well-known theorem of the prime avoidance lemma: suppose that  $N \subseteq \bigcup_{j=1}^n N_j$  and at most two of  $N_j$  are not prime submodules. Then  $N \subseteq N_i$  for some  $1 \leq i \leq n$  if the condition  $(N_i :_R M) \not\subseteq (N_j :_R M)$  holds for every  $i \neq j$  [4, 7]. Now we present a result with a similar prime avoidance lemma for  $r$ -submodules.

**Proposition 8** Let  $N \subseteq \bigcup_{j=1}^n N_j$  for submodules  $N, N_1, N_2, \dots, N_n$  of  $M$ . Suppose that  $N_k$  is an  $r$ -submodule and  $(N_j :_R M) \cap (R - Z(M)) \neq \emptyset$  for every  $j \neq k$ . If  $N \not\subseteq \bigcup_{j \neq k} N_j$ , then  $N \subseteq N_k$ .

**Proof** We may assume that  $k = 1$ . Since  $N \not\subseteq \bigcup_{j=2}^n N_j$ , there exists  $m \in N$  such that  $m \notin \bigcup_{j=2}^n N_j$ , namely  $m \in N_1$ . Let  $n \in N \cap N_2 \cap N_3 \cap \dots \cap N_n$ . Then it is clear that  $m + n \in N - \bigcup_{j=2}^n N_j$ , and thus  $m + n \in N_1$ . This gives  $n \in N_1$ , and so  $N \cap N_2 \cap N_3 \cap \dots \cap N_n \subseteq N_1$ . Since  $(N_j :_R M) \cap (R - Z(M)) \neq \emptyset$ , there exists  $a_j \in (N_j :_R M)$  such that  $ann_M(a_j) = 0_M$  for  $j = 2, 3, \dots, n$ . Then note that  $ann_M(a_2 a_3 \dots a_n) = 0_M$ . Now we take  $I = \bigcap_{j=2}^n (N_j :_R M)$ . Then we have  $a_2 a_3 \dots a_n \in I \cap (R - Z(M))$ . Since  $IN \subseteq N \cap N_2 \cap N_3 \cap \dots \cap N_n \subseteq N_1$  and  $I \cap (R - Z(M)) \neq \emptyset$ , by Theorem 1, we get  $N \subseteq N_1$ .  $\square$

**Definition 2** A nonempty subset  $S$  of  $R$  is said to be an  $r$ -multiplicatively closed subset precisely when  $R - Z(M) \subseteq S$  and  $ab \in S$ , for all  $a \in R - Z(M)$  and  $b \in S$ .

**Example 8** For every  $r$ -submodule  $N$  of  $M$ ,  $R - (N :_R M)$  is an  $r$ -multiplicatively closed subset of  $R$ . We know that if  $N$  is an  $r$ -submodule, then  $(N :_R M) \subseteq Z(M)$  and so  $R - Z(M) \subseteq R - (N :_R M)$ . Let  $a \in R - Z(M)$  and  $b \in R - (N :_R M)$ . Suppose that  $ab \in (N :_R M)$ . Then we have  $abm \in N$  for every  $m \in M$  and  $\text{ann}_M(a) = 0_M$ . Since  $N$  is an  $r$ -submodule, it follows that  $bm \in N$  and thus  $b \in (N :_R M)$ , a contradiction. Hence,  $R - (N :_R M)$  is an  $r$ -multiplicatively closed subset.

**Definition 3** Let  $S$  be an  $r$ -multiplicatively closed subset of  $R$  and  $S^*$  be a nonempty subset of  $M$ . Then  $S^*$  is called an  $S$ -closed subset of  $M$  if  $am \in S^*$  for each  $a \in S$  and  $m \in S^*$ .

**Theorem 4** Let  $S^*$  be an  $S$ -closed subset of  $M$ , where  $S$  is an  $r$ -multiplicatively closed subset of  $R$ . Suppose that  $N$  is a submodule of  $M$  with  $N \cap S^* = \emptyset$ . Then there exists an  $r$ -submodule  $L$  of  $M$  with  $N \subseteq L$  and  $L \cap S^* = \emptyset$ .

**Proof** Let  $\Omega = \{L' : L' \text{ be a submodule of } M \text{ with } N \subseteq L' \text{ and } L' \cap S^* = \emptyset\}$ . Since  $N \in \Omega$ , we have  $\Omega \neq \emptyset$ . By Zorn's lemma,  $\Omega$  has a maximal element  $L$  with  $N \subseteq L$  and  $L \cap S^* = \emptyset$ . Assume that  $L$  is not an  $r$ -submodule of  $M$ . Then there exist  $a \in R, m \in M$  such that  $am \in L$ ,  $\text{ann}_M(a) = 0_M$  and  $m \notin L$ . Since  $m \notin L$  and  $m \in (L :_M a)$ ,  $L \subsetneq (L :_M a)$ . By the maximality of  $L$ , we get  $m' \in (L :_M a) \cap S^*$ . Since  $a \in S$ , we get the result that  $am' \in L \cap S^*$ , a contradiction. Hence,  $L$  is an  $r$ -submodule.  $\square$

**Theorem 5** Let  $M$  be an  $R$ -module. Then every proper submodule of  $M$  is an  $r$ -submodule if and only if for every submodule  $N$  of  $M$ ,  $aN = N$  for every  $a \in R - Z(M)$ .

**Proof** Suppose that every proper submodule of  $M$  is an  $r$ -submodule. Let  $N$  be a submodule and  $a \in R - Z(M)$ . Assume that  $N \neq M$ . If  $aM \neq M$ , then  $aM$  is an  $r$ -submodule of  $M$ . Since  $am \in aM$  for every  $m \in M$  and  $\text{ann}_M(a) = 0_M$ , we conclude that  $m \in aM$ , and thus  $aM = M$ , a contradiction. Thus, we have  $aM = M$ . Now assume that  $N$  is a proper submodule of  $M$ . Then  $aN \subseteq N \neq M$  and so  $aN$  is an  $r$ -submodule of  $M$ . Since  $an \in aN$  for every  $n \in N$ , similarly we get the result that  $aN = N$ . Conversely, suppose that  $aN = N$  for every submodule  $N$  of  $M$  and every  $a \in R - Z(M)$ . Let  $N$  be a proper submodule of  $M$  and  $a \in R - Z(M)$ . Then we have  $aM \cap N = aN$ , and so by Proposition 4,  $N$  is an  $r$ -submodule of  $M$ .  $\square$

Let  $M$  be an  $R$ -module. Recall that the idealization of  $M$  in  $R$ , which is denoted by  $R(+M) = \{(a, m) : a \in R, m \in M\}$ , is a commutative ring with component-wise addition and multiplication  $(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1)$  [10]. In [1,6], the zero divisor set of  $R(+M)$  was characterized as follows:

$$Z(R(+M)) = \{(a, m) : a \in Z(R) \cup Z(M), m \in M\},$$

where  $Z(R) = \{a \in R : \text{ann}(a) \neq 0\}$ .

**Corollary 2** For every  $a \in R$  and  $m \in M$ ,  $\text{ann}_{R(+M)}(a, m) = 0$  if and only if  $\text{ann}(a) = 0$  and  $\text{ann}_M(a) = 0_M$ .

Suppose that  $N$  is a submodule of  $M$  and  $J$  is an ideal of  $R$ . Then it is clear that  $J(+N)$  is an ideal of  $R(+M)$  if and only if  $JM \subseteq N$ . In that case  $J(+N)$  is called a homogeneous ideal.

**Proposition 9** Suppose that  $J$  is an  $r$ -ideal of  $R$ . Then  $J(+M)$  is an  $r$ -ideal of  $R(+M)$ .

**Proof** Let  $J$  be an  $r$ -ideal of  $R$ . Suppose that  $(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1) \in J(+)M$  and  $\text{ann}_{R(+)M}(a_1, m_1) = 0$ . Since  $\text{ann}_{R(+)M}(a_1, m_1) = 0$ , we have  $\text{ann}(a_1) = 0$ . Then we get the result that  $a_2 \in J$ , because  $J$  is an  $r$ -ideal and  $a_1a_2 \in J$ . Thus, we have  $(a_2, m_2) \in J(+)M$ . Consequently,  $J(+)M$  is an  $r$ -ideal.  $\square$

The converse of the previous proposition is not always true. We have a counterexample as follows.

**Example 9** Consider the  $\mathbb{Z}(+)\mathbb{Z}_2$  and the ideal  $2\mathbb{Z}(+)\mathbb{Z}_2$  of  $\mathbb{Z}(+)\mathbb{Z}_2$ . We know that  $2\mathbb{Z}$  is not an  $r$ -ideal of  $\mathbb{Z}$  but  $2\mathbb{Z}(+)\mathbb{Z}_2$  is an  $r$ -ideal of  $\mathbb{Z}(+)\mathbb{Z}_2$ .

**Theorem 6** Suppose that  $J$  is an  $r$ -ideal of  $R$  and  $N$  is an  $r$ -submodule of  $M$  with  $JM \subseteq N$ . Then  $J(+)N$  is an  $r$ -ideal of  $R(+)M$ .

**Proof** Let  $(a_1, m_1)(a_2, m_2) \in J(+)N$  with  $\text{ann}_{R(+)M}(a_1, m_1) = 0$ . Then we have  $\text{ann}(a_1) = 0$  and  $\text{ann}_M(a_1) = 0_M$ . Since  $J$  is an  $r$ -ideal and  $a_1a_2 \in J$ , we have  $a_2 \in J$ . Thus, we have  $a_2m_1 \in N$  and so  $a_1m_2 \in N$ . As  $N$  is an  $r$ -submodule, it follows that  $m_2 \in N$  and so  $(a_2, m_2) \in J(+)N$ . Hence,  $J(+)N$  is an  $r$ -ideal.  $\square$

Example 9 also serves as a counterexample of the previous theorem, but we prove that the converse of Theorem 6 is valid when  $Z(R) = Z(M)$  as follows.

**Theorem 7** Let  $M$  be an  $R$ -module and  $Z(R) = Z(M)$ . If  $J(+)N$  is an  $r$ -ideal of  $R(+)M$  with  $N \neq M$ , then  $J$  is an  $r$ -ideal of  $R$  and  $N$  is an  $r$ -submodule of  $M$ .

**Proof** Suppose that  $J(+)N$  is an  $r$ -ideal. Since  $Z(R) = Z(M)$ ,  $\text{ann}_{R(+)M}(a_1, m_1) = 0$  if and only if  $\text{ann}(a_1) = 0$ . Let  $a, b \in R$  with  $ab \in J$  and  $\text{ann}(a) = 0$ . Then we have  $\text{ann}_{R(+)M}(a, 0_M) = 0$  and so  $(a, 0_M)(b, 0_M) = (ab, 0_M) \in J(+)N$ . Since  $J(+)N$  is an  $r$ -ideal, we get the result that  $(b, 0_M) \in J(+)N$  and thus  $b \in J$ . Hence,  $J$  is an  $r$ -ideal of  $R$ . Suppose that  $am \in N$  with  $\text{ann}_M(a) = 0_M$  for  $a \in R, m \in M$ . Then  $\text{ann}_{R(+)M}(a, 0_M) = 0$ , so we get  $(a, 0_M)(0, m) = (0, am) \in J(+)N$ . As  $J(+)N$  is an  $r$ -ideal, we conclude that  $(0, m) \in J(+)N$  and so  $m \in N$ . Hence,  $N$  is an  $r$ -submodule.  $\square$

Let  $M_1$  be an  $R_1$ -module and  $M_2$  an  $R_2$ -module, where  $R_1$  and  $R_2$  are commutative rings with identity. Suppose that  $R = R_1 \times R_2$  and  $M = M_1 \times M_2$ . Then  $M$  becomes an  $R$ -module with coordinate-wise addition and the scalar multiplication  $(a_1, a_2)(m_1, m_2) = (a_1m_1, a_2m_2)$  for every  $a_1 \in R_1, a_2 \in R_2; m_1 \in M_1$  and  $m_2 \in M_2$ . Also, every submodule  $N$  of  $M$  has the form  $N = N_1 \times N_2$ , where  $N_1$  is a submodule of  $M_1$  and  $N_2$  is a submodule of  $M_2$ . The following theorem characterizes the  $r$ -submodule of Cartesian product of modules.

**Lemma 2** Let  $R = R_1 \times R_2$  and  $M = M_1 \times M_2$ , where  $M_1$  is an  $R_1$ -module and  $M_2$  is an  $R_2$ -module. Suppose that  $N = N_1 \times N_2$  is a submodule of  $M$ . Then the following are equivalent:

- (i)  $N$  is an  $r$ -submodule of  $M$ .
- (ii)  $N_1 = M_1$  and  $N_2$  is an  $r$ -submodule of  $M_2$  or  $N_1$  is an  $r$ -submodule of  $M_1$  and  $N_2 = M_2$  or  $N_1, N_2$  are  $r$ -submodules of  $M_1$  and  $M_2$ , respectively.

**Proof** (i)  $\Rightarrow$  (ii) : First note that  $M/N$  is isomorphic to  $(M_1/N_1) \times (M_2/N_2)$  and  $Z(M/N) = (Z(M_1/N_1) \times Z(M_2/N_2)) \cup (R_1 \times Z(M_2/N_2))$ . Suppose that  $N$  is an  $r$ -submodule of  $M$  and assume that  $N_1 = M_1$ . Since  $N$  is

a proper submodule of  $M$ ,  $N_2 \neq M_2$ . Then  $Z(M/N) = R_1 \times Z(M_2/N_2) \subseteq Z(M) = (Z(M_1) \times R_2) \cup (R_1 \times Z(M_2))$  and so  $Z(M_2/N_2) \subseteq Z(M_2)$ . This implies that  $N_2$  is an  $r$ -submodule of  $M_2$ . In other cases, a similar argument shows that (i) implies (ii).

(ii)  $\Rightarrow$  (i) : Conversely, suppose that (ii) holds. Assume that  $N_1, N_2$  are  $r$ -submodules of  $M_1$  and  $M_2$ , respectively. Then  $Z(M_1/N_1) \subseteq Z(M_1)$  and  $Z(M_2/N_2) \subseteq Z(M_2)$ . This implies that  $Z(M/N) = (Z(M_1/N_1) \times R_2) \cup (R_1 \times Z(M_2/N_2)) \subseteq (Z(M_1) \times R_2) \cup (R_1 \times Z(M_2)) = Z(M)$ , i.e.  $N$  is an  $r$ -submodule of  $M$ . In other cases, one can similarly prove that  $N$  is an  $r$ -submodule.  $\square$

**Theorem 8** Suppose that  $R = R_1 \times R_2 \times \dots \times R_n$  and  $M = M_1 \times M_2 \times \dots \times M_n$ , where  $M_i$  is an  $R_i$ -module for  $n \geq 1$  and  $1 \leq i \leq n$ . Let  $N = N_1 \times N_2 \times \dots \times N_n$  be a submodule of  $M$ . Then the following are equivalent:

(i)  $N$  is an  $r$ -submodule of  $M$ .

(ii)  $N_i = M_i$  for  $i \in \{t_1, t_2, \dots, t_k : k < n\}$  and  $N_i$  is an  $r$ -submodule of  $M_i$  for  $i \in \{1, 2, \dots, n\} \setminus \{t_1, t_2, \dots, t_k\}$ .

**Proof** To prove the claim, we use induction on  $n$ . If  $n = 1$ , then it is clear that (i)  $\Leftrightarrow$  (ii). If  $n = 2$ , by Lemma 2, (i) and (ii) are equal. Assume that  $n \geq 3$  and the claim is valid when  $K = M_1 \times M_2 \times \dots \times M_{n-1}$ . We prove that the claim is true when  $M = K \times M_n$ . Then by Lemma 2 we get the result that  $N$  is an  $r$ -submodule if and only if  $N = K \times N_n$  for some  $r$ -submodule  $N_n$  of  $M_n$  or  $N = L \times M_n$  for some  $r$ -submodule  $L$  of  $K$  or  $N = L \times N_n$  for some  $r$ -submodule  $L$  of  $K$  and some  $r$ -submodule  $N_n$  of  $M_n$ . By induction hypothesis, the result is valid in three cases.  $\square$

### 3. Special $r$ -submodules

In this section, we give another type of generalization of  $r$ -ideals to modules.

**Definition 4** Let  $M$  be an  $R$ -module. Then a submodule  $N$  of  $M$  is said to be a special  $r$ -submodule (briefly  $sr$ -submodule) if  $N \neq M$ , for each  $a \in R, m \in M$  with  $am \in N$  and  $ann_R(m) = 0$ , then  $a \in (N :_R M)$ .

If we consider  $R$ -module  $R$ , the  $sr$ -submodules and  $r$ -submodules coincide. Now we give some examples of  $sr$ -submodules in the following.

**Example 10** By Example 1, we know that all proper submodules of  $\mathbb{Z}$ -module  $\mathbb{Z}_n$  are  $r$ -submodules. One can easily see that all proper submodules of  $\mathbb{Z}_n$  are also  $sr$ -submodules. Now consider the  $\mathbb{Z}$ -module  $E(p)$ . By Example 2, all proper submodules of  $E(p)$  are  $r$ -submodules. Since  $ann_{\mathbb{Z}}\left(\frac{r}{p^t} + \mathbb{Z}\right) \neq 0$  for each  $\frac{r}{p^t} + \mathbb{Z} \in E(p)$ , we conclude that all proper submodules of  $E(p)$  are also  $sr$ -submodules.

In the previous example,  $r$ -submodules and  $sr$ -submodules are equal, but these concepts are different. See the following examples.

**Example 11** (i) By Proposition 1, the subspace  $N = \{(x, 0) : x \in \mathbb{R}\}$  of  $M = \mathbb{R}^2$  is an  $r$ -submodule, but  $2(1, 0) = (2, 0) \in N$ ,  $ann_{\mathbb{R}}(1, 0) = 0$ , and  $2 \notin (N :_{\mathbb{R}} M)$ ; thus, we get the result that  $N$  is not an  $sr$ -submodule.

(ii) Consider the  $R = \mathbb{Z} \times \mathbb{Z}$ -module  $M = \mathbb{Z} \times \mathbb{Z}_2$  and the submodule  $N = 2\mathbb{Z} \times \bar{0}$ . Since  $ann_R(m) \neq 0$  for every  $m \in M$ , it follows that  $N$  is an  $sr$ -submodule of  $M$ . However, it is not an  $r$ -submodule since  $(2, 1)(1, \bar{0}) = (2, \bar{0}) \in N$ ,  $ann_M(2, 1) = 0_M$ , and  $(1, \bar{0}) \notin N$ .

**Lemma 3** *If  $N$  is an  $sr$ -submodule of  $M$ , then  $N \subseteq T(M)$ .*

**Proof** Assume that  $N \not\subseteq T(M)$ . There exists  $m \in N$  with  $ann_R(m) = 0$ . Since  $1.m = m \in N$  and  $N$  is an  $sr$ -submodule, we get the result that  $1 \in (N :_R M)$ , i.e.  $N = M$ , a contradiction. Hence, we have  $N \subseteq T(M)$ .  $\square$

The converse of the previous lemma is not always true. See the following example.

**Example 12** *Consider the  $R = \mathbb{R} \times \mathbb{Z}$ -module  $M = \mathbb{C} \times \mathbb{Z}$  and the submodule  $N = \mathbb{R} \times 0$  of  $M$ . Note that  $T(M) = (0_{\mathbb{C}} \times \mathbb{Z}) \cup (\mathbb{C} \times 0)$  and  $(N :_R M) = 0_R$ . Thus, we have  $N \subseteq T(M)$ . Since  $(2, 0)(2 + 0i, 1) = (4, 0) \in N$ ,  $ann_R(2 + 0i, 1) = 0_R$ , and  $(2, 0) \notin (N :_R M)$ , we get the result that  $N$  is not an  $sr$ -submodule.*

**Example 13** (i) *Every nonzero prime submodule of  $\mathbb{Z}$ -module  $\mathbb{Z}$  is not an  $sr$ -submodule.*

(ii)  *$\langle 4 \rangle$  is an  $sr$ -submodule of  $\mathbb{Z}$ -module  $\mathbb{Z}_{12}$  but it is not prime.*

Now we give a condition for a prime submodule to be an  $sr$ -submodule in the following proposition.

**Proposition 10** *For a prime submodule  $N$  of  $M$ ,  $N$  is an  $sr$ -submodule if and only if  $N \subseteq T(M)$ .*

**Proof** Assume that  $N$  is a prime submodule. If  $N$  is an  $sr$ -submodule, then  $N \subseteq T(M)$  by Lemma 3. Now, suppose  $N \subseteq T(M)$ . Let  $am \in N$  and  $ann_R(m) = 0$  for  $a \in R$  and  $m \in M$ . Since  $ann_R(m) = 0$ ,  $m \notin T(M)$  and so  $m \notin N$ . Since  $N$  is prime submodule, we have  $a \in (N :_R M)$  and hence  $N$  is an  $sr$ -submodule.  $\square$

**Proposition 11** *Let  $M$  be an  $R$ -module. Then the following hold:*

(i) *The zero submodule is an  $sr$ -submodule of  $M$ .*

(ii) *The intersection of an arbitrary nonempty set of  $sr$ -submodules is an  $sr$ -submodule.*

**Proof** (i) Let  $a \in R, m \in M$  with  $am = 0_M$  and  $ann_R(m) = 0$ . Then we have  $a = 0 \in (0_M :_R M)$ . Hence, we get the result that the zero submodule is an  $sr$ -submodule.

(ii) Suppose that  $\{N_i\}_{i \in \Delta}$  is an arbitrary nonempty set of  $sr$ -submodules of  $M$ . Let  $am \in \bigcap_{i \in \Delta} N_i$  and  $ann_R(m) = 0$ . Since  $N_i$  is an  $sr$ -submodule and  $am \in N_i$ , we get  $a \in (N_i :_R M)$  for every  $i \in \Delta$ . Hence, we get  $a \in \bigcap_{i \in \Delta} (N_i :_R M) = \left( \left( \bigcap_{i \in \Delta} N_i \right) :_R M \right)$  and so  $\bigcap_{i \in \Delta} N_i$  is an  $sr$ -submodule.  $\square$

The following example shows that  $(N :_R M)$  need not be an  $r$ -ideal even if  $N$  is an  $sr$ -submodule of  $M$ .

**Example 14** *Consider the  $\mathbb{Z}$ -module  $\mathbb{Z}_6[x]$  and the submodule  $N = \{p(x) \in \mathbb{Z}_6[x] : p(\bar{0}) \in \langle \bar{2} \rangle\}$ . Then  $N$  is an  $sr$ -submodule but  $(N :_{\mathbb{Z}} \mathbb{Z}_6[x]) = 2\mathbb{Z}$  is not an  $r$ -ideal of  $\mathbb{Z}$ .*

**Proposition 12** *Let  $N$  be a proper submodule of  $M$ . Then the following are equivalent:*

(i)  *$N$  is an  $sr$ -submodule of  $M$ .*

(ii)  *$Rm \cap N = (N :_R M)m$  for every  $m \in M - T(M)$ .*

(iii)  *$(N :_R M) = (N :_R m)$  for every  $m \in M - T(M)$ .*

**Proof** (i)  $\Rightarrow$  (ii): Suppose that  $N$  is an  $sr$ -submodule. The inclusion  $(N :_R M)m \subseteq Rm \cap N$  always holds for each  $m \in M$ . Let  $m \in M - T(M)$  and  $x \in Rm \cap N$ . Then we have  $x = am \in N$  for some  $a \in R$ . As  $N$  is an  $sr$ -submodule of  $M$  and  $ann_R(m) = 0$ ,  $a \in (N :_R M)$  and so  $x = am \in (N :_R M)m$ , as desired.

(ii)  $\Rightarrow$  (iii): It is easy to see that  $(N :_R M) \subseteq (N :_R m)$  for every  $m \in M$ . Suppose that  $m \in M - T(M)$  and  $a \in (N :_R m)$ . Then we have  $am \in N$ . Thus, we have  $am \in Rm \cap N = (N :_R M)m$  by assumption. Then  $am = rm$  for some  $r \in (N :_R M)$ . Since  $ann_R(m) = 0$  and  $(a - r)m = 0_M$ , we conclude that  $a \in (N :_R M)$ . Hence, we have  $(N :_R M) = (N :_R m)$ .

(iii)  $\Rightarrow$  (i): Let  $am \in N$  and  $ann_R(m) = 0$ . Then we get  $m \in M - T(M)$  and so  $a \in (N :_R m) = (N :_R M)$  by the assumption. Consequently,  $N$  is an  $sr$ -submodule of  $M$ .  $\square$

**Theorem 9** Let  $f : M_1 \rightarrow M_2$  be an  $R$ -module homomorphism. Then the following hold:

(i) If  $f$  is a monomorphism and  $L$  is an  $sr$ -submodule of  $M_2$  with  $f^{-1}(L) \neq M_1$ , then  $f^{-1}(L)$  is an  $sr$ -submodule of  $M_1$ .

(ii) If  $f$  is an epimorphism and  $K$  is an  $sr$ -submodule of  $M_1$  containing  $Ker(f)$ , then  $f(K)$  is an  $sr$ -submodule of  $M_2$ .

**Proof** (i) Let  $am \in f^{-1}(L)$  with  $ann_R(m) = 0$  for  $a \in R$ ,  $m \in M_1$ . Then  $f(am) = af(m) \in L$  and  $ann_R(f(m)) = 0$ . Since  $L$  is an  $sr$ -submodule of  $M_2$ , we conclude that  $a \in (L :_R M_2) \subseteq (f^{-1}(L) :_R M_1)$ . Hence,  $f^{-1}(L)$  is an  $sr$ -submodule of  $M_1$ .

(ii) Let  $am' \in f(K)$  and  $ann_R(m') = 0$  for  $a \in R, m' \in M_2$ . Since  $f$  is epimorphism, there exists  $m \in M_1$  such that  $f(m) = m'$ . Then we have  $am' = af(m) = f(am) \in f(K)$ . As  $Ker(f) \subseteq K$ , we have  $am \in K$ . Since  $ann_R(m) = 0$ , we conclude that  $a \in (K :_R M_1) \subseteq (f(K) :_R M_2)$ . Consequently,  $f(K)$  is an  $sr$ -submodule.  $\square$

**Corollary 3** Let  $K$  be a submodule of  $M$ . Then the following hold:

(i) For every  $sr$ -submodule  $N$  of  $M$  with  $K \not\subseteq N$ ,  $N \cap K$  is an  $sr$ -submodule of  $K$ .

(ii) For every  $sr$ -submodule  $N$  of  $M$  with  $K \subseteq N$ ,  $N/K$  is an  $sr$ -submodule of  $M/K$ .

**Proof** (i) Consider the injection  $i : K \rightarrow M$  and note that  $i^{-1}(N) = K \cap N$ . Thus,  $N \cap K$  is an  $sr$ -submodule of  $K$  by Theorem 9(i).

(ii) Assume  $\pi : M \rightarrow M/K$  to be the natural homomorphism and note that  $Ker(\pi) = K \subseteq N$ . Thus,  $N/K$  is an  $sr$ -submodule of  $M/K$  by Theorem 9(ii).  $\square$

**Remark 1** For any nonempty subset  $S$  of  $R$  and submodule  $N$  of  $M$ ,  $((N :_M S) :_R M) = ((N :_R M) :_R S)$  always holds.

**Proposition 13** Let  $M$  be an  $R$ -module. Then the following hold:

(i) For every  $sr$ -submodule  $N$  of  $M$  and every subset  $S$  of  $R$  with  $S \not\subseteq (N :_R M)$ ,  $(N :_M S)$  is an  $sr$ -submodule of  $M$ . In particular,  $(0_M :_M S)$  is always an  $sr$ -submodule if  $S \not\subseteq Ann_R(M)$ .

(ii)  $ann_M(a)$  is an  $sr$ -submodule of  $M$  for every  $a \notin Ann_R(M)$ .

**Proof** (i) Let  $am \in (N :_M S)$  with  $ann_R(m) = 0$  for  $a \in R, m \in M$ . Then  $asm \in N$  for every  $s \in S$ . Since  $N$  is an  $sr$ -submodule, we get the result that  $as \in (N :_R M)$  for every  $s \in S$  and so  $a \in ((N :_R M) :_R S)$ . By Remark 1,  $a \in ((N :_M S) :_R M)$ , and thus  $(N :_M S)$  is an  $sr$ -submodule.

(ii) Follows from (i) and Proposition 11. □

**Theorem 10** For a proper submodule  $N$  of  $M$ , the following hold:

(i)  $N$  is an  $sr$ -submodule of  $M$  if and only if whenever  $L$  is a submodule of  $M$  with  $L \cap (M - T(M)) \neq \emptyset$  and  $J$  is an ideal of  $R$  with  $JL \subseteq N$ , then  $J \subseteq (N :_R M)$ .

(ii) If  $N$  is not an  $sr$ -submodule with  $N \subseteq T(M)$ , then there is an ideal  $J$  of  $R$  and submodule  $L$  of  $M$  with  $L \cap (M - T(M)) \neq \emptyset$ ,  $N \subsetneq L$ ,  $(N :_R M) \subsetneq J$ , and  $JL \subseteq N$ .

**Proof** (i) Suppose  $N$  is an  $sr$ -submodule. For submodule  $L$  of  $M$  with  $L \cap (M - T(M)) \neq \emptyset$  and ideal  $J$  of  $R$ , assume that  $JL \subseteq N$ . Since  $L \cap (M - T(M)) \neq \emptyset$ ,  $ann_R(m) = 0$  for some  $m \in L$ . By assumption,  $am \in N$  for every  $a \in J$ , and thus  $a \in (N :_R M)$ . We get the result that  $J \subseteq (N :_R M)$ . Conversely, let  $am \in N$  and  $ann_R(m) = 0$  for  $a \in R, m \in M$ . Now we take  $J = aR$  and  $L = Rm$ . Then we have  $JL \subseteq N$  for submodule  $L$  of  $M$  with  $L \cap (M - T(M)) \neq \emptyset$  and ideal  $J$  of  $R$ . By assumption,  $J = aR \subseteq (N :_R M)$  so that  $a \in (N :_R M)$ . Consequently,  $N$  is an  $sr$ -submodule.

(ii) If  $N$  is not an  $sr$ -submodule, then  $am \in N$  with  $ann_R(m) = 0$  but  $a \notin (N :_R M)$  for some  $a \in R, m \in M$ . Now we take  $L = (N :_M a)$ . Since  $m \in L - N$ ,  $N \subsetneq L$ . Also, we take  $J = (N :_R L)$ . Since  $a \in J - (N :_R M)$ , we get  $(N :_R M) \subsetneq J$ . Then we get  $JL = (N :_R L)L \subseteq N$ , as desired. □

As a consequence of Theorem 10, we have the following result.

**Theorem 11** Let  $L$  be a submodule of  $M$  with  $L \cap (M - T(M)) \neq \emptyset$ . Then the following hold:

(i) If  $N_1, N_2$  are  $sr$ -submodules of  $M$  with  $(N_1 :_R M)L = (N_2 :_R M)L$ , then  $(N_1 :_R M) = (N_2 :_R M)$ .

(ii) If  $JL$  is an  $sr$ -submodule for an ideal  $J$  of  $R$ , then  $JL = JM$ . Particularly,  $JM$  is an  $sr$ -submodule of  $M$ .

**Theorem 12** Suppose that  $N_1, N_2, \dots, N_n$  are prime submodules of  $M$  with  $(N_i :_R M)$ s not comparable. If

$\bigcap_{i=1}^n N_i$  is an  $sr$ -submodule, then  $N_i$  is an  $sr$ -submodule for each  $i \in \{1, 2, \dots, n\}$ .

**Proof** The proof is similar to Theorem 3. □

The following theorem characterizes the torsion-free modules by  $sr$ -submodule.

**Theorem 13** For any  $R$ -module  $M$ , the following are equivalent:

(i)  $M$  is torsion-free.

(ii)  $M$  is faithful and the zero submodule is the only  $sr$ -submodule.

**Proof** (i)  $\Rightarrow$  (ii): It is obvious that  $M$  is faithful. For every  $sr$ -submodule  $N$  of  $M$ ,  $N \subseteq T(M) = 0_M$  and so  $N = 0_M$  by Lemma 3. However, the zero submodule is always an  $sr$ -submodule.

(ii)  $\Rightarrow$  (i): Let  $m \in T(M)$ . Then we have  $0 \neq r \in R$  such that  $rm = 0_M$ . We know that  $ann_M(r)$  is an  $sr$ -submodule by Proposition 13(ii), and we have  $m \in ann_M(r) = 0_M$  by assumption. Hence, we have  $T(M) = 0_M$ . □

**Proposition 14** *If  $N$  is a maximal  $sr$ -submodule of  $M$ , then  $N$  is prime submodule.*

**Proof** Let  $am \in N$  and  $a \notin (N :_R M)$ ; we show that  $m \in N$ . Then  $(N :_M a)$  is an  $sr$ -submodule by Proposition 13(i). Since  $N$  is maximal  $sr$ -submodule,  $m \in (N :_M a) = N$ . Consequently,  $N$  is prime submodule.  $\square$

**Theorem 14** *Let  $M$  be an  $R$ -module. Then every proper submodule is an  $sr$ -submodule of  $M$  if and only if  $T(M) = M$  or  $Rm = M$  for every  $m \in M - T(M)$ .*

**Proof** Suppose every proper submodule of  $M$  is an  $sr$ -submodule and  $T(M) \neq M$ . Let  $m \in M - T(M)$ . If  $Rm \neq M$ , then we get the result that  $Rm$  is an  $sr$ -submodule. Since  $rm \in Rm$  for every  $r \in R$  and  $ann_R(m) = 0$ ,  $(Rm :_R M) = R$ . Thus, we have  $Rm = RM = M$ , which contradicts the assumption. Hence, we have  $Rm = M$  for all  $m \in M - T(M)$ . Conversely, if  $T(M) = M$ , then every proper submodule is an  $sr$ -submodule. Now assume that  $Rm = M$  for all  $m \in M - T(M)$ . Suppose  $N$  is a proper submodule of  $M$ . Let  $am \in N$  and  $ann_R(m) = 0$  for  $a \in R, m \in M$ . Then we get the result that  $Rm = M$ , because  $m \in M - T(M)$ . Thus,  $a \in (N :_R m) = (N :_R M)$ . Consequently,  $N$  is an  $sr$ -submodule.  $\square$

**Lemma 4** *For every  $R_1$ -module  $M_1$  and  $R_2$ -module  $M_2$ ,  $T(M_1 \times M_2) = (T(M_1) \times M_2) \cup (M_1 \times T(M_2))$  always holds.*

**Proof** Let  $(m_1, m_2) \in T(M_1 \times M_2)$ . Then there exists  $(0_{R_1}, 0_{R_2}) \neq (a_1, a_2) \in R_1 \times R_2$  such that  $(a_1, a_2)(m_1, m_2) = (0_{M_1}, 0_{M_2})$  and so  $a_1 m_1 = 0_{M_1}$ ,  $a_2 m_2 = 0_{M_2}$ . Since  $a_1 \neq 0_{R_1}$  or  $a_2 \neq 0_{R_2}$ , we conclude that  $m_1 \in T(M_1)$  or  $m_2 \in T(M_2)$ . Hence, we have  $(m_1, m_2) \in (T(M_1) \times M_2) \cup (M_1 \times T(M_2))$ . Conversely, let  $(m_1, m_2) \in (T(M_1) \times M_2) \cup (M_1 \times T(M_2))$ . Without loss of generality, we may assume that  $(m_1, m_2) \in T(M_1) \times M_2$ . There exists  $0_{R_1} \neq a_1 \in R_1$  such that  $a_1 m_1 = 0_{M_1}$  since  $m_1 \in T(M_1)$ . Thus, we have  $(0_{R_1}, 0_{R_2}) \neq (a_1, 0_{R_2}) \in R_1 \times R_2$  such that  $(a_1, 0_{R_2})(m_1, m_2) = (0_{M_1}, 0_{M_2})$  and so  $(m_1, m_2) \in T(M_1 \times M_2)$ . Hence, we have  $T(M_1 \times M_2) = (T(M_1) \times M_2) \cup (M_1 \times T(M_2))$ .  $\square$

**Corollary 4** *If  $T(M_1) = M_1$  or  $T(M_2) = M_2$ , then we have  $T(M_1 \times M_2) = M_1 \times M_2$  and so every proper submodule of  $M_1 \times M_2$  is an  $sr$ -submodule of  $M_1 \times M_2$ .*

Now we characterize the  $sr$ -submodules of Cartesian products of modules in case  $T(M_1) \neq M_1$  and  $T(M_2) \neq M_2$ .

**Lemma 5** *Let  $R = R_1 \times R_2$  and  $M = M_1 \times M_2$ , where  $M_i$  is an  $R_i$ -module with  $T(M_i) \neq M_i$  for  $i = 1, 2$ . Suppose that  $N = N_1 \times N_2$  is a submodule of  $M$ . Then the following are equivalent:*

- (i)  $N$  is an  $sr$ -submodule.
- (ii)  $N_1 = M_1$  and  $N_2$  is an  $sr$ -submodule of  $M_2$  or  $N_1$  is an  $sr$ -submodule of  $M_1$  and  $N_2 = M_2$  or  $N_1, N_2$  are  $sr$ -submodules of  $M_1$  and  $M_2$ , respectively.

**Proof** (i)  $\Rightarrow$  (ii): Assume that  $N = N_1 \times N_2$  is an  $sr$ -submodule and  $N_1 = M_1$ . Since  $N$  is proper, we conclude that  $N_2 \neq M_2$ . Now we show that  $N_2$  is an  $sr$ -submodule of  $M_2$ . Suppose not. Then there exist  $a_2 \in R_2, m_2 \in M_2$  such that  $a_2 m_2 \in N_2$  with  $ann_{R_2}(m_2) = 0_{R_2}$  but  $a_2 \notin (N_2 :_{R_2} M_2)$ . Since  $T(M_1) \neq M_1$ , we get  $ann_{R_1}(m_1) = 0_{R_1}$  for some  $m_1 \in M_1$ . Thus, we have  $ann_R(m_1, m_2) = 0_R$  and

$(0_{R_1}, a_2)(m_1, m_2) = (0_{M_1}, a_2 m_2) \in N$  but  $(0_{R_1}, a_2) \notin (N :_R M)$ , which contradicts  $N$  being an  $sr$ -submodule of  $M$ . Hence, we have that  $N_2$  is an  $sr$ -submodule of  $M_2$ . If  $N_2 = M_2$ , in a similar way we can see that  $N_1$  is an  $sr$ -submodule of  $M_2$ . If  $N_1 \neq M_1$  and  $N_2 \neq M_2$ , it can be proved that  $N_1, N_2$  are  $sr$ -submodules of  $M_1$  and  $M_2$ , respectively.

(ii)  $\Rightarrow$  (i): Assume  $N_1, N_2$  are  $sr$ -submodules of  $M_1$  and  $M_2$ , respectively. Let  $(a_1, a_2) \in R_1 \times R_2$  and  $(m_1, m_2) \in M_1 \times M_2$  such that  $(a_1, a_2)(m_1, m_2) = (a_1 m_1, a_2 m_2) \in N$  with  $ann_R(m_1, m_2) = (0_{R_1}, 0_{R_2})$ . Then we have  $ann_{R_i}(m_i) = 0_{R_i}$  and  $a_i m_i \in N_i$  for  $i = 1, 2$ . Since  $N_i$  is an  $sr$ -submodule of  $M_i$ , we conclude that  $a_i \in (N_i :_{R_i} M_i)$  and so  $(a_1, a_2) \in (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2) = (N :_R M)$ . Hence, we get the result that  $N$  is an  $sr$ -submodule. In other cases, one can easily prove the result.  $\square$

**Theorem 15** *Suppose that  $R = R_1 \times R_2 \times \dots \times R_n$  and  $M = M_1 \times M_2 \times \dots \times M_n$ , where  $M_i$  is an  $R_i$ -module with  $T(M_i) \neq M_i$  for  $n \geq 1$  and  $1 \leq i \leq n$ . For a submodule  $N = N_1 \times N_2 \times \dots \times N_n$  of  $M$ , the following are equivalent:*

(i)  $N$  is an  $sr$ -submodule.

(ii)  $N_i = M_i$  for  $i \in \{t_1, t_2, \dots, t_k : k < n\}$  and  $N_i$  is an  $sr$ -submodule of  $M_i$  for  $i \in \{1, 2, \dots, n\} \setminus \{t_1, t_2, \dots, t_k\}$ .

**Proof** We use induction on  $n$ . If  $n = 1$ , of course (i)  $\Leftrightarrow$  (ii). If  $n = 2$ , by Lemma 5, (i) and (ii) are equal. Assume  $n \geq 3$  and (i)  $\Leftrightarrow$  (ii) holds when  $K = M_1 \times M_2 \times \dots \times M_{n-1}$ . Now we prove that (i) and (ii) are equal when  $M = K \times M_n$ . Then, by Lemma 5,  $N$  is an  $sr$ -submodule of  $M$  if and only if  $N = K \times N_n$  for some  $sr$ -submodule  $N_n$  of  $M_n$  or  $N = L \times M_n$  for some  $sr$ -submodule  $L$  of  $K$  or  $N = L \times N_n$  for some  $sr$ -submodule  $L$  of  $K$  and some  $sr$ -submodule  $N_n$  of  $M_n$ . By induction hypothesis, the result is true in three cases.  $\square$

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