


Normalizer Maps Modulo N

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Abstract: The present paper is devoted to studying the maps corresponding to the suborbital graphs for the normalizer $\Gamma_B(N)$ of $\Gamma_0(N)$ modulo N , where N denotes a positive integer. We reveal the complete structure of these maps, finding their vertices, edges, darts, and faces explicitly. The maps we investigated in the present paper were all regular maps of large genus except for some low values of N .

Keywords: regular maps; normalizer; modular group

MSC: 11G32; 14H57; 30F35

1. Introduction

We will denote the group of all linear fractional transformations by $PSL(2, \mathbb{R})$. By a linear fractional transformation we mean a function of a complex variable z , defined by

$$T : z \rightarrow \frac{az + b}{cz + d}, \text{ where } a, b, c \text{ and } d \text{ are } \mathbb{R} \text{ and } ad - bc = 1.$$

It is well-known that the linear fractional transformations can be represented by matrices. Hence, we can regard the elements of $PSL(2, \mathbb{R})$ as the matrices

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c \text{ and } d \text{ are } \mathbb{R} \text{ and } ad - bc = 1.$$

As $PSL(2, \mathbb{R})$ acts on the upper half plane

$\mathbb{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$, $PSL(2, \mathbb{R})$ turns out to be the automorphism group of \mathbb{H} . We also denote the extended rationals $\mathbb{Q} \cup \{\infty\}$ by $\hat{\mathbb{Q}}$ and $\mathbb{H} \cup \hat{\mathbb{Q}}$ by \mathcal{U} .

The modular group Γ is the subgroup of $PSL(2, \mathbb{R})$ such that a, b, c , and d are integers. The modular group has well-studied congruence subgroups. The principal congruence subgroup $\Gamma(N)$, where N denotes a positive integer, of the modular group consists of the transformations corresponding to the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $a \equiv d \equiv 1 \pmod{N}$ and $b \equiv c \equiv 0 \pmod{N}$. The other congruence subgroups are $\Gamma_1(N)$ and $\Gamma_0(N)$. $\Gamma_1(N)$ consists of the transformations corresponding to the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $a \equiv d \equiv 1 \pmod{N}$ and $c \equiv 0 \pmod{N}$, and $\Gamma_0(N)$ consists of the transformations corresponding to the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $c \equiv 0 \pmod{N}$. The relations between these subgroups are well-known. Γ is the normalizer of $\Gamma(N)$ in $PSL(2, \mathbb{R})$ and the normalizer of $\Gamma_1(N)$ in Γ is $\Gamma_0(N)$. The indices of these groups in relation to each other are given by the formulas

- i. $|\Gamma : \Gamma(N)| = \frac{N^3}{3} \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$,
- ii. $|\Gamma_1(N) : \Gamma(N)| = N$,



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$$\text{iii. } |\Gamma_0(N) : \Gamma_1(N)| = \frac{N}{2} \prod_{p|N} \left(1 - \frac{1}{p}\right),$$

and

- i. $|\Gamma : \Gamma(2)| = 2,$
- ii. $|\Gamma_1(2) : \Gamma(2)| = 2,$
- iii. $|\Gamma_0(N) : \Gamma_1(N)| = 1.$

The study by Jones, Singerman and Wicks [1] is a pioneering study concerning these groups and has enabled the analytical examination of graphs. Many researchers have conducted studies [2–7] that reveal the relationship of many groups of graphs with the methods and results presented in this study. In particular, because of the interesting nature of the normalizer $\Gamma_B(N)$ of $\Gamma_0(N)$ in $PSL(2, \mathbb{R})$ [8–10] and its complexity relative to the modular group, researchers have studied normalizer-related graphs under various conditions [2,11–13].

Singerman, in his two studies [14,15], investigated the regular maps corresponding to the principal congruence subgroups of the modular group, arithmetically using the theory of maps [16] and universal tessellations [17]. The natural chain consisting of the modular group and its congruence subgroups allowed the author to construct regular maps, dividing the Farey map according to the principal congruence subgroup. Briefly, a map on an orientable surface is a decomposition of the surface into simply-connected polygonal cells called faces. Thus, a map is considered to have vertices and edges formed by the underlying graph and faces formed by the polygonal cells. It is known that a similar natural chain does not exist for the normalizer. However, the authors in [18] described two subgroups of the normalizer containing $\Gamma_0(N)$ to obtain a chain, and investigated regular maps corresponding to the subgroups $\Gamma_0(N)$ for the values of N , which make the normalizer a triangle group. Their first study [18] concerned the investigation of maps with triangular faces and their subsequent study [19] concerned maps with quadrilateral and hexagonal faces. Their results appear to be similar to those of the relation between Γ and $\Gamma(N)$.

The main purpose of the present paper is to describe a modulo- N subgroup of the normalizer $\Gamma_B(N)$ and to investigate regular maps corresponding to this modulo- N subgroup. In this manner we first construct a natural-like chain, and then construct regular maps, dividing the normalizer maps by the modulo- N subgroup $\Gamma^h(N)$. This paper unifies maps with triangular, quadrilateral, and hexagonal faces. The results show that the regular maps that are constructed in this paper are quite interesting because, with the exception of some low values of N , they are all of large genus. When we deal with regular maps, we define three parametrizations, namely, $N_1, N_2,$ and N_3 , each of which corresponds to a range of values that makes $\Gamma_B(N)$ a triangular group. In this way we guarantee that all the maps are regular and we can reveal their arithmetic structure.

2. The Structure of the Normalizer and Some Subgroups

As described in [9], the normalizer $\Gamma_B(N)$ of $\Gamma_0(N)$ consists of the transformations corresponding to the matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix} \tag{1}$$

where all symbols represent integers, h is the largest divisor of 24 for which $h^2 \mid N$, $e > 0$ is an exact divisor of N/h^2 , and the determinant is e . (We say that r is an exact divisor of s if $r \mid s$ and $(r, s/r) = 1$).

Here we define some subgroups of the normalizer, which helps us to form and investigate the maps.

The first group we are going to define is $\Gamma^h(N)$. It consists of the transformations corresponding to the matrices

$$\begin{pmatrix} a & b/h \\ cN/h & d \end{pmatrix}, a \equiv d \equiv 1 \pmod N \text{ and } b \equiv c \equiv 0 \pmod (N),$$

where h is the largest divisor of 24 for which $h^2 \mid N$.

Here we present a special subgroup of $\Gamma(N)$ in order to calculate the index of $\Gamma^h(N)$ in $\Gamma_B(N)$. We denote this subgroup of $\Gamma(N)$ by $\Gamma^*(N)$, which consists of the transformations corresponding to the matrices $\begin{pmatrix} a & bN \\ cN & d \end{pmatrix} \in \Gamma(N)$ such that $c \equiv 0 \pmod N/h^2$.

Proposition 1. $|\Gamma(N) : \Gamma^*(N)| = N/h^2$

Proof. The proof follows from the homomorphism $\varphi : \Gamma(N) \rightarrow \mathbb{Z}_{N/h^2}$ defined by $\varphi\left(\begin{pmatrix} a & bN \\ cN & d \end{pmatrix}\right) = c \pmod N/h^2$ and the first isomorphism Theorem. \square

Remark 1. $\Gamma^h(N)$ is a conjugation of $\Gamma^*(N)$ by $\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$. One can easily verify this conjugation via the following equality

$$\Gamma^*(N) = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \Gamma^h(N) \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}. \tag{2}$$

The following remark directly follows from Remark 1.

Remark 2. $|\Gamma(N) : \Gamma^h(N)| = N/h^2$

The following proposition is one of the most important propositions in the paper because it will ensure that the maps constructed in the paper are all regular.

Proposition 2. $\Gamma^h(N)$ is a normal subgroup of $\Gamma_B(N)$.

Proof. Let $T = \begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix} \in \Gamma_B(N)$ and $S = \begin{pmatrix} x & y/h \\ zN/h & t \end{pmatrix} \in \Gamma^h(N)$. Consider

$$\begin{aligned} T^{-1}ST &= \begin{pmatrix} de & -b/h \\ -cN/h & ae \end{pmatrix} \begin{pmatrix} x & y/h \\ zN/h & t \end{pmatrix} \begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix} \\ &= \begin{pmatrix} adex - abzN/h^2 + cdyN/h^2 - bctN/eh^2 & (bdx - b^2zN/eh^2 + d^2ey - bdt)/h \\ (a^2ez - acx + act - c^2yN/eh^2)N/h & abzN/h^2 - bcxN/eh^2 + adet - cdyN/h^2 \end{pmatrix}, \end{aligned}$$

where the resulting matrix is divided by e in order to obtain a matrix with determinant 1. Since $x \equiv t \equiv 1 \pmod N, y \equiv z \equiv 0 \pmod N$, using the determinant $ade - bcN/eh^2 = 1$ of T , we have

$$\begin{aligned} adex - abzN/h^2 + cdyN/h^2 - bctN/eh^2 &\equiv 1 \pmod N \\ abzN/h^2 - bcxN/eh^2 + adet - cdyN &\equiv 1 \pmod N \\ bdx - b^2zN/eh^2 + d^2ey - bdt &\equiv 0 \pmod N \\ a^2ez - acx + act - c^2yN/eh^2 &\equiv 0 \pmod N. \end{aligned}$$

This completes the proof. \square

Theorem 1 ([20]). $|\Gamma_B(N) : \Gamma_0(N)| = 2^\rho h^2 \tau$, where ρ is the number of distinct prime factors of N/h^2 and

$$\tau = \frac{\prod_{p|N} \left(1 + \frac{1}{p}\right)}{\prod_{p|N/h^2} \left(1 + \frac{1}{p}\right)}.$$

Finally, using Theorem 1 and Remark 2, we can present the following Proposition.

Proposition 3. For $N > 2$, $|\Gamma_B(N) : \Gamma^h(N)| = 2^{\rho-1} \eta N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$, and for $N = 2$, $|\Gamma_B(2) : \Gamma^1(2)| = 8$, where ρ is the number of distinct prime factors of N/h^2 and

$$\eta = \frac{1}{\prod_{p|N/h^2} \left(1 + \frac{1}{p}\right)}.$$

Proof. This is straightforward due to the relation

$$|\Gamma_B(N) : \Gamma^h(N)| = |\Gamma_B(N) : \Gamma_0(N)| |\Gamma_0(N) : \Gamma(N)| |\Gamma(N) : \Gamma^h(N)|.$$

If $N = 2$, then since $|\Gamma_0(2) : \Gamma(2)| = 2$, we have $|\Gamma_B(2) : \Gamma^1(2)| = 8$. □

The second important subgroup of $\Gamma_B(N)$ that we described is $\Gamma_1^h(N)$, which consists of the transformations corresponding to the matrices

$$\begin{pmatrix} a & b/h \\ cN/h & d \end{pmatrix}, a \equiv d \equiv 1 \pmod N \text{ and } c \equiv 0 \pmod N,$$

where h is the largest divisor of 24 for which $h^2 | N$.

Proposition 4. $\Gamma^h(N)$ is a normal subgroup of $\Gamma_1^h(N)$.

Proof. By definition, it is easily seen that $\Gamma^h(N)$ is the kernel of the homomorphism $\psi : \Gamma_1^h(N) \rightarrow \mathbb{Z}_N$ defined by

$$\psi \begin{pmatrix} a & b/h \\ cN/h & d \end{pmatrix} = b \pmod N.$$

□

Proposition 5. $|\Gamma_1^h(N) : \Gamma^h(N)| = N$

Proof. This is straightforward due to the homomorphism in the proof of Proposition 4 and the first isomorphism theorem. □

By means of Proposition 3 and Proposition 5, the following corollary is obtained.

Corollary 1. For $N > 2$, $|\Gamma_B(N) : \Gamma_1^h(N)| = 2^{\rho-1} \eta N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$, where ρ is the number of distinct prime factors of N/h^2 and

$$\eta = \frac{1}{\prod_{p|N/h^2} \left(1 + \frac{1}{p}\right)}.$$

For $N = 2$, $|\Gamma_B(2) : \Gamma_1^1(2)| = 4$.

The last subgroup of $\Gamma_B(N)$ that we described is $\Gamma_B^h(N)$, which consists of the transformations corresponding to the matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}, c \equiv 0 \pmod{eh^2},$$

where h is the largest divisor of 24 for which $h^2 \mid N$ and the determinant of the matrix is e .

The purpose of defining this group is to be able to form a chain. In fact, this group corresponds to the axes of the maps, but axes will not be examined in this article since finding the axes is not requisite for revealing the structure of the map. So, just to complete the chain, we will give only the following theorem.

Proposition 6. *The normalizer of $\Gamma_1^h(N)$ in $\Gamma_B(N)$ is $\Gamma_B^h(N)$.*

Proof. Let $N(\Gamma_1^h(N))$ denote the normalizer of $\Gamma_1^h(N)$ in $\Gamma_B(N)$. Consider an element

$T = \begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}$ of $\Gamma_B^h(N)$; then, for $U = \begin{pmatrix} x & y/h \\ zN/h & t \end{pmatrix} \in \Gamma_1^h(N)$, we have

$$\begin{aligned} T^{-1}UT &= \begin{pmatrix} de & -b/h \\ -cN/h & ae \end{pmatrix} \begin{pmatrix} x & y/h \\ zN/h & t \end{pmatrix} \begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix} \\ &= \begin{pmatrix} ade^2x - abezN/h^2 + dceyN/h^2 - bctN/h^2 & \frac{bdex - bdet + d^2ye^2 - b^2zN/h^2}{h} \\ (acet - acex + a^2e^2z - c^2yN/h^2)N/h & ade^2t - bcxN/h^2 + abzeN/h^2 - cdeyN/h^2 \end{pmatrix}. \end{aligned}$$

The resulting matrix has determinant e^2 . Thus, dividing all terms by e , we have a matrix with determinant 1

$$\begin{pmatrix} adex - abzN/h^2 + dcyN/h^2 - bctN/eh^2 & \frac{bdx - bdt + d^2ye - b^2zN/eh^2}{h} \\ (act - acx + a^2ez - c^2yN/eh^2)N/h & adet - bcxN/eh^2 + abzN/h^2 - cdyN/h^2 \end{pmatrix}.$$

Since $x \equiv t \equiv 1 \pmod{N}$, $c \equiv 0 \pmod{eh^2}$, $z \equiv 0 \pmod{N}$ and $ade - bcN/eh^2 = 1$, we have

$$adex - abzN/h^2 + dcyN/h^2 - bctN/eh^2 \equiv adet - bcxN/eh^2 + abzN/h^2 - cdyN/h^2 \equiv 1 \pmod{N}. \tag{3}$$

Furthermore, $c \equiv 0 \pmod{eh^2}$ and $z \equiv 0 \pmod{N}$ yield

$$act - acx + a^2ez - c^2yN/eh^2 \equiv 0 \pmod{N}. \tag{4}$$

Thus, using (3) and (4), we conclude $TUT^{-1} \in \Gamma_1^h(N)$, that is, $T \in N(\Gamma_1^h(N))$.

Conversely, let $S = \begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix} \in N(\Gamma_1^h(N))$. Then, for an arbitrary element

$U = \begin{pmatrix} x & y/h \\ zN/h & t \end{pmatrix} \in \Gamma_1^h(N)$, we have $SUS^{-1} \in \Gamma_1^h(N)$. Therefore,

$$\begin{aligned} SUS^{-1} &= \begin{pmatrix} de & -b/h \\ -cN/h & ae \end{pmatrix} \begin{pmatrix} x & y/h \\ zN/h & t \end{pmatrix} \begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix} \\ &= \begin{pmatrix} ade^2x - abezN/h^2 + dceyN/h^2 - bctN/h^2 & \frac{bdex - bdet + d^2ye^2 - b^2zN/h^2}{h} \\ (acet - acex + a^2e^2z - c^2yN/h^2)N/h & ade^2t - bcxN/h^2 + abzeN/h^2 - cdeyN/h^2 \end{pmatrix}, \end{aligned}$$

where the resulting matrix has determinant e^2 . Dividing each element by e , we have

$$\left(\begin{array}{cc} adex - abzN/h^2 + dcyN/h^2 - bctN/eh^2 & \frac{bdx - bdt + d^2ye - b^2zN/eh^2}{h} \\ (act - acx + a^2ez - c^2yN/eh^2)N/h & adet - bcxN/eh^2 + abzN/h^2 - cdyN/h^2 \end{array} \right).$$

Since the resulting matrix is in $\Gamma_1^h(N)$, we have

$$\begin{aligned} adex - abzN/h^2 + dcyN/h^2 - bctN/eh^2 &\equiv 1 \pmod{N} \\ adet - bcxN/eh^2 + abzN/h^2 - cdyN/h^2 &\equiv 1 \pmod{N}, \\ act - acx + a^2ez - c^2yN/eh^2 &\equiv 0 \pmod{N}. \end{aligned}$$

Consider the congruences $adex - abzN/h^2 + dcyN/h^2 - bctN/eh^2 \equiv 1 \pmod{N}$ and $adet - bcxN/eh^2 + abzN/h^2 - cdyN/h^2 \equiv 1 \pmod{N}$. Using $ade - bcN/eh^2 = 1$, $x \equiv t \equiv 1 \pmod{N}$ and $z \equiv 0 \pmod{N}$, we obtain $cd \equiv 0 \pmod{h^2}$. Moreover, consider the congruence $act - acx + a^2ez - c^2yN/eh^2 \equiv 0 \pmod{N}$. Since $x \equiv t \equiv 1 \pmod{N}$, we have $x - t \equiv 0 \pmod{N}$. Thus, $z \equiv 0 \pmod{N}$ yields $c^2N/eh^2 \equiv 0 \pmod{N}$. As $h^2 \mid N$, we have $h^2 \mid c^2N/eh^2$. Since $h^2 \mid cd$ and $h^2 \mid c^2N/eh^2$, for integers de and b we have $h^2 \mid adec - bc^2N/eh^2 = c$. Finally, by $h^2 \mid c$ and $N \mid c^2N/eh^2$, one can easily obtain $eh^2 \mid c$. This means that $S \in \Gamma_B^h(N)$. \square

We now present two important theorems of [20] regarding the structure of $\Gamma_B(N)$.

Theorem 2. Let $N = 2^\alpha 3^\beta$ and $\beta = 0$ or 2 . Then, $\Gamma_B(N)$ is a triangle group if and only if $\alpha \leq 8$. In these cases

$$\Gamma_B(N) \text{ has signature } \begin{cases} (0; 2, 3, \infty) & \text{if } \alpha = 0, 2, 4, 6 \\ (0; 2, 4, \infty) & \text{if } \alpha = 1, 3, 5, 7 \\ (0; 2, \infty, \infty) & \text{if } \alpha = 8. \end{cases}$$

Theorem 3. Let $N = 2^\alpha 3^\beta$ and $\beta = 1$ or 3 . Then $\Gamma_B(N)$ is a triangle group if and only if $\alpha = 0, 2, 4, 6$. In these cases $\Gamma_B(N)$ has signature $(0; 2, 6, \infty)$.

Based on the above mentioned theorems we conclude

Remark 3. Let $N = 2^\alpha 3^\beta$,

- i. if $\beta = 0, 2$ and $\alpha = 0, 2, 4, 6$, then regular 3-valent maps correspond to normal subgroups of $\Gamma_B(N)$,
- ii. if $\beta = 0, 2$ and $\alpha = 1, 3, 5, 7$, then regular 4-valent maps correspond to normal subgroups of $\Gamma_B(N)$,
- iii. if $\beta = 1, 3$ and $\alpha = 0, 2, 4, 6$, then regular 6-valent maps correspond to normal subgroups of $\Gamma_B(N)$.

In the present study we investigate the regular 3-valent, 4-valent, and 6-valent maps corresponding to normal subgroups of $\Gamma_B(N)$.

Let us identify parameter N corresponding to each statement of Remark 3.

Notation 1. We denote N by

- i. N_1 if N satisfies i of Remark 3,
- ii. N_2 if N satisfy ii of Remark 3,
- iii. N_3 if N satisfy iii of Remark 3.

Let us identify the elements of $\Gamma_B(N)$ for each parameter in Notation 1 in the following Remark.

Remark 4.

i. According to the definition of N_1 , we have $N_1 = h^2$ and hence $e = 1$. Thus, $\Gamma_B(N_1)$ consists of the transformations corresponding to the matrices

$$\begin{pmatrix} a & b/h \\ ch & d \end{pmatrix}, ad - bc = 1, \tag{5}$$

where h is the largest divisor of 24 for which $h^2 \mid N_1$.

ii. According to the definition of N_2 , we have $N_2 = 2h^2$ and hence $e = 1, 2$. Thus, $\Gamma_B(N_2)$ consists of two types of elements, namely, even elements and odd elements [19]. Even elements are the transformations corresponding to the matrices

$$\begin{pmatrix} a & b/h \\ 2ch & d \end{pmatrix}, ad - 2bc = 1, \tag{6}$$

and odd elements are the transformations corresponding to the matrices

$$\begin{pmatrix} 2a & b/h \\ 2ch & 2d \end{pmatrix}, 2ad - bc = 1, \tag{7}$$

where h is the largest divisor of 24 for which $h^2 \mid N_2$.

iii. According to definition of N_3 , we have $N_2 = 3h^2$ and hence $e = 1, 3$. Similarly to ii, $\Gamma_B(N_3)$ consists of even elements and odd elements [19]. Even elements are the transformations corresponding to the matrices

$$\begin{pmatrix} a & b/h \\ 3ch & d \end{pmatrix}, ad - 3bc = 1, \tag{8}$$

and odd elements are the transformations corresponding to the matrices

$$\begin{pmatrix} 3a & b/h \\ 3ch & 3d \end{pmatrix}, 3ad - bc = 1, \tag{9}$$

where h is the largest divisor of 24 for which $h^2 \mid N_3$.

3. The Normalizer Maps

In this section, we construct three universal maps, namely, the normalizer maps \mathcal{M}_3^h , \mathcal{M}_4^h , and \mathcal{M}_6^h . These universal maps are investigated in [18,19]. On the other hand, Akbaş showed that $\Gamma_B(N)$ acts transitively on $\hat{\mathbb{Q}}$ for N_1, N_2 , and N_3 . Now we are ready to construct the normalizer maps.

First let us introduce the universal map \mathcal{M}_3^h corresponding to $\Gamma_B(N_1)$ (see Notation 1 for the parameters N_1, N_2 , and N_3). Vertices of \mathcal{M}_3^h are the fractions $\frac{a}{ch}$ with $(a, c) = 1$ and two vertices, $\frac{a}{ch}$ and $\frac{b}{dh}$, are joined by an edge if and only if $ad - bc = \pm 1$. All these edges are just hyperbolic lines, and in this way \mathcal{M}_3^h has the following properties:

1. There is a triangle with vertices $\frac{1}{0}, \frac{1}{h}, \frac{0}{h}$.
2. $\Gamma_B(N_1)$ acts as a group of homomorphisms of \mathcal{M}_3^h .
3. There is a triangle with vertices $\frac{a}{ch}, \frac{a+b}{(c+d)h}, \frac{b}{dh}$.

\mathcal{M}_3^h is a triangular tessellation of the upper half plane. The triangles in 3 are the images of the triangle in 1 under the elements of $\Gamma_B(N)$.

Remark 5. One can easily see that when $h = 1$, \mathcal{M}_3^1 is just the Farey map. When $h > 1$, the normalizer map \mathcal{M}_3^h is the Farey map scaled by a factor of $1/h$ (see Figure 1 for $N_1 = 4$ and $h = 2$).

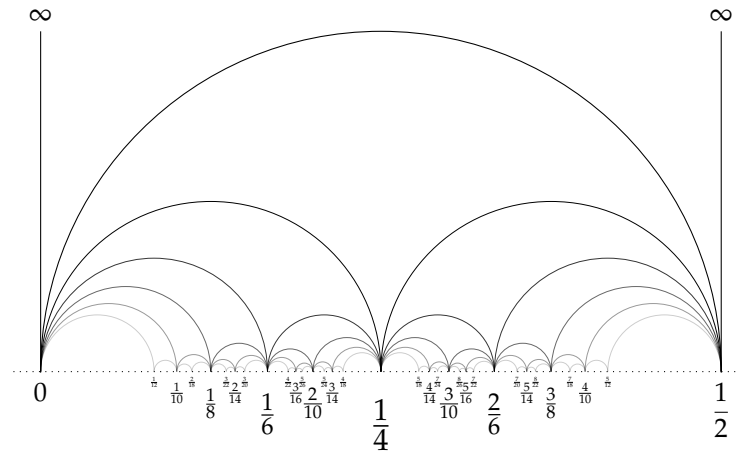


Figure 1. Part of the normalizer map \mathcal{M}_3^2 .

In a similar way, we can construct the normalizer map \mathcal{M}_4^h (\mathcal{M}_6^h) corresponding to $\Gamma_B(N_2)$ ($\Gamma_B(N_3)$). However, there is a slight difference: since $\Gamma_B(N_2)$ ($\Gamma_B(N_3)$) has two types of elements, namely, the even and odd elements, vertices of \mathcal{M}_4^h (\mathcal{M}_6^h) have two categories, namely, even and odd vertices. Even vertices are the fractions $\frac{a}{2ch}$ ($\frac{a}{3ch}$) with $(a, c) = 1$ and $2 \nmid a$ ($(a, c) = 1$ and $3 \nmid a$). Odd vertices are the fractions $\frac{b}{dh}$ ($\frac{b}{dh}$) with $(b, d) = 1$ and $2 \nmid d$ ($(b, d) = 1$ and $3 \nmid d$).

We can denote 0 and ∞ by $\frac{0}{1.h}$ (resp. $\frac{0}{1.h}$) and $\frac{1}{2.0.h}$ (resp. $\frac{1}{3.0.h}$), respectively. In this manner, we can see that ∞ is an even vertex and 0 is an odd vertex. Thus the edges of \mathcal{M}_4^h (resp. \mathcal{M}_6^h) are the images of the edge joining 0 and ∞ under the elements of $\Gamma_B(N_2)$ (resp. $\Gamma_B(N_3)$). Hence all the edges of \mathcal{M}_4^h (resp. \mathcal{M}_6^h) join an even and odd vertex. Finally, the vertices $\frac{a}{2ch}$ (resp. $\frac{a}{3ch}$) and $\frac{b}{dh}$ (resp. $\frac{b}{dh}$) are joined by an edge if and only if $ad - 2bc = \pm 1$ (resp. $ad - 3bc = \pm 1$). We know that \mathcal{M}_4^h (resp. \mathcal{M}_6^h) is quadrilateral (resp. hexagonal), and its principal quadrilateral (resp. hexagon) is $\infty, 0, \frac{1}{2h}, \frac{1}{h}$ (resp. $\infty, 0, \frac{1}{3h}, \frac{1}{2h}, \frac{2}{3h}, \frac{1}{h}$). One can find other quadrilaterals (resp. hexagons) applying the elements of $\Gamma_B(N_2)$ (resp. $\Gamma_B(N_3)$) to the vertices of the principle quadrilateral (resp. hexagon).

In [17], the author showed that any triangular map on a surface is the quotient of the universal triangular map by a subgroup of $\Gamma(2, \infty, 3)$ and any regular triangular map is the quotient of the universal triangular map by a normal subgroup of $\Gamma(2, \infty, 3)$. As $\Gamma^h(N)$ is a normal subgroup of $\Gamma_B(N)$ and $\Gamma_B(N)$ is a triangle group for all values of N_1, N_2 , and N_3 , we can form the regular maps $\mathcal{M}_3^h/\Gamma^h(N_1)$, $\mathcal{M}_4^h/\Gamma^h(N_2)$, and $\mathcal{M}_6^h/\Gamma^h(N_3)$. Since $\Gamma^h(N)$ is a modulo- N -subgroup of $\Gamma_B(N)$, we will call these regular maps “normalizer maps modulo N ”.

4. Normalizer Maps Modulo N

Definition 1. The map $\mathcal{M}_i^h(N)$ is defined as the map $\mathcal{M}_i^h/\Gamma^h(N)$ for $i = 3, 4, 6$.

Before investigating regular maps $\mathcal{M}_i^h(N)$, $i = 3, 4, 6$ analytically, we present some results concerning the normalizer $\Gamma_B(N)$ and its subgroups that we already described in Section 2.

A straightforward calculation yields that the stabilizer S_∞ of ∞ in $\Gamma_B(N)$ is the cyclic group generated by $\begin{pmatrix} 1 & 1/h \\ 0 & 1 \end{pmatrix}$. Let $[\infty]$ be the orbit of ∞ under the action of $\Gamma^h(N)$ and $S_{[\infty]}$ be the set-wise stabilizer of $[\infty]$ in $\Gamma_B(N)$.

Proposition 7. $S_{[\infty]} = S_\infty \Gamma^h(N)$.

Proof. Let $T \in S_{[\infty]}$. For an arbitrary element U of $\Gamma^h(N)$, since $U\infty \in [\infty]$, T stabilizes $U\infty$. So we have $TU\infty \in [\infty]$. That is, there exists an element $V \in \Gamma^h(N)$ such that $TU\infty = V\infty$. This means that $V^{-1}TU \in S_\infty$. As $\Gamma^h(N)$ is a normal subgroup of $\Gamma_B(N)$; we thus have $UT = TU$. Then $R^{-1}T \in S_\infty$, where $R = U^{-1}V \in \Gamma^h(N)$. Finally, we conclude that $T \in \Gamma^h(N)S_\infty = S_\infty \Gamma^h(N)$, as $\Gamma^h(N)$ is normal in $\Gamma_B(N)$.

For the converse, let $T \in S_\infty \Gamma^h(N)$. Then, there exist $U \in S_\infty$ and $V \in \Gamma^h(N)$ such that $T = UV$. Consider the action of UV on $[\infty]$. As $V \in \Gamma^h(N)$ and U stabilizes ∞ , we have $T[\infty] = UV[\infty] = [\infty]$. \square

Proposition 8. $S_\infty \Gamma^h(N) = \Gamma_1^h(N)$.

Proof. Let $T \in S_\infty \Gamma^h(N)$. Then, there exist $U = \begin{pmatrix} 1 & u/h \\ 0 & 1 \end{pmatrix} \in S_\infty$ and $V = \begin{pmatrix} a & b/h \\ cN/h & d \end{pmatrix} \in \Gamma^h(N)$ such that $T = UV$. Consider the following matrix

$$T = UV = \begin{pmatrix} 1 & u/h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b/h \\ cN/h & d \end{pmatrix} = \begin{pmatrix} a + cuN/h^2 & (b + du)/h \\ cN/h & d \end{pmatrix}.$$

As $a \equiv 1 \pmod N$ and $c \equiv 0 \pmod N$, we have $a + cuN/h^2 \equiv 1 \pmod N$. Furthermore, $d \equiv 1 \pmod N$ yields that the resulting matrix is in $\Gamma_1^h(N)$.

For the converse, let $T = \begin{pmatrix} a & b/h \\ cN/h & d \end{pmatrix} \in \Gamma_1^h(N)$. Consider the following equality

$$\begin{pmatrix} 1 & ab/h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - abcN/h^2 & b(1 - ad)/h \\ cN/h & d \end{pmatrix} = \begin{pmatrix} a & b/h \\ cN/h & d \end{pmatrix} = T.$$

Since $a \equiv 1 \pmod N$ and $c \equiv 0 \pmod N$, $a - abcN/h^2 \equiv 1 \pmod N$. Furthermore, $a \equiv d \equiv 1 \pmod N$ yields $b(1 - ad) \equiv 0 \pmod N$. Thus,

$$\begin{pmatrix} a - abcN/h^2 & b(1 - ad)/h \\ cN/h & d \end{pmatrix} \in \Gamma_1^h(N).$$

This completes the proof. \square

Now we are ready to find the number of the vertices of $\mathcal{M}_i^h(N)$, $i = 3, 4, 6$. We give the theorem only for $\mathcal{M}_3^h(N_1)$. This theorem also holds for $\mathcal{M}_4^h(N_2)$ and $\mathcal{M}_6^h(N_2)$.

Theorem 4. *There exists a one-to-one correspondence between the left cosets of $\Gamma_1^h(N_1)$ in $\Gamma_B(N_1)$ and the vertices of $\mathcal{M}_3^h(N_1)$.*

Proof. Due to the transitivity of the action of $\Gamma_B(N_1)$ on $\hat{\mathbb{Q}}$ and $\Gamma_1^h(N_1) \leq \Gamma_B(N_1)$, we can choose the fixed vertex $[\infty]$ of $\mathcal{M}_3^h(N_1)$. In accordance with Propositions 7 and 8, stabilizer of the vertex $[\infty]$ in $\Gamma_B(N_1)$ is $\Gamma_1^h(N_1)$. Therefore, the theorem follows from the orbit-stabilizer theorem. \square

Using Theorem 4 and Corollary 1, we can give the following corollary.

Corollary 2. *The number of vertices is*

- i. $V_3^h(N_1) = 2^{p-1} \eta N_1^2 \prod_{p|N_1} \left(1 - \frac{1}{p^2}\right)$ for $\mathcal{M}_3^h(N_1)$;
- ii. $V_4^h(N_2) = 2^{p-1} \eta N_2^2 \prod_{p|N_2} \left(1 - \frac{1}{p^2}\right)$ for $\mathcal{M}_4^h(N_2)$;
- iii. $V_6^h(N_3) = 2^{p-1} \eta N_3^2 \prod_{p|N_3} \left(1 - \frac{1}{p^2}\right)$ for $\mathcal{M}_6^h(N_3)$.

Now, we find that the set of left cosets has a one-to-one correspondence with the set of vertices of the regular maps. So if we want to study the vertices, we can study the left cosets. Therefore, the following propositions will allow us to identify the vertices uniquely.

Proposition 9. *Let $T_1 = \begin{pmatrix} a_1 e_1 & b_1/h \\ c_1 N/h & d_1 e_1 \end{pmatrix}, T_2 = \begin{pmatrix} a_2 e_2 & b_2/h \\ c_2 N/h & d_2 e_2 \end{pmatrix} \in \Gamma_B(N)$. Then, T_1 and T_2 determine the same left coset of $\Gamma_1^h(N)$ in $\Gamma_B(N)$ if and only if $e_1 = e_2$ and $\begin{pmatrix} a_1 e \\ c_1 N/eh^2 \end{pmatrix} \equiv \pm \begin{pmatrix} a_2 e \\ c_2 N/eh^2 \end{pmatrix} \pmod N$.*

Proof. If $e_1 \neq e_2$ then the determinant of $T_2^{-1}T_1$ cannot be 1 such that $T_2^{-1}T_1 \in \Gamma_1^h(N)$. Thus, let $e_1 = e_2 = e$. Thus, $T_2^{-1}T_1 \in \Gamma_1^h(N)$ if and only if $\begin{pmatrix} a_1 \\ c_1 \end{pmatrix} \equiv \pm \begin{pmatrix} a_2 \\ c_2 \end{pmatrix} \pmod N$ which follows from the definition of $\Gamma_1^h(N)$. \square

Proposition 9 states that each e determines different types of vertices. For the map $\mathcal{M}_3^h(N_1)$, since $e = 1$, we have just one type of vertex. So we simply call them vertices. However, for $\mathcal{M}_4^h(N_2)$ ($\mathcal{M}_6^h(N_3)$), since $e = 1$ or $e = 2$ ($e = 1$ or $e = 3$), there are two types of vertices. So we call them, as before, the even vertices and odd vertices of the map. Here, let us identify these vertices using Proposition 9.

We denote the left coset of $\Gamma_1^h(N)$ corresponding to the matrix $\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}$ by the row vector $(ae, cN/eh^2)$ for the corresponding e , as the equivalence of cosets depends only on the integers a, c , and e . Since the determinant of the matrix is written in the form of $ade - bcn/eh^2 = 1$, we have that $(ae, cN/eh^2) = 1$. So we denote the vertices of the maps by $(ae, cN/eh^2)$, as any coset corresponds to a vertex of the map. In the case of N_1, N_2, N_3 , i.e., the case of $\mathcal{M}_3^h(N_1), \mathcal{M}_4^h(N_2)$, and $\mathcal{M}_6^h(N_3)$, using (5)–(9), we have the following cases.

- i. For the case N_1 , since $N_1 = h^2$, we have $e = 1$. Thus, using Proposition 9, the set of vertices of $\mathcal{M}_3^h(N_1)$ is

$$\{(a, c) \mid a, c \in \mathbb{Z}_{N_1}, (a, c, N_1) = 1\} / \sim,$$

where $(a, c) \sim (N_1 - a, N_1 - c)$.

- ii. For the case N_2 , since $N_2 = 2h^2$, we have $e = 1$ or $e = 2$. Thus, there are two types of vertices, namely, the odd vertices and even vertices. Using Proposition 9, the set of odd vertices of $\mathcal{M}_4^h(N_2)$ is

$$\{(2a, c) \mid a, c \in \mathbb{Z}_{N_2}, (a, c, N_2) = 1, 2 \nmid c\} / \sim,$$

where $(2a, c) \sim (2(N_2 - a), N_2 - c)$.

The set of even vertices of $\mathcal{M}_4^h(N_2)$ is

$$\{(a, 2c) \mid a, c \in \mathbb{Z}_{N_2}, (a, c, N_2) = 1, 2 \nmid a\} / \sim,$$

where $(a, 2c) \sim (N_2 - a, 2(N_2 - c))$.

- iii. For the case N_3 , since $N_3 = 3h^2$, we have $e = 1$ or $e = 3$. Thus, there are two types of vertices, namely, the odd vertices and even vertices. Using Proposition 9, the set of odd vertices of $\mathcal{M}_6^h(N_3)$ is

$$\{(3a, c) \mid a, c \in \mathbb{Z}_{N_3}, (a, c, N_3) = 1, 3 \nmid c\} / \sim,$$

where $(3a, c) \sim (3(N_3 - a), N_3 - c)$.

The set of even vertices of $\mathcal{M}_6^h(N_3)$ is

$$\{(a, 3c) \mid a, c \in \mathbb{Z}_{N_3}, (a, c, N_3) = 1, 3 \nmid a\} / \sim,$$

where $(a, 3c) \sim (N_3 - a, 3(N_3 - c))$.

A directed edge of a map will be called a dart. Since $\Gamma_B(N)$ for all parameters N_1, N_2 , and N_3 acts transitively on $\hat{\mathbb{Q}}$, it acts transitively on the darts of \mathcal{M}_i^h for $i = 3, 4, 6$. Thus, $\Gamma_B(N)/\Gamma^h(N)$ acts transitively on the darts of $\mathcal{M}_i^h(N)$ for $i = 3, 4, 6$ and N_1, N_2, N_3 because $\Gamma^h(N)$ is a normal subgroup of $\Gamma_B(N)$. According to the results of [16], a map is regular if its automorphism group acts transitively on its darts, which makes $\mathcal{M}_i^h(N)$ regular for $i = 3, 4, 6$ and N_1, N_2, N_3 .

Here we present a theorem to find the number of darts of regular maps. We give the theorem only for $\mathcal{M}_3^h(N_1)$. Again, the theorem also holds for $\mathcal{M}_4^h(N_2)$ and $\mathcal{M}_6^h(N_3)$.

Theorem 5. *There exists a one-to-one correspondence between the left cosets of $\Gamma^h(N_1)$ in $\Gamma_B(N_1)$ and the darts of $\mathcal{M}_3^h(N_1)$.*

Proof. Due to the transitivity of the action of $\Gamma_B(N_1)$ on $\hat{\mathbb{Q}}$ and $\Gamma^h(N_1) \trianglelefteq \Gamma_B(N_1)$, we can choose the fixed dart $[0] \rightarrow [\infty]$ of $\mathcal{M}_3^h(N_1)$. We will show that the stabilizer of the dart $[0] \rightarrow [\infty]$ is $\Gamma^h(N_1)$. Let $T \in \Gamma_B(N_1)$ stabilize the dart $[0] \rightarrow [\infty]$. Thus, it stabilizes both $[0]$ and $[\infty]$. If T stabilizes $[\infty]$, by Propositions 7 and 8, $T \in \Gamma_1^h(N_1)$. On the other hand, if T stabilizes $[0]$, then for an arbitrary $S \in \Gamma^h(N_1)$ we have $TS \in \Gamma^h(N_1)$. Let $T = \begin{pmatrix} a_1 & b_1/h \\ c_1 N_1/h & d_1 \end{pmatrix} \in \Gamma_1^h(N_1)$ and $S = \begin{pmatrix} a_2 & b_2/h \\ c_2 N_1/h & d_2 \end{pmatrix} \in \Gamma^h(N_1)$. Consider the following equation

$$TS = \begin{pmatrix} a_1 & b_1/h \\ c_1 N_1/h & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2/h \\ c_2 N_1/h & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 N_1/h^2 & (a_1 b_2 + b_1 d_2)/h \\ (a_2 c_1 + c_2 d_1) N_1/h & c_1 b_2 N_1/h^2 + d_1 d_2 \end{pmatrix}. \tag{10}$$

The matrix on the right-hand side of Equation (10) must be in $\Gamma^h(N_1)$. According to the definition of $\Gamma^h(N_1)$ and $\Gamma_1^h(N_1)$ we have $a_1 a_2 + b_1 c_2 N_1/h^2 \equiv c_1 b_2 N_1/h^2 + d_1 d_2 \equiv 1 \pmod{N_1}$, and $a_2 c_1 + c_2 d_1 \equiv 0 \pmod{N_1}$. Thus, $a_1 b_2 + b_1 d_2 \equiv 0 \pmod{N_1}$ must hold. Since $b_2 \equiv 0 \pmod{N_1}$ and S is arbitrary, we have $b_1 \equiv 0 \pmod{N_1}$. This yields $T \in \Gamma^h(N_1)$. Again, the orbit-stabilizer theorem applies. \square

Based on Theorem 5 and Proposition 3, the following corollary follows.

Corollary 3. *The number of darts is*

- i. $D_3^h(N_1) = 2^{\rho-1} \eta N_1^3 \prod_{p|N_1} \left(1 - \frac{1}{p^2}\right)$ for $\mathcal{M}_3^h(N_1)$;
- ii. $D_4^h(N_2) = 2^{\rho-1} \eta N_2^3 \prod_{p|N_2} \left(1 - \frac{1}{p^2}\right)$ for $\mathcal{M}_4^h(N_2)$;
- iii. $D_6^h(N_3) = 2^{\rho-1} \eta N_3^3 \prod_{p|N_3} \left(1 - \frac{1}{p^2}\right)$ for $\mathcal{M}_6^h(N_3)$.

Using the regularity of the maps, the definition of a dart and Corollary 3, with the following corollary, we can determine the number of edges and the number of faces of the maps. The number of edges is always the half of the number of darts, and the number of faces can be found by dividing the number of darts by the number of edges in each face of the map.

Corollary 4. *The number of edges is*

- i. $E_3^h(N_1) = \frac{D_3^h(N_1)}{2}$ for $\mathcal{M}_3^h(N_1)$;
- ii. $E_4^h(N_2) = \frac{D_4^h(N_2)}{2}$ for $\mathcal{M}_4^h(N_2)$;
- iii. $E_6^h(N_3) = \frac{D_6^h(N_3)}{2}$ for $\mathcal{M}_6^h(N_3)$.

Corollary 5. *The number of faces is*

- i. $F_3^h(N_1) = \frac{D_3^h(N_1)}{3}$ for $\mathcal{M}_3^h(N_1)$;
- ii. $F_4^h(N_2) = \frac{D_4^h(N_2)}{4}$ for $\mathcal{M}_4^h(N_2)$;
- iii. $F_6^h(N_3) = \frac{D_6^h(N_3)}{6}$ for $\mathcal{M}_6^h(N_3)$.

Finally, using the Euler formula $V - E + F = 2 - 2g$ for a regular map, where V, E, F is the numbers of vertices, edges, and faces of the map, respectively, we can find the genus of the map as follows.

Corollary 6. *The genus is computed using the formula*

- i. $g_3^h(N_1) = N_1^2 \frac{N_1 - 6}{3} 2^{\rho-3} \eta \prod_{p|N_1} \left(1 - \frac{1}{p^2}\right) + 1$ for $\mathcal{M}_3^h(N_1)$;
- ii. $g_4^h(N_2) = N_2^2 (N_2 - 4) 2^{\rho-4} \eta \prod_{p|N_2} \left(1 - \frac{1}{p^2}\right) + 1$, $N > 2$ and $g_4^1(2) = 0$, $N = 2$ for $\mathcal{M}_4^h(N_2)$;
- iii. $g_6^h(N_3) = N_3^2 \frac{N_3 - 3}{3} 2^{\rho-2} \eta \prod_{p|N_3} \left(1 - \frac{1}{p^2}\right) + 1$ for $\mathcal{M}_6^h(N_3)$.

In the following section, we present some examples to show the details.

Examples for Low N

Example 1 (Triangular case). For $N_1 = 4$, we have $h = 2$ and the regular map $\mathcal{M}_3^2(4)$. The numbers of vertices, edges, , and faces are $V_3^2(4) = 6$, $E_3^2(4) = 12$, and $F_3^2(4) = 8$, respectively. Genus of $\mathcal{M}_3^2(4)$ is $g_3^2(4) = 0$. The vertices are

$$(1, 0), (0, 1), (1, 1), (1, 2), (2, 1), (3, 1).$$

This is an octahedron lying on the sphere $\mathcal{U} / \Gamma^2(4)$ with Schläfli $\{3, 4\}$ (see Figure 2).

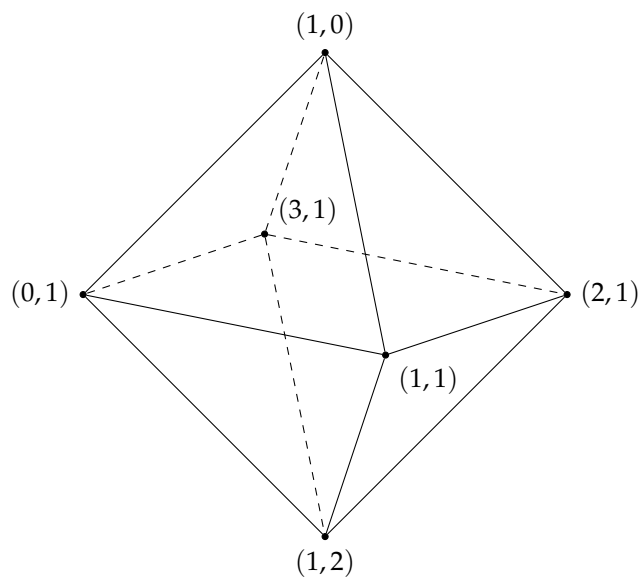


Figure 2. $\mathcal{M}_3^2(4)$: Octahedron.

Example 2 (Quadrilateral case 1). For $N_2 = 2$, we have $h = 1$ and the regular map $\mathcal{M}_4^1(2)$. The numbers of vertices, edges, and faces are $V_4^1(2) = 4$, $E_4^1(2) = 4$, and $F_4^1(2) = 2$, respectively. The genus of $\mathcal{M}_4^1(2)$ is $g_4^1(2) = 0$. The odd vertices are

$$(0, 1), (2, 1),$$

and the even vertices are

$$(1, 0), (1, 2).$$

This is a di-square lying on the sphere $\mathcal{U} / \Gamma^1(2)$ with Schläfli $\{4, 2\}$ (see Figure 3).

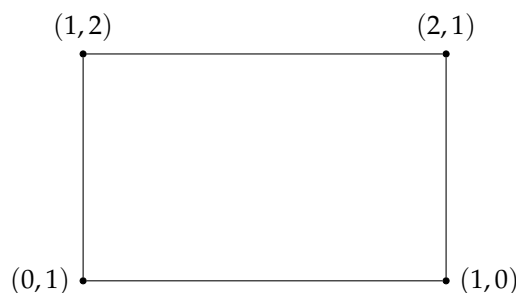


Figure 3. $\mathcal{M}_4^1(2)$: Di-square.

Example 3 (Quadrilateral case 2). For $N_2 = 8$, we have $h = 2$ and the regular map $\mathcal{M}_4^2(8)$. The numbers of vertices, edges, and faces are $V_4^2(8) = 32$, $E_4^2(8) = 128$, and $F_4^2(8) = 64$, respectively. The genus of $\mathcal{M}_4^2(8)$ is $g_4^2(8) = 17$. The odd vertices are

$$(0, 1), (0, 3), (2, 1), (2, 3), (2, 5), (2, 7), (4, 1), (4, 3), \\ (4, 5), (4, 7), (6, 1), (6, 3), (6, 5), (6, 7), (8, 1), (8, 3),$$

and the even vertices are

$$(1, 0), (1, 2), (1, 4), (1, 6), (1, 8), (1, 10), (1, 12), (1, 14), \\ (3, 0), (3, 2), (3, 4), (3, 6), (3, 8), (3, 10), (3, 12), (3, 14).$$

This is the map R17.6 lying on $\mathcal{U} / \Gamma^2(8)$ with Schläfli $\{4, 8\}$.

Example 4 (Hexagonal case). For $N_3 = 3$, we have $h = 1$ and the regular map $\mathcal{M}_6^1(3)$. The numbers of vertices, edges, and faces are $V_6^1(3) = 6$, $E_6^1(3) = 9$, and $F_6^1(3) = 3$, respectively. The genus of $\mathcal{M}_6^1(3)$ is $g_6^1(3) = 1$. The odd vertices are

$$(0, 1), (3, 1), (3, 2),$$

and the even vertices are

$$(1, 0), (1, 3), (1, 6).$$

This is the map $\{6, 3\}_{(0,2)}$ lying on the torus $\mathcal{U}/\Gamma^1(3)$ with Schläfli $\{6, 3\}$ (see Figure 4).

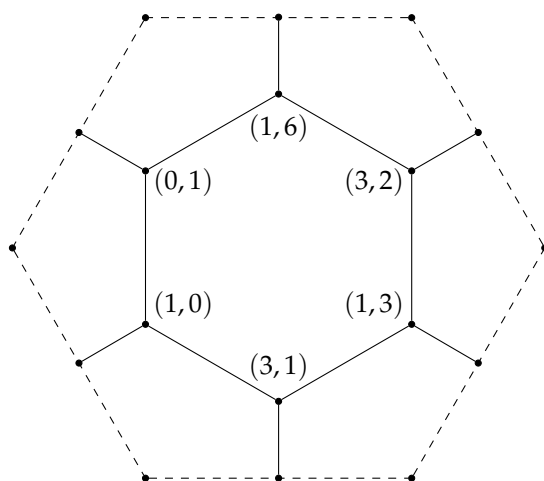


Figure 4. $\mathcal{M}_6^1(3) : \{6, 3\}_{0,2}$.

5. Complete Tables of the Regular Maps

In this final section we provide the complete tables of regular maps corresponding to N_1, N_2, N_3 . The triangular maps are provided in Table 1.

Table 1. The complete table of regular maps for N_1 .

N_1	D	E	V	F	g	Map (Schläfli)
2^2	$2^3 \cdot 3$	$2^2 \cdot 3$	$2 \cdot 3$	2^3	0	{3, 4}
3^2	$2^2 \cdot 3^4$	$2 \cdot 3^4$	$2^2 \cdot 3^2$	$2^2 \cdot 3^3$	10	{3, 9}
2^4	$2^9 \cdot 3$	$2^8 \cdot 3$	$2^5 \cdot 3$	2^9	81	{3, 16}
$2^2 \cdot 3^2$	$2^6 \cdot 3^5$	$2^5 \cdot 3^5$	$2^4 \cdot 3^3$	$2^6 \cdot 3^4$	$2^9 \cdot 3^8 \cdot 5 + 1$	{3, 36}
2^6	$2^{15} \cdot 3$	$2^{14} \cdot 3$	$2^9 \cdot 3$	2^{15}	$2^8 \cdot 29 + 1$	{3, 64}
$2^4 \cdot 3^2$	$2^{12} \cdot 3^5$	$2^{11} \cdot 3^5$	$2^8 \cdot 3^3$	$2^{12} \cdot 3^4$	$2^7 \cdot 3^3 \cdot 23 + 1$	{3, 144}
$2^6 \cdot 3^2$	$2^{18} \cdot 3^5$	$2^{17} \cdot 3^5$	$2^{12} \cdot 3^3$	$2^{18} \cdot 3^4$	$2^{11} \cdot 3^3 \cdot 5 \cdot 19 + 1$	{3, 576}

The quadrilateral maps are provided in Table 2.

Table 2. The complete table of regular maps for N_2 .

N_2	D	E	V	F	g	Map (Schläfli)
2	2^3	2^2	2^2	2	0	{4, 2}
2^3	2^8	2^7	2^5	2^6	17	{4, 8}
$2 \cdot 3^2$	$2^5 \cdot 3^4$	$2^4 \cdot 3^4$	$2^4 \cdot 3^2$	$2^3 \cdot 3^4$	$2^2 \cdot 3^2 \cdot 7 + 1$	{4, 18}
2^5	2^{14}	2^{13}	2^9	2^{12}	$2^8 \cdot 7 + 1$	{4, 32}
$2^3 \cdot 3^2$	$2^{11} \cdot 3^4$	$2^{10} \cdot 3^4$	$2^8 \cdot 3^2$	$2^9 \cdot 3^4$	$2^7 \cdot 3^2 \cdot 17 + 1$	{4, 72}
2^7	2^{20}	2^{19}	2^{13}	2^{18}	$2^{12} \cdot 31 + 1$	{4, 128}
$2^5 \cdot 3^2$	$2^{17} \cdot 3^7$	$2^{16} \cdot 3^7$	$2^{12} \cdot 3^5$	$2^{15} \cdot 3^7$	$2^{11} \cdot 3^4 \cdot 71 + 1$	{4, 288}
$2^7 \cdot 3^2$	$2^{23} \cdot 3^4$	$2^{22} \cdot 3^4$	$2^{16} \cdot 3^2$	$2^{21} \cdot 3^4$	$2^{15} \cdot 3^2 \cdot 277 + 1$	{4, 1152}

The hexagonal maps are provided in Table 3.

Table 3. The complete table of regular maps for N_3 .

N_3	D	E	V	F	g	Map (Schläfli)
3	$2 \cdot 3^2$	3^2	$2 \cdot 3$	3	1	{6, 3}
$2^2 \cdot 3$	$2^5 \cdot 3^3$	$2^4 \cdot 3^3$	$2^3 \cdot 3^2$	$2^4 \cdot 3^2$	$2^2 \cdot 3^3 + 1$	{6, 12}
3^3	$2 \cdot 3^8$	3^8	$2 \cdot 3^5$	3^7	$3^5 \cdot 2^3 + 1$	{6, 27}
$2^4 \cdot 3$	$2^{11} \cdot 3^3$	$2^{10} \cdot 3^3$	$2^7 \cdot 3^2$	$2^{10} \cdot 3^2$	$2^6 \cdot 3^3 \cdot 5 + 1$	{6, 48}
$2^2 \cdot 3^3$	$2^5 \cdot 3^9$	$2^4 \cdot 3^9$	$2^3 \cdot 3^6$	$2^4 \cdot 3^8$	$2^2 \cdot 3^6 \cdot 11 \cdot 13 + 1$	{6, 108}
$2^6 \cdot 3$	$2^{17} \cdot 3^3$	$2^{16} \cdot 3^3$	$2^{11} \cdot 3^2$	$2^{16} \cdot 3^2$	$2^{10} \cdot 3^2 \cdot 31 + 1$	{6, 192}
$2^4 \cdot 3^3$	$2^{11} \cdot 3^9$	$2^{10} \cdot 3^9$	$2^7 \cdot 3^6$	$2^{10} \cdot 3^8$	$2^6 \cdot 3^6 \cdot 11 \cdot 13 + 1$	{6, 432}
$2^6 \cdot 3^3$	$2^{17} \cdot 3^9$	$2^{16} \cdot 3^9$	$2^{11} \cdot 3^6$	$2^{16} \cdot 3^8$	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 23 + 1$	{6, 1728}

6. Conclusions

In the present paper, we construct a method that reveals the structure of the maps modulo N corresponding to a modulo N subgroup of the normalizer. It is known that the structure of the normalizer is complicated compared to known modular groups. For this reason, the structure of the normalizer has been an active area of study for researchers in this field. When it comes to investigating regular maps, it is again interesting to investigate the normalizer as a universal map because of its variability. For instance, if one uses the modular group Γ as a universal map to form regular maps, the regular maps will be all triangular, but the normalizer admits triangular, quadrilateral, and hexagonal maps corresponding to its normal subgroups. In this article, we investigated triangular, quadrilateral, and hexagonal maps corresponding to a modulo N subgroup thanks to this rich structure. The results demonstrate that all of the resulting regular maps are of large genus except for the lower values of N . We formulated all of the elements of the regular maps, such as the number of vertices, edges, and darts, and the genus.

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