Available online at www.sciencedirect.com
ScienceDirect
Procedia - Social and Behavioral Sciences 152 (2014) 563 – 568
Procedia
Social and Behavioral Sciences

ERPA 2014

University students' solution processes in systems of linear equation

Deniz Kardeş Birinci[a], Ali Delice[b*], Emin Aydın[b]

[a]*Department of Mathematics Education, Fatih University, Buyukcekmece Campus,Buyukcekmece, Istanbul-34500, Turkey*
[b]*Department of Mathematics Education, Marmara University, Goztepe Campus, Kadikoy, Istanbul-34722, Turkey*

Abstract

The focus of the study is system of linear equations which is one of the important topics of linear algebra. The aim is to investigate how university students' skills in the process of solving systems of linear equations by focusing on how they perform operations with matrices. This study is a case study based on a non-positivist paradigm with interpretivist approach. Findings indicate that candidates face differing levels difficulties in conceptualizing and doing operations in their solutions.

Peer-review under responsibility of the Organizing Committee of the ERPA Congress 2014.
Keywords: systems of linear equation; matrix; solution process

1. Introduction

Linear algebra is one of the areas of mathematics in which students face many learning difficulties (Hillel and Sierpinska, 1993; Dorier and Sierpinska, 2001). Hence it is important to seek answers to questions such as how it is learnt and taught. Concepts and procedures contained in linear algebra are widely in many other areas of mathematics including algebra, analytical geometry, calculus, numerical analysis and areas outside mathematics such as anatomy, genetics, chemistry, physics, engineering and economics. Matrices, linear transformations and vector spaces are subjects which most linear algebra textbooks contain. Haddad (1999), who separated learning difficulties faced in linear algebra in three categories, nature of linear algebra and its teaching & learning, attributed the cause of difficulties to inability of the learners to think abstractly, their lacking of the fundamentals of mathematics and the axiomatic nature of the subject. These dimensions also coincide with the epistemological,

* Corresponding author. Tel.: +90-505-799-3324
E-mail address: alidelice@marmara.edu.tr

Peer-review under responsibility of the Organizing Committee of the ERPA Congress 2014.
doi:10.1016/j.sbspro.2014.09.244

pedagogical psychological factors that are the main reasons of conceptual misunderstandings (Cornu, 1991). According to Harel (1989a) the reason for learning difficulties about fundamental notations is the attempt to build an abstract structure on a weak conceptual basis of the fundamentals of mathematics. On the other hand, students can do calculations without a conceptual understanding (Harel, 1989b; Wang, 1989). Carlson (1993) stated that students generally are successful in tasks involving simple calculation algorithms (e.g. matrix multiplication and simple linear equation systems) are making errors in tasks related to linear independence and transformations.

A focal point of many studies on algebraic understanding is the solution procedures used by the students. Sharma (1987), for example, categorized the reasons for errors used in solution procedures as pertaining to (a) arithmetic's, (b) characteristics of number systems (e.g. associative law), (c) procedures (e.g. misuse of the equation sign & change of sign), (d) concepts (e.g. variable-constant complication), (e) human physiology (e.g. carelessness).

According to Tall (1993) students' difficulties are generally related to insufficient comprehension of the fundamental concepts and being unable to express the word problems mathematically. Findings of the Tall and Razali (1993: 209, 219) study suggested that most learning difficulties are accumulated on using the concepts and coordinating procedures and that those understand procedurally seem to experience more difficulties than those who understand conceptually.

Procedures, by definition, have algorithmic structures which need to be comprehended as a whole. In following a procedure the operations are handled stepwise in a logical manner and a conclusion is reached. For example in a task containing a system of linear equations the solution is found by using elementary row operations. This procedure necessitates the use of a limited number of rules each of which should be shown using mathematical symbols. In this context, we believe, system of linear equations (SLE) and matrices can be considered as an application area for symbols, language and operations. It is important to understand students' knowledge and skills in matrices and SLE. The aim of the study is to investigate the skills of solving SLE and their processes of doing operations.

2. Methodology

The paradigm of the study (Guba and Lincoln, 1994) is a non-positivist, interpretivist one. The focus of the study is students' solution processes in solving SLE and specifically their use of matrices in their solutions and the intention is to explore the influence of their skills of solving SLE on these processes. The case study method was chosen as the research design for the potential it presents for deep analysis of events, situations and individuals in a limited time in their natural environment (Yin, 1994). According to Cohen, Manion and Morrison (2000, s. 92) the choice of the sampling strategy is as important as the paradigm and method decisions. The sample is chosen from amongst the 41 second year mathematics departments' students taking the linear algebra course were selected using an appropriate sampling technique (ibid., s. 104; Patton 1990). As data collection tool, The Linear Algebra Test (LAT) is used. The categorization method (Robson, 1993) and descriptive statistics are the methods of data analysis.

3. Findings

Firstly in this section the answers given to LAT were analyzed. For this, firstly, the answers were categorized with respect to their being correct, partial, wrong and void which yielded a general performance description. This is followed by a descriptive analysis of the processes in the context of each of these categories.

3.1. Students' performances of doing operations with matrices

The results of the LAT analysis are summarized in the Table 1 below. The smallest correct answer percentage is in question 1 (1.27%). The highest is in question 7 (74.68%), followed by question 5 (59.99%). In partial answers, the highest value is in question 1 (50.63%). In wrong answer category the highest percentage is in question 2 (32.91%) followed by question 1 (31.65%). The smallest percentage of wrong answers is in question 7 (9.37%). The highest 'no answer' rate is observed in question 6 (26.58%). Other 'no answer' questions have close values.

Table 1. Percentage Values for Performance Results.

Question Number	Correct Answer	Partial Answer	Wrong Answer	Unanswered
1	1,27	50,63	31,65	16,46
2	15,19	39,24	32,91	12,66
3a	31,65	35,44	16,46	16,46
3b	35,44	29,11	18,99	16,46
3c	48,10	20,25	15,19	16,46
4	24,05	39,24	24,05	12,66
5	59,99	-	22,93	17,09
6	6,33	21,52	45,57	26,58
7	74,68	-	9,37	15,95

3.2. Students' matrix operation processes

For this analysis six categories were observed (Table 2) and the processes are analyzed under these headings.

Table 2: Percentage Values for Operational Processes

	Concept	Rule	Operation	Interpretation	Language	
					Verbal Expression	Symbol
1	76,19	0,00	0,00	0,00	71,43	42,86
2	95,24	0,00	0,00	0,00	90,48	66,67
3a	0,00	0,00	71,43	14,29	19,05	71,43
3b	0,00	4,76	80,95	42,86	0,00	80,95
3c	0,00	9,52	80,95	57,14	33,33	80,95
4	66,67	0,00	95,24	23,81	33,33	95,24
5	100,00	80,95	0,00	0,00	28,57	100,00
6	0,00	14,29	47,62	52,38	38,10	57,14
7	0,00	85,71	0,00	0,00	0,00	80,95

3.3. Process with respect to concept

LAT findings indicate that the highest concept use is in questions 1, 2, 4 and 5. In these questions students were asked to define the matrix concept, explain the characteristics of some special matrices & ways in which they are used and apply the some of matrix terminology. Only 1.27% were able to define matrix correctly and 50.63% gave partial answer. Partial answers came in the form of their interpretation of lecturer's descriptions of the concept such as "a tool used for solving m equations with n unknowns". Wrong answers generally are comprised of the definition of the concept as a method.

Characteristics of lower and upper triangular matrices were asked under the 'special matrices' heading. In most of correct and partial answers, the tendency was to give examples instead of knowledge (Fig. 1). Moreover, conceptual complications were observed between matrix and determinant. Wrong answers, on the other hand, seemed generally to have resulted from the lack of knowledge.

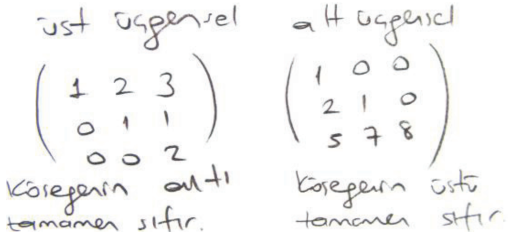

Fig. 1: An example given by a student

The matrix terminology used in the study is limited to augmented matrix and row reduced form and students were asked to solve questions using them. Most of the students were not able to transform a given matrix into the row reduced form while writing the augmented form (Fig. 2). A reason for this may be the lack of knowledge.

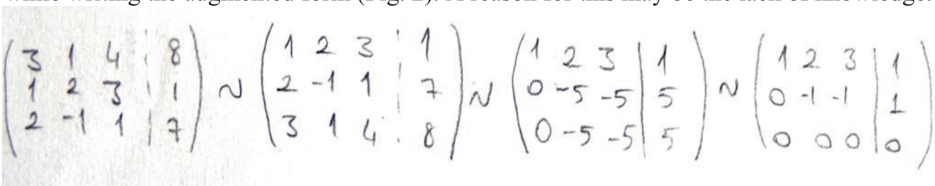

Fig. 2: A student answer which shows the inability to transform into a row reduced form.

When asked to write down the dimensions of given matrices students did not have much difficulty but when it comes to finding the related characteristics of the matrix from the list they were not very successful. For example, in questions about upper and lower triangular matrices they made mistakes despite the correctness of their explanations.

3.4. Process with respect to rule

In the question that necessitates the use of the rules related to matrix operations, no process was observed. Most of the answers (74.68%) were correct. The most common mistake seems to have resulted from being unable to relate $(AB)^{-1}$ and $(AB)^{T}$ (Fig. 3). Moreover, most of the students could not understand the meaning of the "σ" symbol in the $(\sigma A)^{-1}$ expression.

Fig. 3: A student answer in which the relation was not seen

3.5. Process with respect to operation

In questions 3 and 6 SLE were asked for which the use of elementary row operations is required. In question 3c for which the correct solution can be found by the help of a 2x2 coefficient matrix the rate of correct answer was 48.10%. Some correct answers were reached by using the elimination method in this question which they had learned in their secondary education. The rate of 'correct answer was 35.44% in question 3b. Most of wrong answer might have resulted from not recognizing that the order of the unknowns changed in different lines (Fig. 4). Most of errors seem to be related to not paying attention to the signs of the coefficients in the construction of the coefficient matrix. For these students a correct procedure leads to an incorrect answer because of such errors which is also the cause of misinterpretations. Some of the students while writing the coefficients in the AX=B form tried to operate with the $A^{-1}B$ augmented matrix and were not able to reach a correct answer.

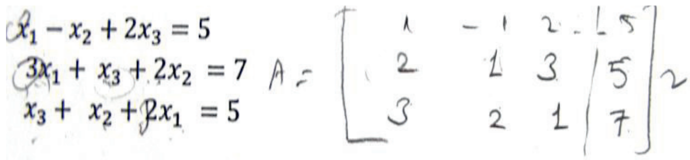

Fig. 4: An example of error while writing the coefficient matrix.

The rate of correct response for question 6 was 24.05%. An analysis of partial and wrong answers revealed that the answers were incorrect while the use of procedures was generally appropriate. The cause for this was attributed to (1) mistakes made in constructing the coefficient matrix, (2) using the A matrix instead of the augmented matrix and (3) operational errors. Moreover, in this question some students wrongly assumed to have completed the operations when they found a whole zero row while 2 or 3 steps still needed to be done for the exact row reduced form.

3.5. Process with respect to interpretation

The questions 3c and 6 were asked to assess students' interpretive capabilities. However, only 6.33% of the answers were correct in question 6. Findings revealed that many errors pertained to students' inability to interpret their correct answers. These students also seemed to assume a homogenous SLE using the AX=B formula, made use of special cases giving values for the variables and tried to make interpretations with respect to the coefficient ratios. They also tried to use their self-referenced and mostly wrong assumptions (e.g. "There are infinitely many solutions when the there is whole zero row." and "If the determinant is zero then there is an infinitely many solutions.").

3.6. Process with respect to language

The use of language was analyzed under the headings of verbal expressions and symbol use. Findings revealed that symbols were used in all of the questions and verbal expressions were given in all questions except question 7. The most frequent symbolic error is the confusion of the matrix brackets ([ ]) with the determinant brackets (| |). Moreover some of students used the "=" and $\rightarrow$ symbols instead of the "~" symbol in relating two subsequent steps in the row reduction procedure.

4. Discussions and conclusion

Informal definitions were very frequent in students' responses. Findings indicate that students developed their matrix concept images under the influence of the solved question and matrix notations that they faced a lot. Their concept images based on their visual memories which were fed by the solutions they 'watched' in the lectures and with their experience in solving problems explain why the definitions tended to be informal. Effective use of language necessitates a good blend of verbal expressions and symbols. Findings also suggest that some concepts and rules were used without having been internalized. This and insufficient practice or even insufficient memorization seemed to lead to mistakes in rule writing. Hence a new rule categorization can be made: (1) rules that memorized directly (matrix characteristics) and (2) indirectly with the help of the process (Cramer rule & row reduction).

Ercerman (2008) found out that high school students with poor conceptual basis had difficulty in solving questions involving operational knowledge. The process-operation analysis revealed that some rules were memorized through the process. Students that used this 'technique' seemed to prefer elementary row operations. The reason for this may be to do with its easiness to remember and handle (arithmetical knowledge is enough). Despite being easily remembered, students' operational errors did not vanish. Among these errors, there were 'to which row the resulting value after the "+,- & x" operations would be written' and 'forgetting some entities in the operations'. It is also observed that students think in analytical-arithmetical way because of they try to reach a solution by using elementary row operations (Sierpinska, 2000). As a result, if the the solution process of SLE are related to elementary operations, then generalized, some misconceptions may vanish (Harel ve Tall, 1989).

In general students use their informal definitions stemmed from their concept images rather than the accepted meaning of the matrix concepts. They also tend to use their pre-university knowledge and seem to have difficulties in making use of the symbolization given the lectures. In their solution students tend to go for the easy option which are the methods in which using "+,- & x" operations are sufficient rather than using the 'new' methods.

References

Carlson, D. (1993). Teaching Linear Algebra: Must the Fog Always Roll In?, *College Mathematics Journal*, 12 (1) 29-40.

Cohen, L., Manion, L. and Morrison, K. (2000). *Research Methods in Education,* (5th edition). London: Routledge.

Cornu, B. (1991). Limits. In. Tall, D. (Eds), *Advanced mathematical thinking*. p. 158. Kluwer, Boston.

Dorier, J. L., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. In D. Holton, M. Artigue, U. Krichgraber, J. Hillel, M. Niss & A. Schoenfeld (Eds.), The *Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 255-273). Dordrecht, Netherlands: Kluwer Academic Publishers.

Erçerman, B. (2008). *Kavramsal ve İşlemsel Bilgi Bağlamında Lise Öğrencilerinin Lineer Cebir Bilgilerinin Değerlendirilmesi*. Unpublished Master Dissertation. Yüzüncü Yıl Üniversitesi, Van, Türkiye.

Guba, E. G. ve Lincoln, Y. S. (1994). Competing paradigms in qualitative research. In N. Denzin & Y. Lincoln (Eds.), *Handbook of qualitative research*, Sage Publications.

Haddad, M., (1999). *Difficulties in the learning and teaching of linear algebra – a personal experience*. Unpublished Master Dissertation, Concordia University, Montreal, Quebec, Canada.

Harel, G., (1989a). Learning and Teaching Linear Algebra: Difficulties and an Alternative Approach to Visualizing Concepts and Processes, *Focus on Learning Problems in Mathematics*, 11(2), s.139-148

Harel, G., (1989b). Applying the principle of multible embodiment in teaching linear algebra: Aspect of familiarity and mode of representation, *Schools Science and Mathematics*, 89(1), 40-57

Harel, G. & Tall, D. (1989). Teh general, the abstract, and the generic ,n advanced mathematics. *For the Learning of Mathematics*, 11(1), 38-42.

Hillel, J., & Sierpinska, A. (1993). On one persistent mistake in linear algebra. *Proceedings of the 18th International Conference for the Psychology of Mathematics Education*, Lisbon, Portugal, 3, 65-72.

Patton, M. Q. (1990). *Qualitative evaluation and research methods* (2nd Ed). Newbury Park, Calif: Sage Publication.

Robson, C. (1993). *Real world research: A resource for social scientists and practitioner-researchers* (1st Ed). Oxford: Blackwell.

Sharma, M. C. (1987). Math notebook (From theory to practice). *Information for teachers and parents of children with learning problems in mathematics*. Vol. 5&6 (Parts 2&3).

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J-L. Dorier (Eds.) *On the teaching of Linear Algebra* (s. 209-246). Dordrecht, Netherlands: Kluwer Academic Publishers.

Tall, D., (1993), Students' difficulties in calculus, *Proceedings of Working Group 3 on Students' Difficulties in Calculus*. ICME-7, Quebec, Canada, 13– 28.

Tall, D. O. and Razali, M. R., (1993). Diagnosing students' difficulties in learning mathematics. *International Journal of Mathematical Education in Science and Technology*, 24(2), 209–222.

Wang, Tse-Wei, 1989. A Course on Applied Linear Algebra, *Chemical Engineering Education*, 23(4), 236-241.

Yin, R.K. (1994). *Case study research: Design and methods*, (2nd Ed). London: Sage Publication.