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Research Article

A Generalization of the Krätzel Function and Its Applications

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In this paper, we introduce new functions $Y^{\nu}_{\rho,r}(x)$ as a generalization of the Krätzel function. We investigate recurrence relations, Mellin transform, fractional derivatives, and integral of the function $Y^{\nu}_{\rho,r}(x)$. We show that the function $Y^{\nu}_{\rho,r}(x)$ is the solution of differential equations of fractional order.

1. Introduction

The Krätzel function is defined for $x > 0$ by the integral

$$Z^{\nu}_{\rho}(x) = \int_0^{\infty} t^{\nu-1} e^{-t^{\rho}-x/t}\,dt, \tag{1}$$

where $\rho \in \mathbb{R}$ and $\nu \in \mathbb{C}$, such that $\Re(\nu) < 0$ for $\rho \le 0$ (cf. [1]). For $\rho \ge 1$ the function (1) was introduced by Krätzel as a kernel of the integral transform as follows:

$$\left(\mathscr{K}^{\rho}_{\nu} f\right)(x) = \int_0^{\infty} Z^{\nu}_{\rho}(xt)\, f(t)\,dt \quad (x > 0). \tag{2}$$

The Krätzel function $Z^{\nu}_{\rho}(x)$ is related to the modified Bessel function of the second kind K_{ν} by the relationship

$$Z^{\nu}_{1}(x) = 2x^{\nu/2} K_{\nu}\left(2\sqrt{x}\right). \tag{3}$$

The generalized Krätzel function $D^{\nu,\alpha}_{\rho,r}(x)$ is given in [2, 3] by the following relation:

$$D^{\nu,\alpha}_{\rho,r}(x) = \int_0^{\infty} t^{\nu-1}\left[1 + a(\alpha-1)t^{\rho}\right]^{1/(\alpha-1)} e^{-xt^{-r}}\,dt, \tag{4}$$

where $\rho \in \mathbb{R}$, $r \in \mathbb{R}^{+}$, $\nu \in \mathbb{C}$, and $\alpha > 1$. Kilbas and Kumar considered the special case for $r = 1$ in [2], calculated fractional derivatives and fractional integrals of $D^{\nu,\alpha}_{\rho,1}(x)$, and obtained a representation using Wright hypergeometric functions. On the other hand the general case of (1) is given in [2, (54), p. 845].

We consider the generalized Krätzel function $Y^{\nu}_{\rho,r}(x)$ defined by the integral

$$Y^{\nu}_{\rho,r}(x) = \int_0^{\infty} t^{\nu-1} e^{-t^{\rho}-xt^{-r}}\,dt, \tag{5}$$

for $x > 0$, $\rho \in \mathbb{R}$, $r \in \mathbb{R}^{+}$, and $\nu \in \mathbb{C}$. The function $Y^{\nu}_{\rho,r}(x)$ is a generalization of the Krätzel function $Z^{\nu}_{\rho}(x)$ since

$$\lim_{r\to 1} Y^{\nu}_{\rho,r}(x) = Z^{\nu}_{\rho}(x). \tag{6}$$

If $a = 1$ in (4), then

$$\lim_{\alpha\to 1} D^{\nu,\alpha}_{\rho,r}(x) = Y^{\nu}_{\rho,r}(x). \tag{7}$$

We give some definitions and inequalities that will be needed. The Turán type inequalities

$$f_n(x)\cdot f_{n+2}(x) - \left[f_{n-1}(x)\right]^2 \ge 0, \quad n = 0, 1, 2, \ldots \tag{8}$$

are important and well known in many fields of mathematics (cf. [4]). A function $f(x)$ is completely monotonic on $(0, \infty)$, if f has derivatives of all orders and satisfies the inequality

$$(-1)^m f^{(m)}(x) \ge 0 \tag{9}$$

for all $x > 0$ and $m \in \mathbb{N}$ (cf. [5, Section IV]). A function $f(x)$ is said to be log-convex on $(0, \infty)$, if

$$f\left[\alpha x_1 + (1-\alpha) x_2\right] \le \left[f\left(x_1\right)\right]^{\alpha} \left[f\left(x_2\right)\right]^{1-\alpha} \tag{10}$$

for all $x_1, x_2 > 0$ and $\alpha \in [0, 1]$ (cf. [5, p. 167]).

Let $p, q \in \mathbb{R}$ such that $p > 1$ and $1/p + 1/q = 1$. If f and g are real valued functions defined on a closed interval and $|f|^p$, $|g|^q$ are integrable in this interval, then we have

$$\begin{aligned} &\int_a^b \left|f(t)\, g(t)\right| dt \\ &\quad \le \left[\int_a^b \left|f(t)\right|^p dt\right]^{1/p} \left[\int_a^b \left|g(t)\right|^q dt\right]^{1/q}. \end{aligned} \tag{11}$$

The following inequality is due to Mitrinović et al. (cf. [6, p. 239]). Let f and g be two functions which are integrable and monotonic in the same sense on $[a, b]$ and p is a positive and integrable function on the same interval, then the following inequality holds true:

$$\begin{aligned} &\int_a^b p(t)\, f(t)\, dt \int_a^b p(t)\, g(t)\, dt \\ &\quad \le \int_a^b p(t)\, dt \int_a^b p(t)\, f(t)\, g(t)\, dt, \end{aligned} \tag{12}$$

if and only if one of the functions f and g reduces to a constant.

The Mellin transform of the function f is defined by

$$\mathscr{M}\{f(x); s\} = \int_0^\infty x^{s-1} f(x)\, dx \tag{13}$$

when $\mathscr{M}\{f(x); s\}$ exists. The Mellin transform of the generalized Krätzel function (5) is given by Kilbas and Kumar in [2].

The Laplace transform of the function f is defined by

$$\mathscr{L}\{f(x); s\} = \int_0^\infty e^{-sx} f(x)\, dx \tag{14}$$

provided that the integral on the right-hand side exists.

The Liouville fractional integral is defined by

$$\left(\mathscr{I}_-^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\, dt \tag{15}$$

and its derivatives $\mathscr{I}_-^{\alpha}$ and $\mathscr{D}_-^{\alpha}$ are

$$\begin{aligned} \left(\mathscr{D}_-^{\alpha} f\right)(x) &= \left(-\frac{d}{dx}\right)^{[\Re(\alpha)]+1} \left(\mathscr{I}_-^{1-\alpha+[\Re(\alpha)]} f\right)(x) \\ &= \frac{1}{\Gamma\left(1-\alpha+[\Re(\alpha)]\right)} \int_x^\infty (t-x)^{-\alpha+[\Re(\alpha)]} f(t)\, dt, \end{aligned} \tag{16}$$

where $x > 0$, $\alpha \in \mathbb{C}$, and $\Re(\alpha) > 0$ (cf. [7, Section 5.1]).

We introduce new operators

$$\mathscr{L}_{\lambda}^{\nu} \coloneqq -rx\mathscr{D}_-^{\lambda+1} + (\lambda r - \nu)\, \mathscr{D}_-^{\lambda}, \tag{17}$$

$$\begin{aligned} \mathscr{T}_{\lambda}^{\nu} \coloneqq{}& r^2 x^2 \mathscr{D}_-^{2\lambda+2} + rx\left(2\nu - 3\lambda r - r\right) \mathscr{D}_-^{2\lambda+1} \\ &+ (\nu - r\lambda)(\nu - 2r\lambda)\, \mathscr{D}_-^{2\lambda}, \end{aligned} \tag{18}$$

where $\nu \in \mathbb{C}$ and $\lambda > 0$.

A standard source in the theory of fractional calculus is the book [8]. For applications of fractional calculus to science and engineering, we refer the reader to the articles [9–11].

In this paper, we investigate the properties of the functions $Y_{\rho,r}^{\nu}(x)$ and prove their composition of $Y_{\rho,r}^{\nu}(x)$ with fractional integral and derivatives $(\mathscr{I}_-^{\alpha} f)(x)$, $(\mathscr{D}_-^{\alpha} f)(x)$ given by (15) and (16) (cf. [2, 6, 12, 13]). In Section 3, we show that $Y_{\rho,r}^{\nu}(x)$ is the solution of differential equations of fractional order.

2. The Main Theorems

In this section, we will give some properties of generalized Krätzel functions $Y_{\rho,r}^{\nu}$.

Lemma 1. *Let $\rho \in \mathbb{R}$ $(\rho \ne 0)$, $r \in \mathbb{R}^+$, $\nu \in \mathbb{C}$, $\Re(s) > 0$ be such that $\Re(\nu + rs) > 0$ when $\rho > 0$ and $\Re(\nu + rs) < 0$ when $\rho < 0$. The Mellin transform of the function $Y_{\rho,r}^{\nu}(x)$ is given by*

$$\mathscr{M}\left\{Y_{\rho,r}^{\nu}; s\right\} = \frac{1}{|\rho|} \Gamma(s)\, \Gamma\left(\frac{\nu + rs}{\rho}\right) \quad x > 0. \tag{19}$$

Proof. Using (13) and (5), we have

$$\mathscr{M}\left\{Y_{\rho,r}^{\nu}; s\right\} = \int_0^\infty x^{s-1} \left(\int_0^\infty t^{\nu-1} e^{-t^{\rho} - xt^{-r}}\, dt\right) dx. \tag{20}$$

Changing the order of integration and using the substitution of $xt^{-r} = u$, we have

$$\begin{aligned} \mathscr{M}\left\{Y_{\rho,r}^{\nu}; s\right\} &= \int_0^\infty t^{\nu-1} e^{-t^{\rho}} \left(\int_0^\infty x^{s-1} e^{-xt^{-r}}\, dx\right) dt \\ &= \Gamma(s) \int_0^\infty t^{\nu+rs-1} e^{-t^{\rho}}\, dt. \end{aligned} \tag{21}$$

Making the change of variable the integral $t^{\rho} = z$, and using the known formula (1) from [14, p. 145], we find that

$$\begin{aligned} \mathscr{M}\left\{Y_{\rho,r}^{\nu}; s\right\} &= \frac{\Gamma(s)}{\rho} \int_0^\infty z^{(\nu+rs)/\rho-1} e^{-z}\, dz \\ &= \frac{\Gamma(s)}{\rho} \Gamma\left(\frac{\nu + rs}{\rho}\right), \end{aligned} \tag{22}$$

when $\rho > 0$ and

$$\begin{aligned} \mathscr{M}\left\{Y_{\rho,r}^{\nu}; s\right\} &= \frac{\Gamma(s)}{\rho} \int_0^\infty z^{(\nu+rs)/\rho-1} e^{-z}\, dz \\ &= -\frac{\Gamma(s)}{\rho} \Gamma\left(\frac{\nu + rs}{\rho}\right), \end{aligned} \tag{23}$$

when $\rho < 0$. □

Theorem 2. *We have the following relationship for the function $Y_{\rho,r}^{\nu}(x)$:*

$$Y_{\rho,r}^{\nu}(x) = L\left\{\frac{1}{r} t^{-\nu/r-1} e^{-t^{-\rho/r}}; x\right\}, \tag{24}$$

where $\rho \in \mathbb{R}$, $r \in \mathbb{R}^+$, $\nu \in \mathbb{C}$, and $x > 0$.

Proof. Using (5) and making the change of $t^{-r} = u$, we obtain

$$\begin{aligned} Y_{\rho,r}^{\nu}(x) &= \int_0^\infty t^{\nu-1} e^{-t^\rho - xt^{-r}} dt \\ &= \int_0^\infty \frac{1}{r} u^{-\nu/r-1} e^{-u^{-\rho/r}} e^{-xu} du. \end{aligned} \tag{25}$$

Now the assertion (24) follows from the definition (14) of the Laplace transform. □

Using the known formula (29) from [14, p. 146], we find that

$$\begin{aligned} Y_{\rho,\rho}^{\nu}(x) &= \frac{1}{\rho} \mathscr{L} \left\{ t^{-\nu/r-1} e^{-t^{-\rho/r}}; x \right\} \\ &= \frac{2}{\rho} x^{\nu/2\rho} K_{\nu/\rho}\left(2\sqrt{x}\right), \end{aligned} \tag{26}$$

for $\rho = 1$:

$$\begin{aligned} Y_{1,1}^{\nu}(x) &= Z_1^{\nu}(x) = 2x^{\nu/2} K_\nu\left(2\sqrt{x}\right), \\ Y_{1/r,1/r}^{\nu}(x) &= 2rx^{\nu/2} K_\nu\left(2\sqrt{x}\right). \end{aligned} \tag{27}$$

Theorem 3. *If $\rho \in \mathbb{R}$, $r \in \mathbb{R}^+$, $\nu \in \mathbb{C}$ and $x > 0$, then the following assertions are true:*

(a) *The function $Y_{\rho,r}^{\nu}(x)$ satisfies the recurrence relation*

$$\nu Y_{\rho,r}^{\nu}(x) = \rho Y_{\rho,r}^{\nu+\rho}(x) - rxY_{\rho,r}^{\nu-r}(x). \tag{28}$$

(b) *The function $x \to Y_{\rho,r}^{\nu}(x)$ is completely monotonic on $(0,\infty)$.*

Proof. (a) The above recurrence relation could be verified by using integration by parts as follows:

$$\begin{aligned} Y_{\rho,r}^{\nu}(x) &= \frac{1}{\nu} t^\nu e^{-t^\rho - xt^{-r}} \Big|_0^\infty \\ &\quad - \int_0^\infty \frac{1}{\nu} t^\nu \left[-\rho t^{\rho-1} + rxt^{-r-1} \right] e^{-t^\rho - xt^{-r}} dt \\ &= \frac{\rho}{\nu} \int_0^\infty t^{\nu+\rho-1} e^{-t^\rho - xt^{-r}} dt \\ &\quad - \frac{rx}{\nu} \int_0^\infty t^{\nu-r-1} e^{-t^\rho - xt^{-r}} dt \\ &= \frac{\rho}{\nu} Y_{\rho,r}^{\nu+\rho}(x) - \frac{rx}{\nu} Y_{\rho,r}^{\nu-r}(x). \end{aligned} \tag{29}$$

(b) From Bernstein-Widder theorem (see Theorem 1, [5, p. 145]), the function $Y_{\rho,r}^{\nu}(x)$ is completely monotonic on $(0,\infty)$ for all $x > 0$. This could be verified directly as follows:

$$\frac{d^n}{dx^n} Y_{\rho,r}^{\nu}(x) = (-1)^{n-1} Y_{\rho,r}^{\nu-nr}(x) > 0 \qquad (n = 0, 1, 2, \ldots), \tag{30}$$

which follows via mathematical induction from (5) provided that $\rho \in \mathbb{R}$, $r \in \mathbb{R}^+$, $\nu \in \mathbb{C}$ and $x > 0$. From Bernstein-Widder theorem, generalized forms of Krätzel function are completely monotonic on $(0,\infty)$ for all $x > 0$. Due to (30), the functions are completely monotonic on $(0,\infty)$ for all $x > 0$. □

Setting $r \to 1$ and using (28), the equation yields

$$Y_{\rho,1}^{\nu}(x) = \frac{\rho}{\nu} Y_{\rho,1}^{\nu+\rho}(x) - \frac{x}{\nu} Y_{\rho,1}^{\nu-1}(x). \tag{31}$$

Then using (31) and (6), we obtain the relation

$$\nu Z_\rho^{\nu}(x) = \rho Z_\rho^{\nu+\rho}(x) - xZ_\rho^{\nu-1}(x), \tag{32}$$

(cf. 2.1 of Theorem 1 from [12]).

Theorem 4. *Let $\nu_1, \nu_2, \rho \in \mathbb{R}$, $0 < \lambda < 1$, and $x > 0$, then the following assertions hold true:*

(a) *The function $\nu \to Y_{\rho,r}^{\nu}(x)$ is log-convex on $\mathbb{R}$:*

$$Y_{\rho,r}^{\lambda\nu_1+(1-\lambda)\nu_2,\alpha}(x) \le \left[Y_{\rho,r}^{\nu_1}(x) \right]^\lambda \left[Y_{\rho,r}^{\nu_2}(x) \right]^{1-\lambda}. \tag{33}$$

(b) *The function $x \to Y_{\rho,r}^{\nu}(x)$ is log-convex on $(0,\infty)$:*

$$Y_{\rho,r}^{\nu}(\lambda x_1 + (1-\lambda) x_2) \le \left[Y_{\rho,r}^{\nu}(x_1) \right]^\lambda \left[Y_{\rho,r}^{\nu}(x_2) \right]^{1-\lambda}. \tag{34}$$

(c) *The function $Y_{\rho,r}^{\nu}(x)$ satisfies the following relation:*

$$Y_{\rho,r}^{\nu}(t^r) = t^\nu Y_{r,\rho}^{-\nu}(t^\rho). \tag{35}$$

Proof. (a) Using (5) and (11), we obtain

$$\begin{aligned} Y_{\rho,r}^{\lambda\nu_1+(1-\lambda)\nu_2}(x) &= \int_0^\infty t^{\lambda\nu_1+(1-\lambda)\nu_2-1} e^{-t^\rho - xt^{-r}} dt \\ &= \int_0^\infty \left(t^{\nu_1-1} e^{-t^\rho} e^{-xt^{-r}} \right)^\lambda \left(t^{\nu_2-1} e^{-t^\rho} e^{-xt^{-r}} \right)^{1-\lambda} dt \\ &\le \left[\int_0^\infty t^{\nu_1-1} e^{-t^\rho - xt^{-r}} dt \right]^\lambda \left[\int_0^\infty t^{\nu_2-1} e^{-t^\rho - xt^{-r}} dt \right]^{1-\lambda} \\ &= \left[Y_{\rho,r}^{\nu_1}(x) \right]^\lambda \left[Y_{\rho,r}^{\nu_2}(x) \right]^{1-\lambda}, \end{aligned} \tag{36}$$

where $\lambda \in [0,1]$, $\nu_1, \nu_2, \rho \in \mathbb{R}$, $\alpha > 1$, and $x > 0$. Thus, $\nu \to Y_{\rho,r}^{\nu}(x)$ is log-convex on $\mathbb{R}$.

(b) The integrand in (5) is a log-linear convex function of x. By using (11), we have

$$\begin{aligned} &Y_{\rho,r}^{\nu}(\lambda x_1 + (1-\lambda) x_2) \\ &= \int_0^\infty t^{\nu-1} e^{-t^\rho} e^{-(\lambda x_1 + (1-\lambda) x_2) t^{-r}} dt \\ &= \int_0^\infty \left[t^{\nu-1} e^{-t^\rho} e^{-x_1 t^{-r}} \right]^\lambda \left[t^{\nu-1} e^{-t^\rho} e^{-x_2 t^{-r}} \right]^{1-\lambda} dt \\ &\le \left[\int_0^\infty t^{\nu-1} e^{-t^\rho} e^{-x_1 t^{-r}} dt \right]^\lambda \left[\int_0^\infty t^{\nu-1} e^{-t^\rho} e^{-x_2 t^{-r}} dt \right]^{1-\lambda} \\ &= \left[Y_{\rho,r}^{\nu}(x_1) \right]^\lambda \left[Y_{\rho,r}^{\nu}(x_2) \right]^{1-\lambda}, \end{aligned} \tag{37}$$

where $\lambda \in [0,1]$, $\nu, \rho \in \mathbb{R}$, $r > 0$, and $x_1, x_2 > 0$. Thus, $x \to Y^{\nu}_{\rho,r}(x)$ is log-convex on $(0,\infty)$.

(c) Again using (5), we conclude that

$$\begin{aligned} Y^{\nu}_{\rho,r}(x) &= \int_0^{\infty} t^{\nu-1} e^{-t^{\rho}-xt^{-r}} dt, \quad \left(t = x^{1/r}u^{-1}\right) \\ &= \int_0^{\infty} \left(x^{1/r}u^{-1}\right)^{\nu-1} e^{-(x^{1/r}u^{-1})^{\rho}} e^{-x(x^{1/r}u^{-1})^{-r}} x^{1/r}u^{-2} du \\ &= x^{\nu/r} \int_0^{\infty} u^{-\nu-1} e^{-u^{r}-x^{\rho/r}u^{-\rho}} du \\ &= x^{\nu/r} Y^{-\nu}_{r,\rho}\left(x^{\rho/r}\right) \end{aligned} \tag{38}$$

or for the change of $x = t^{r}$, we obtain (35).

Moreover, since $Y^{\nu}_{\rho,r}(x)$ is log-convex on $\mathbb{R}$, we have Turán type inequality

$$\left[Y^{(\nu_1+\nu_2)/2}_{\rho,r}(x)\right]^2 \leq Y^{\nu_1}_{\rho,r}(x) Y^{\nu_2}_{\rho,r}(x) \tag{39}$$

for $\nu_1, \nu_2, \rho \in \mathbb{R}$, $\alpha > 1$, and $x > 0$. Making the change of variable $\nu_1 = \nu - 2$ and $\nu_2 = \nu$, the equation yields

$$f^{\nu,\alpha}_{\rho}(x) = \left[Y^{\nu-1}_{\rho,r}(x)\right]^2 - Y^{\nu-2}_{\rho,r}(x) Y^{\nu}_{\rho,r}(x) \leq 0 \tag{40}$$

which is valid for $\nu, \rho \in \mathbb{R}$, $\alpha > 1$, and $x > 0$.

Using (39) and making the change of variables $\nu_1 = \nu - n - 1$ and $\nu_2 = \nu - n + 1$, we have

$$\left[Y^{\nu-n}_{\rho,r}(x)\right]^2 \leq Y^{\nu-n-1}_{\rho,r}(x) Y^{\nu-n+1}_{\rho,r}(x). \tag{41}$$

□

Theorem 5. *If $\nu, \rho \in \mathbb{R}$, $r \in \mathbb{R}^{+}$ and $x > 0$, then the following inequality holds true:*

$$\Gamma\left(\frac{\nu}{r}\right) Y^{-1/r}_{\rho/r,1}(x) \leq x^{(\nu-1)/r} \Gamma\left(\frac{1}{r}\right) Y^{-\nu}_{\rho,r}(x). \tag{42}$$

Proof. Let $p(t) = e^{-xt^{r}}$, $f(t) = t^{\nu-1}$ and $g(t) = e^{-t^{-\rho}}$. The function $f(t)$ is increasing on $(0,\infty)$ for $\nu \geq 1$ and is decreasing for $\nu \leq 1$. On the other hand, we observe that, for all $\rho > 0$,

$$\frac{g'(t)}{g(t)} = \rho t^{-\rho-1} > 0. \tag{43}$$

Thus, $g(t)$ is increasing if and only if $\rho > 0$. Moreover, making the change of $t^{r} = u$ and using the known formula (1) from [14, p. 137], we have

$$\int_0^{\infty} p(t)\, dt = \int_0^{\infty} e^{-xt^{r}} dt = \frac{1}{r}\Gamma\left(\frac{1}{r}\right) x^{-1/r}. \tag{44}$$

Making the change of $t^{-r} = u$, we find

$$\int_0^{\infty} p(t) f(t)\, dt = \int_0^{\infty} e^{-xt^{r}} t^{\nu-1} dt = \frac{1}{r}\Gamma\left(\frac{\nu}{r}\right) x^{-\nu/r}. \tag{45}$$

Making the change of variable $t = u^{-1/r}$ and using (6), we have

$$\begin{aligned} \int_0^{\infty} p(t) g(t)\, dt &= \int_0^{\infty} e^{-xt^{r}} e^{-t^{-\rho}} dt \\ &= \frac{1}{r} \int_0^{\infty} u^{-1/r-1} e^{-u^{\rho/r}} e^{-x/u} du \\ &= \frac{1}{r} Y^{-1/r}_{\rho/r,1}(x). \end{aligned} \tag{46}$$

Using (5) and making the change of variable $t = u^{-1}$, we find

$$\begin{aligned} \int_0^{\infty} p(t) f(t) g(t)\, dt &= \int_0^{\infty} e^{-xt^{r}} t^{\nu-1} e^{-t^{-\rho}} dt \\ &= \int_0^{\infty} u^{-\nu-1} e^{-u^{\rho}-xu^{-r}} du = Y^{-\nu}_{\rho,r}(x). \end{aligned} \tag{47}$$

Finally, by using the relation (12), we obtain the inequality (42):

$$\begin{aligned} &\int_0^{\infty} p(t) f(t)\, dt \int_0^{\infty} p(t) g(t)\, dt \\ &\quad \leq \int_0^{\infty} p(t)\, dt \int_0^{\infty} p(t) f(t) g(t)\, dt \frac{1}{r}\Gamma\left(\frac{\nu}{r}\right) x^{-\nu/r} \\ &\quad \cdot \frac{1}{r} Y^{-1/r}_{\rho/r,1}(x) \\ &\quad \leq \frac{1}{r}\Gamma\left(\frac{1}{r}\right) x^{-1/r} \cdot Y^{-\nu}_{\rho,r}(x) x^{(\nu-1)/r} \Gamma\left(\frac{1}{r}\right) Y^{-\nu}_{\rho,r}(x) \\ &\quad \geq \Gamma\left(\frac{\nu}{r}\right) Y^{-1/r}_{\rho/r,1}(x). \end{aligned} \tag{48}$$

□

If we choose $r \to 1$ in (42), then we have

$$x^{\nu-1} Y^{-\nu}_{\rho,1}(x) \geq \Gamma(\nu) Y^{-1}_{\rho,1}(x). \tag{49}$$

As a result, we find the following inequality by using (6):

$$Z^{-\nu}_{\rho}(x) \geq x^{1-\nu} \Gamma(\nu) Z^{-1}_{\rho}(x). \tag{50}$$

3. Differential Equations of Fractional Order

In this section, we show that $Y^{\nu}_{\rho,r}(x)$ is the solution of differential equations of fractional order.

Theorem 6. *If $\alpha, \nu \in \mathbb{C}$, $\Re(\alpha) > 0$, and $\rho > 0$, then the following identity holds true:*

$$\left(\mathcal{I}^{\alpha}_{-} Y^{\nu}_{\rho,r}\right)(x) = Y^{\nu+r\alpha}_{\rho,r}(x). \tag{51}$$

Proof. Applying (15), (5), and relation (11) of [15, p. 202], we obtain

$$\begin{aligned} &\left(\mathcal{I}^{\alpha}_{-} Y^{\nu}_{\rho,r}\right)(x) \\ &\quad = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} dt \int_0^{\infty} u^{\nu-1} e^{-u^{\rho}-tu^{-r}} du \\ &\quad = \int_0^{\infty} u^{\nu-1} e^{-u^{\rho}} \left(\frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} e^{-tu^{-r}} dt\right) du \end{aligned}$$

$$\begin{aligned} &= \int_0^\infty u^{\nu-1} e^{-u^\rho} \left(\mathscr{I}_-^\alpha e^{-tu^{-r}} \right)(x)\, du \\ &= \int_0^\infty u^{\nu-1} e^{-u^\rho} e^{-xu^{-r}} u^{r\alpha} du \\ &= \int_0^\infty u^{\nu+r\alpha-1} e^{-u^\rho} e^{-xu^{-r}} du = Y_{\rho,r}^{\nu+r\alpha}(x). \end{aligned} \tag{52}$$

□

Theorem 7. *If* $\alpha, \nu \in \mathbb{C}$, $\Re(\alpha) > 0$, *and* $\rho > 0$ *then we have*

$$\left(\mathscr{D}_-^\alpha Y_{\rho,r}^\nu \right)(x) = Y_{\rho,r}^{\nu-r\alpha}(x). \tag{53}$$

Proof. Using (16), (5), and (51), we obtain

$$\begin{aligned} \left(\mathscr{D}_-^\alpha Y_{\rho,r}^\nu \right)(x) &= \left(-\frac{d}{dx} \right)^{[\Re(\alpha)]+1} \left(\mathscr{I}_-^{1-\alpha+[\Re(\alpha)]} Y_{\rho,r}^\nu \right)(x) \\ &= \left(-\frac{d}{dx} \right)^{[\Re(\alpha)]+1} Y_{\rho,r}^{\nu+r(1-\alpha+[\Re(\alpha)])}(x) \\ &= \left(-\frac{d}{dx} \right)^{[\Re(\alpha)]+1} \int_0^\infty t^{\nu+r(1-\alpha+[\Re(\alpha)])-1} e^{-t^\rho-xt^{-r}} dt \\ &= \int_0^\infty t^{\nu+r(1-\alpha+[\Re(\alpha)])-1} e^{-t^\rho} \left(-\frac{d}{dx} \right)^{[\Re(\alpha)]+1} \\ &\cdot \left(e^{-xt^{-r}} \right) dt = \int_0^\infty t^{\nu+r(1-\alpha+[\Re(\alpha)])-1} e^{-t^\rho} \\ &\cdot \frac{1}{t^{r([\Re(\alpha)]+1)}} e^{-xt^{-r}} dt = \int_0^\infty t^{\nu-\alpha r-1} e^{-t^\rho} e^{-xt^{-r}} dt \\ &= Y_{\rho,r}^{\nu-r\alpha}(x). \end{aligned} \tag{54}$$

□

Corollary 8. *If* α, β, *and* $\nu \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, *and* $\rho > 0$, *then we have*

$$\begin{aligned} \left(\mathscr{D}_-^\alpha \mathscr{I}_-^\beta Y_{\rho,r}^\nu \right)(x) &= \left(\mathscr{I}_-^\beta \mathscr{D}_-^\alpha Y_{\rho,r}^\nu \right)(x) \\ &= Y_{\rho,r}^{\nu+\beta-\alpha+(1-r)(1+[\mathrm{Re}(\alpha)])}. \end{aligned} \tag{55}$$

Theorem 9. *If* $\nu \in \mathbb{C}$ *and* $\rho > 0$, *then the following identity holds true:*

$$\mathscr{L}_\rho^\nu Y_{\rho,r}^\nu(x) = -\rho Y_{\rho,r}^{\nu+(1-r)\rho}(x). \tag{56}$$

Proof. Applying (17) to (5), we get

$$\begin{aligned} \mathscr{L}_\rho^\nu Y_{\rho,r}^\nu(x) &= -rx \mathscr{D}_-^{\rho+1} Y_{\rho,r}^\nu(x) + (\rho r - \nu) \mathscr{D}_-^\rho Y_{\rho,r}^\nu(x) \\ &= -rx Y_{\rho,r}^{\nu-(\rho+1)r}(x) + (\rho r - \nu) Y_{\rho,r}^{\nu-\rho r}(x) \\ &= -rx \int_0^\infty t^{\nu-(\rho+1)r-1} e^{-t^\rho-xt^{-r}} dt \end{aligned}$$

$$\begin{aligned} &+ (\rho r - \nu) \int_0^\infty t^{\nu-\rho r-1} e^{-t^\rho-xt^{-r}} dt \\ &= -\int_0^\infty t^{\nu-\rho r-1} \left(r\frac{x}{t^r} + \nu - \rho r \right) e^{-t^\rho-xt^{-r}} dt. \end{aligned} \tag{57}$$

Using the formula

$$\left(t^{\nu-\rho r} e^{-xt^{-r}} \right)' = t^{\nu-\rho r-1} \left(\nu - \rho r + xrt^{-r} \right) e^{-xt^{-r}} \tag{58}$$

and applying the integration by parts, we find

$$\begin{aligned} \mathscr{L}_\rho^\nu Y_{\rho,r}^\nu(x) &= -\int_0^\infty \left(t^{\nu-\rho r} e^{-xt^{-r}} \right)' e^{-t^\rho} dt \\ &= -\, t^{\nu-\rho r} e^{-xt^{-r}} e^{-t^\rho} \Big|_0^\infty \\ &+ \int_0^\infty t^{\nu-\rho r} e^{-xt^{-r}} \left(-\rho t^{\rho-1} e^{-t^\rho} \right) dt \\ &= 0 - \rho \int_0^\infty t^{\nu-\rho r+\rho-1} e^{-xt^{-r}} e^{-t^\rho} dt \\ &= -\rho Y_{\rho,r}^{\nu+(1-r)\rho}(x). \end{aligned} \tag{59}$$

□

Corollary 10. *If* $\nu \in \mathbb{C}$ *and* $\rho > 0$, *then the function* $Y_{\rho,r}^\nu(x)$ *is a solution of the differential equation of fractional order*

$$\begin{aligned} rx \mathscr{D}_-^{\rho+1} Y_{\rho,r}^\nu(x) + (\nu - \rho r) \mathscr{D}_-^\rho Y_{\rho,r}^\nu(x) \\ - \rho Y_{\rho,r}^{\nu+(1-r)\rho}(x) = 0. \end{aligned} \tag{60}$$

Remark 11. If $\nu \in \mathbb{C}$, and $\rho = r = 1$, then the function $Y_{1,1}^\nu(x) = Z_1^\nu(x)$ is a solution of the following differential equation:

$$xy'' + (\nu - 1) y' - y = 0 \tag{61}$$

(cf. [13, (30), p. 20]).

Theorem 12. *If* $\nu \in \mathbb{C}$ *and* $\rho > 0$, *then the function* $Y_{\rho,r}^\nu(x)$ *is a solution of the differential equation of fractional order*

$$\begin{aligned} \left(\mathscr{T}_\rho^\nu Y_{\rho,r}^\nu \right)(x) + \rho^2 Y_{\rho,r}^{\nu-2(r-1)\rho}(x) \\ - \rho^2 (r-1) Y_{\rho,r}^{\nu-(2r-1)\rho}(x) = 0. \end{aligned} \tag{62}$$

Proof. Using (18), (5), and (53), we get

$$\begin{aligned} \left(\mathscr{T}_\rho^\nu Y_{\rho,r}^\nu \right)(x) &= \int_0^\infty \left\{ \frac{r^2 x^2}{t^{2r}} + (2\nu - 3\rho r - r) \frac{rx}{t^r} \right\} \\ &\cdot t^{\nu-2r\rho-1} e^{-xt^{-r}} e^{-t^\rho} dt + \int_0^\infty (\nu - r\rho)(\nu - 2r\rho) \\ &\cdot t^{\nu-2r\rho-1} e^{-xt^{-r}} e^{-t^\rho} dt. \end{aligned} \tag{63}$$

If we take the derivative as the proof of Theorem 9, then we arrive at

$$\begin{aligned}\left(\left(\nu - r\rho + \frac{rx}{t^r}\right) t^{\nu-2r\rho} e^{-xt^{-r}}\right)' &= \left\{\frac{r^2x^2}{t^{2r}}\right.\\ &\quad \left. + \left(2\nu - 3\rho r - r\right)\frac{rx}{t^r} + \left(\nu - r\rho\right)\left(\nu - 2r\rho\right)\right\}\\ &\quad \cdot t^{\nu-2r\rho-1} e^{-xt^{-r}}.\end{aligned} \tag{64}$$

Substituting (64), (16), into (63) and applying the integration by parts, we get

$$\begin{aligned}&\left(\mathcal{T}_\rho^\nu Y_{\rho,r}^\nu\right)(x)\\ &= \int_0^\infty \left(\left(\nu - r\rho + \frac{rx}{t^r}\right) t^{\nu-2r\rho} e^{-xt^{-r}}\right)' \cdot e^{-t^\rho} dt\\ &= \left(\nu - r\rho + \frac{rx}{t^r}\right) t^{\nu-2r\rho} e^{-xt^{-r}} \cdot e^{-t^\rho}\Big|_0^\infty\\ &\quad + \rho\int_0^\infty \left(\nu - r\rho + \frac{rx}{t^r}\right) t^{\nu-2r\rho+\rho-1} e^{-xt^{-r}} e^{-t^\rho} dt\\ &= \rho\int_0^\infty \left(\nu - r\rho + \frac{rx}{t^r}\right) t^{\nu-(2r-1)\rho-1} e^{-xt^{-r}} e^{-t^\rho} dt.\end{aligned} \tag{65}$$

If we rewrite the expression in (65) relation as

$$\nu - r\rho + \frac{rx}{t^r} = \nu - (2r-1)\rho + \frac{rx}{t^r} + (r-1)\rho, \tag{66}$$

then we have

$$\begin{aligned}\left(\mathcal{T}_\rho^\nu Y_{\rho,r}^\nu\right)(x) &= \rho\int_0^\infty \left(\nu - (2r-1)\rho + \frac{rx}{t^r}\right)\\ &\quad \cdot t^{\nu-(2r-1)\rho-1} e^{-xt^{-r}} e^{-t^\rho} dt + \rho^2 (r-1)\\ &\quad \cdot \int_0^\infty t^{\nu-(2r-1)\rho-1} e^{-xt^{-r}} e^{-t^\rho} dt.\end{aligned} \tag{67}$$

If we evaluate the integral on the right-hand side of relation (64) and apply the integration by parts, we arrive at (62) as follows:

$$\begin{aligned}\left(\mathcal{T}_\rho^\nu Y_{\rho,r}^\nu\right)(x) &= \rho\int_0^\infty \left(t^{\nu-(2r-1)\rho} e^{-xt^{-r}}\right)' e^{-t^\rho} dt\\ &\quad + \rho^2 (r-1) Y_{\rho,r}^{\nu-(2r-1)\rho}\\ &= \rho \cdot t^{\nu-(2r-1)\rho} e^{-xt^{-r}} e^{-t^\rho}\Big|_0^\infty\\ &\quad - \rho^2 \int_0^\infty t^{\nu-2(r-1)\rho-1} e^{-xt^{-r}} e^{-t^\rho} dt\\ &\quad + \rho^2 (r-1) Y_{\rho,r}^{\nu-(2r-1)\rho}\\ &= -\rho^2 Y_{\rho,r}^{\nu-2(r-1)\rho} + \rho^2 (r-1) Y_{\rho,r}^{\nu-(2r-1)\rho},\end{aligned} \tag{68}$$

where

$$\begin{aligned}&\left(t^{\nu-(2r-1)\rho} e^{-xt^{-r}}\right)'\\ &\quad = \left(\nu - (2r-1)\rho + \frac{rx}{t^r}\right) t^{\nu-(2r-1)\rho-1} e^{-xt^{-r}}.\end{aligned} \tag{69}$$

□

Remark 13. If $\nu \in \mathbb{C}$ and $\rho = 1$, then the function $Y_{1,1}^\nu(x) = Z_1^\nu(x)$ is a solution of the differential equation of fourth order

$$x^2 y^{(IV)} + (2\nu - 4) xy''' + (\nu - 1)(\nu - 2) y'' + y = 0 \tag{70}$$

(cf. [13, p. 21]).

4. Conclusion

Mejer's G functions, which are generalization of hypergeometric functions, are Mellin-Barnes integrals. Generalized Krätzel functions, $Y_{\rho,r}^\nu(x)$ could be written in terms of H-functions, which are generalization of G-function, as a Mellin-Barnes integral. Furthermore, the integral transform with the kernel $Y_{\rho,r}^\nu(x)$ could be investigated.

Competing Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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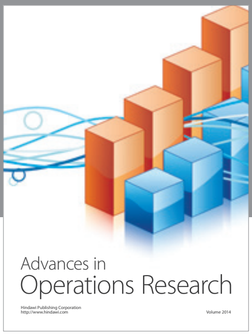
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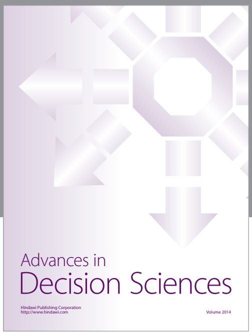
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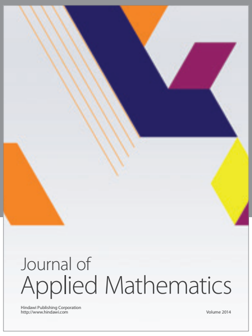
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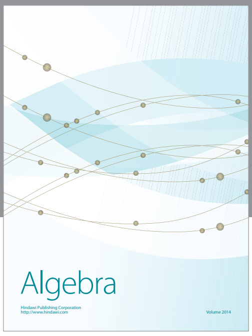
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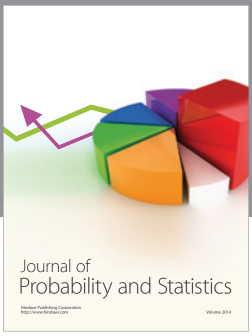
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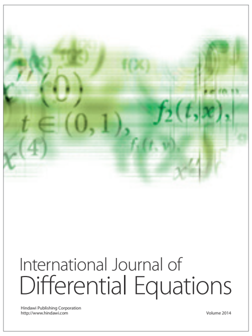
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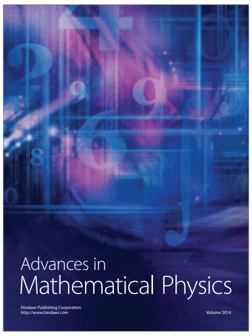
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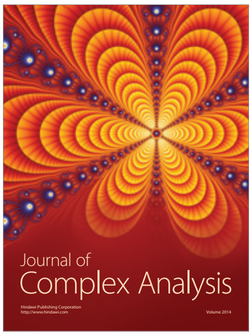
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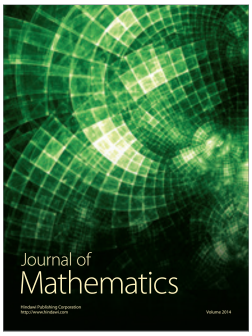
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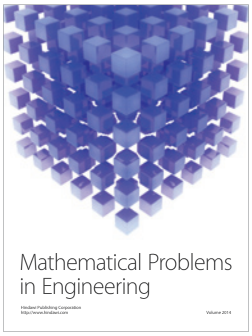
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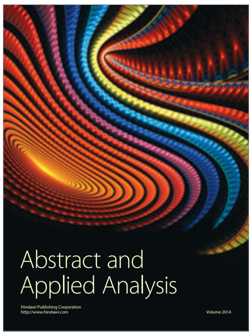
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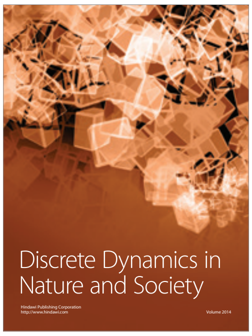
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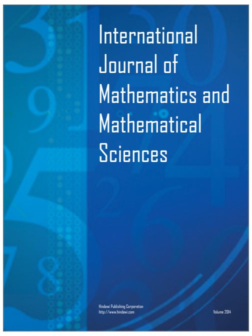
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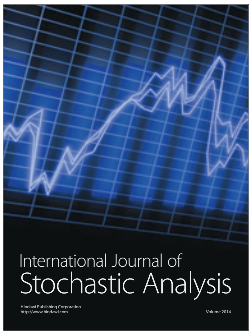
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