Neurocomputing 529 (2023) 33–47

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Cluster consensus with first and higher-order antagonistic interaction dynamics

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ARTICLE INFO

Article history:
Received 28 April 2022
Revised 17 November 2022
Accepted 8 January 2023
Available online 11 January 2023
Communicated by Zidong Wang

Keywords:
Signed digraphs
Consensus protocols
Cluster consensus
Antagonistic interactions
Multi-agent systems

ABSTRACT

This paper studies the cluster consensus problem for networks with antagonistic interactions modelled by adjacency matrices with negative weights. By introducing an extended digraph representation consisting of purely cooperative interactions with lifting approach, we relate the trajectories of the system to those of an extended system with a positive digraph. The behaviours of agents in a signed network are extracted from its extended digraph. Consequently, the number of clusters and the cluster members are explicitly determined for any signed digraph by using primary and secondary layer subgraph concepts. The relation between the dynamics of a signed system and its extended representation is investigated for first and higher-order systems. The conditions that make both systems stable are stated. Additionally, the control parameters to achieve cluster consensus are derived explicitly for first, second and third order systems. The obtained results for continuous-time networks are subsequently extended to discrete-time networks. Finally, theoretical results are illustrated via numerical examples.

1. Introduction

The agreement problem of networked agents, referred to as *consensus*, has been extensively studied in diverse fields including control engineering, sensor networks, social systems, robotics and opinion dynamics. In the literature, the focus is mainly on *complete consensus* where all members of a network attain a single steady state [1–7]. Due to the structure or the topology of a given network, agreeing on a common value may not always be possible. In such a case, the members of the network converge to different final states, called *clusters*. This phenomenon is known as *cluster consensus* [8–12].

The complete or the cluster consensus problem is mostly discussed in the context of cooperative systems. In such systems, communication links between agents are assumed to be positive weighting. However, this assumption may not be realistic for some real-world systems such as opinion dynamics, decision making, etc. In these systems, also called competitive systems, interaction flow from an agent to another may be weighted positive or negative based on agents' opinions. Positive links represent collaboration of agents while negative links represent antagonistic interaction of agents. Such a topology is represented by *signed digraphs*. In this paper, our main focus is to investigate the effect of negative links on the cluster agreement problem and to determine the clusters and their members for any given higher-order network associated with a signed digraph. While the literature consists of studies that consider the consensus problem in signed networks, no research investigates the number of clusters and the agents in each cluster for a network of agents with antagonistic interactions. Research on cluster consensus has been mostly restricted to the bipartite consensus and interval bipartite consensus problems [13–17].

Altafini investigates *bipartite consensus* where all members of a signed network converge to the same final value in modulus but not in sign [13]. Bipartite consensus is a special type of cluster consensus such that the number of clusters equals two. A necessary and sufficient condition to achieve bipartite consensus for strongly connected networks is that the signed digraph is structurally balanced and digon sign-symmetric.

In [14], the authors study *interval bipartite consensus* where the root nodes of networks reach bipartite consensus while the non-root nodes converge to some value which is smaller than the bipartite consensus value in modulus. To achieve interval bipartite consensus, signed digraphs associated with networks are required to have a spanning tree. However, in cases where interval bipartite consensus is achieved, the number of clusters and their members are unknown.

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https://doi.org/10.1016/j.neucom.2023.01.025
0925-2312/

Nomenclature

$\mathcal{S}_c$	Continuous-time signed system
$\bar{\mathcal{S}}_c$	Continuous-time extended unsigned system
$\mathcal{S}_d$	Discrete-time signed system
$\bar{\mathcal{S}}_d$	Discrete-time extended unsigned system
L	$n \times n$ signed Laplacian matrix for $\mathcal{S}_c$
$\bar{L}$	$2n \times 2n$ unsigned Laplacian matrix for $\bar{\mathcal{S}}_c$
$\hat{L}$	$n \times n$ unsigned Laplacian matrix for $\hat{\mathcal{S}}_c$
$\mathcal{L}_m$	$mn \times mn$ continuous-time system matrix for m-th order $\mathcal{S}_c$
$\bar{\mathcal{L}}_m$	$2mn \times 2mn$ continuous-time system matrix for m-th order $\bar{\mathcal{S}}_c$
$\hat{L}_m$	$mn \times mn$ continuous-time system matrix for m-th order $\hat{\mathcal{S}}_c$
Γ_m	$mn \times mn$ discrete-time system matrix for m-th order $\mathcal{S}_d$
$\bar{\Gamma}_m$	$2mn \times 2mn$ discrete-time system matrix for m-th order $\bar{\mathcal{S}}_d$
$\hat{\Gamma}_m$	$mn \times mn$ discrete-time system matrix for m-th order $\hat{\mathcal{S}}_d$

Opinion dynamics in social networks are investigated by constructing structurally balanced signed digraphs in [15]. The authors consider the case where the opinions are the same (consensus) or totally opposite (bipartite). Signed digraphs containing a spanning tree are used to achieve bipartite consensus. However, in real life, there may be more than two opposite opinions. This case is not taken into account in [15].

Bipartite consensus is achieved for structurally balanced systems using binary cooperative-competitive interactions. In [16], the authors argue that the level of collaboration or competition may be different from binary. It is shown that multi-partite consensus can be achieved by using weighted cooperative-competitive interactions. Interactively balanced or sub-balanced connectivity of digraph is a necessary condition to reach cluster consensus. In [17], two necessary and sufficient conditions are derived for structurally balanced and unbalanced systems with weighted interactions. For structurally unbalanced systems, pinning control is required.

In [18], the authors show that the solution of a signed network dynamics with n agents can be obtained from the solution of the corresponding extended unsigned network dynamics with $2n$ agents. This approach is used in [19] for consensus analysis of the signed graph containing spanning tree.

As discussed above, research on the cluster consensus problem is mainly limited to bipartite consensus and interval bipartite consensus for networks evolving over signed digraphs. Except for the bipartite consensus case (where there are only two clusters), there is no study that computes the number of clusters an arbitrary network converges to. Furthermore, structurally balanced digraph structure is a necessary assumption to reach bipartite or interval bipartite consensus. The existing results on networks characterized by signed digraphs are restricted to limited digraph structures i.e., structurally balanced graphs and graphs consisting of a spanning tree [13–17]. For signed digraphs with no spanning tree, there is no available result in the literature. Moreover, the existing results [13–17] cannot be directly extended to cover such cases. The main difficulty is that the graph theoretical concepts used in [13–17] are restricted to digraphs with spanning tree.

This paper is the first study in the literature that addresses signed digraphs that have no spanning tree. We consider a general network characterized by positive and negative interactions to overcome the limitations in [13–17] to address the following fundamental questions:

- Given a multi-agent network evolving over a signed digraph with no assumption on the graph structure, does the network achieve cluster consensus?
- If so, how many clusters are formed and what are the members of each cluster?

To address the above fundamental questions, we introduce the extended digraph representation for the analysis of the corresponding signed digraph. In this context, the contributions of this paper can be highlighted as follows:

- By exploiting the lifting approach in [18], we first relate the dynamics of any signed digraph (possibly with no spanning tree) to those of an unsigned digraph and then study the convergence properties of the associated system that contains agents with higher-order dynamics in continuous-time.
- For any given first or higher-order system with a signed digraph, we compute the number of clusters and explicitly determine cluster members.
- For higher-order systems, controller parameter selection for signed networks is investigated and conditions that guarantee the cluster consensus are stated.
- The above results are extended to discrete-time systems.

The rest of the paper is organized as follows. In Section 2, we review some crucial properties of signed digraphs, primary/secondary layer subgraph concepts and introduce the extended digraph representation. Our main results on any given higher-order system associated with a signed digraph are provided in Section 3. In Section 4, we extend the results of Section 5 to discrete-time networks. A numerical example is provided in Section 6 to illustrate the main results of the paper. Finally, there are some concluding remarks in Section 7.

2. Mathematical preliminaries

In this section, we review the properties of signed digraphs, the definitions of primary and secondary layer subgraphs; and give the mathematical formulation of the consensus protocol.

2.1. Signed digraphs and primary/secondary layer subgraphs

A weighted signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ represents interaction between agents. For n agents, $\mathcal{V} = \{v_1, \ldots, v_n\}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. A directed edge, (v_j, v_i), denotes information flow from node v_j to node v_i. The adjacency element, a_{ij}, is the weight of the corresponding edge, (v_j, v_i), is defined such that $a_{ij} \neq 0 \iff (v_j, v_i) \in \mathcal{E}$. A signed digraph, $\mathcal{G}$, is called digon sign-symmetric if $a_{ij}a_{ji} \geqslant 0$. $\mathcal{G}$ is called an unsigned digraph, if all adjacency elements are non-negative, i.e., $a_{ij} \geqslant 0$.

A path $\mathcal{P}$ between two nodes (from node v_i to node v_j) is defined as a finite sequence of nodes such that $(v_{l_1}, v_{l_2}, \ldots, v_{l_{m-1}}, v_{l_m})$ where $l_1 = i, l_m = j$ and $(v_{l_k}, v_{l_{k+1}}) \in \mathcal{E}$ for $k = 1, \ldots, m-1$. A cycle $\mathcal{C}$ is a directed path whose starting and ending nodes are the same. A cycle is positive if $a_{l_2 l_1} a_{l_3 l_2} \ldots a_{l_1 l_m} > 0$ and is negative if $a_{l_2 l_1} a_{l_3 l_2} \ldots a_{l_1 l_m} < 0$.

For unsigned digraphs, there are novel concepts in the literature that are used to analyze the general structure of digraphs: primary and secondary layer subgraphs. The definitions are given below:

Definition 1. [20] (PRIMARY LAYER SUBGRAPHS) In any given weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, there exist unique l_p ($l_p \geqslant 1$) subgraphs $\mathcal{G}_{p,i} = (\mathcal{V}_{p,i}, \mathcal{E}_{p,i}), i = 1, 2, \ldots, l_p$, such that

- the node set of each subgraph $\mathcal{V}_{p,i}, i = 1, 2, \ldots, l_p$, is the largest possible subset that has a spanning tree and
- nodes in a subgraph, $\mathcal{V}_{p,i}$, do not receive information from out of the subgraph, $\mathcal{V} \setminus \mathcal{V}_{p,i}$.

Definition 2. [20] (SECONDARY LAYER SUBGRAPHS) Let $\hat{\mathcal{V}} = \mathcal{V} \setminus \cup_{i=1}^{l_p} \mathcal{V}_{p,i}$. Then, there exist l_s unique subgraphs $\mathcal{G}_{s,i} = (\mathcal{V}_{s,i}, \mathcal{E}_{s,i}), i = 1, 2, \ldots, l_s$, such that $\cup_{i=1}^{l_s} \mathcal{V}_{s,i} = \hat{\mathcal{V}}$, and

- each subgraph $\mathcal{G}_{s,i}, i = 1, 2, \ldots, l_s$, has a spanning tree and
- the root node of a subgraph receives information from at least two nodes in two different subgraphs and
- all nodes except the root in a subgraph do not receive information from out of the subgraph.

Remark 1. The primary and secondary layer subgraphs in Definition 1 and 2 can be determined by using the detection algorithms in [20]. Given an unsigned digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, we first utilize the primary layer subgraph detection algorithm in [20] to determine the primary layer subgraph node sets $\mathcal{V}_{p,i}, i = 1, \ldots, l_p$. The remaining nodes $\hat{\mathcal{V}} = \mathcal{V} \setminus \cup_{i=1}^{l_p} \mathcal{V}_{p,i}$ constitute members of secondary layer subgraphs. The secondary layer detection algorithm in [20] can then be applied on $\hat{\mathcal{V}}$ to determine the secondary layer subgraph node sets $\mathcal{V}_{s,i}, i = 1, \ldots, l_s$.

2.2. Consensus protocol

Consider a multi-agent system consisting of n agents with the following linear consensus protocol

$$\begin{aligned} \mathcal{S}_c: \quad \dot{x}_i^{(1)}(t) &= x_i^{(2)}(t), \\ \dot{x}_i^{(2)}(t) &= x_i^{(3)}(t), \\ &\vdots \\ \dot{x}_i^{(m-1)}(t) &= x_i^{(m)}(t), \\ \dot{x}_i^{(m)}(t) &= \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} a_{ij} \left(x_j^{(q)}(t) - \mathrm{sgn}(a_{ij}) x_i^{(q)}(t) \right) \end{aligned} \tag{1}$$

for $i = 1, 2, \ldots, n$, where $x_i^{(q)}(t) \in \mathbb{R}$ $(q = 1, \ldots, m)$ are the states of node v_i at time t, $x_i^{(q)}$ $(q = 2, \ldots, m)$ denotes the $(q-1)$st derivative of $x_i^{(1)}$, and $\gamma_q > 0, q = 1, \ldots, m$, are the controller parameters.

The dynamics of system (1) are equivalent to

$$\begin{aligned} \dot{x}_i^{(1)}(t) &= x_i^{(2)}(t), \\ &\vdots \\ \dot{x}_i^{(m-1)}(t) &= x_i^{(m)}(t), \\ \dot{x}_i^{(m)}(t) &= -\sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} l_{ij} x_j^{(q)}(t), \quad i = 1, 2, \ldots, n \end{aligned} \tag{2}$$

where l_{ij} is the element of the signed Laplacian matrix, L, which is defined by

$$l_{ij} = \begin{cases} \sum_{k=1, k \neq i}^{n} |a_{ik}|, & \text{if } i = j \\ -a_{ij}, & \text{otherwise}. \end{cases} \tag{3}$$

System $\mathcal{S}_c$ will be investigated under first and higher-order ($m \geqslant 2$) dynamics separately. In this context, system $\mathcal{S}_c$ can be expressed for first-order dynamics as

$$\dot{x}(t) = -Lx(t) \tag{4}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ with $x_i(t) \in \mathbb{R}, i = 1, \ldots, n$ is the state vector of the first-order system at time t. For $m \geqslant 2$, system $\mathcal{S}_c$ can be rewritten in the following form:

$$\begin{bmatrix} \dot{x}^{(1)}(t) \\ \dot{x}^{(2)}(t) \\ \vdots \\ \dot{x}^{(m)}(t) \end{bmatrix} = \mathcal{L}_m \begin{bmatrix} x^{(1)}(t) \\ x^{(2)}(t) \\ \vdots \\ x^{(m)}(t) \end{bmatrix}$$

where $x^{(q)}(t) \in \mathbb{R}^n, q = 1, 2, \ldots, m$ is a state vector with components $x_i^{(q)}(t), i = 1, 2, \ldots, n$, at time t and

$$\mathcal{L}_m = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & I_n \\ -\gamma_1 L & -\gamma_2 L & -\gamma_3 L & \ldots & -\gamma_m L \end{bmatrix}; \tag{5}$$

is the $mn \times mn$ system matrix where $0_{n\times n}$ denotes the $n \times n$ matrix with all zero entries.

The eigenvalues of $\mathcal{L}_m$ can be computed by solving the characteristic equation $\det(\mu I_{mn} - \mathcal{L}_m) = 0$ where

$$\begin{aligned} \det(\mu I_{mn} - \mathcal{L}_m) &= \det\left(\begin{bmatrix} \mu I_n & -I_n & 0_{n\times n} & \ldots & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \mu I_n & -I_n & \ldots & 0_{n\times n} & 0_{n\times n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & \mu I_n & -I_n \\ \gamma_1 L & \gamma_2 L & \gamma_3 L & \ldots & \gamma_{m-1} L & \mu I_n + \gamma_m L \end{bmatrix} \right) \\ &= \det(\mu^m I_n + (\gamma_1 + \gamma_2 \mu + \ldots + \gamma_m \mu^{m-1}) L). \end{aligned}$$

The above expression can be rewritten with respect to the eigenvalues of the Laplacian matrix, L, as

$$\det(\mu I_{mn} - \mathcal{L}_m) = \prod_{i=1}^{n} (\mu^m + (\gamma_1 + \gamma_2 \mu + \ldots + \gamma_m \mu^{m-1}) \lambda_i) = 0 \tag{6}$$

where λ_i is the i-th eigenvalue of L. From (6), the eigenvalues of $\mathcal{L}_m$ can be obtained from

$$\mu^m + \gamma_m \lambda_i \mu^{m-1} + \ldots + \gamma_2 \lambda_i \mu + \gamma_1 \lambda_i = 0 \tag{7}$$

for $i = 1, 2, \ldots, n$. Each eigenvalue of L corresponds to m eigenvalues of $\mathcal{L}_m$.

2.3. Extended digraph

In this paper, the evolution of a higher-order with a signed digraph $\mathcal{G}$ is investigated by using its extended digraph representation introduced in [18]. We define this representation as follows:

Definition 3. For a given signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{v_1, \ldots, v_n\}$ is the node set, $\mathcal{E}$ is the edge set and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the adjacency matrix; the corresponding extended digraph is $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$ with the node set $\bar{\mathcal{V}} = \{v_1, \ldots, v_n, v'_1, \ldots, v'_n\}$, the edge set $\bar{\mathcal{E}}$ and the adjacency matrix $\bar{\mathcal{A}} = [\bar{a}_{ij}] \in \mathbb{R}^{2n\times 2n}$. The edge set, $\bar{\mathcal{E}}$, and the adjacency matrix, $\bar{\mathcal{A}}$, are defined as follows:
If $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} > 0$, then

- $(v_j, v_i) \in \bar{\mathcal{E}}$ and $\bar{a}_{ij} = a_{ij}$, and
- $\left(v_j', v_i'\right) \in \bar{\mathcal{E}}$ and $\bar{a}_{i+n,j+n} = a_{ij}$ for $i,j = 1,\dots,n$.

If $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} < 0$, then

- $(v_j, v_i') \in \bar{\mathcal{E}}$ and $\bar{a}_{i+n,j} = -a_{ij}$, and
- $\left(v_j', v_i\right) \in \bar{\mathcal{E}}$ and $\bar{a}_{i,j+n} = -a_{ij}$ for $i,j = 1,\dots,n$.

Remark 2. The extended adjacency elements $\bar{a}_{ij}$ in Definition 3 can be also formulated as

$$\bar{a}_{ij} = \bar{a}_{i+n,j+n} = \frac{|a_{ij}| + a_{ij}}{2} \qquad \text{and} \qquad \bar{a}_{i+n,j} = \bar{a}_{i,j+n} = \frac{|a_{ij}| - a_{ij}}{2},$$

for $i,j = 1,2,\dots,n$. From the adjacency elements $\bar{a}_{ij}$, the extended adjacency matrix $\bar{\mathcal{A}}$ is obtained as

$$\bar{\mathcal{A}} = \frac{1}{2}\begin{bmatrix} \mathbb{A} + \mathcal{A} & \mathbb{A} - \mathcal{A} \\ \mathbb{A} - \mathcal{A} & \mathbb{A} + \mathcal{A} \end{bmatrix}. \tag{8}$$

where $\mathbb{A} = [|a_{ij}|] \in \mathbb{R}^{n\times n}$. Note that an extended digraph has only non-negative adjacency elements, i.e., $\bar{\mathcal{G}}$ is an unsigned digraph consisting of $2n$ nodes.

The extended digraph concept is illustrated by the following example.

Example 1. Consider the signed digraph in Fig. 1. Black and red arrows denote positive and negative edges, respectively. Note that the digraph has no spanning tree. In order to construct the extended digraph of the given network, the following procedure can be used:

- A new node, v_i', is created for each node, $v_i, i = 1,\dots,5$. In this way, the number of nodes is doubled.
- While the positive edges in Fig. 1 are preserved in Fig. 2, the corresponding new positive edges are created according to Definition 3. For example, the positive edge (v_1, v_3) is maintained and the corresponding new edge (v_1', v_3') is created with the same weight i.e., $a_{31} = 0.7$.
- The negative edges in Fig. 1 are removed in Fig. 2 and new positive edges are generated in its place based on Definition 3. For example, the negative edge (v_1, v_2) is removed and the corresponding new edges are created as (v_1, v_2') and (v_1', v_2) with the weight of $|a_{21}| = 0.1$.

By following the steps above, the corresponding extended digraph is obtained as shown in Fig. 2. Notice that the weights of all edges in the extended digraph are positive.

3. Cluster consensus in signed continuous-time networks

Given $\mathcal{S}_c$ in (1) evolving over $\mathcal{G}$, the extended system $\bar{\mathcal{S}}_c$ associated with extended digraph $\bar{\mathcal{G}}$ is defined as

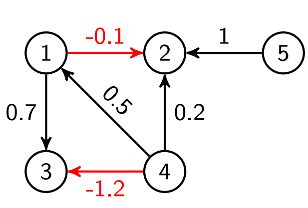

Fig. 1. Signed digraph in Example 1.

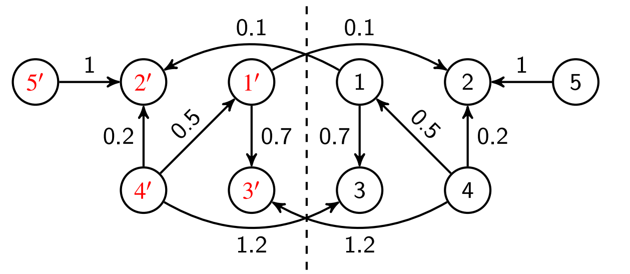

Fig. 2. Extended digraph in Example 1.

$$\begin{aligned} \bar{S}_c: \quad \dot{\bar{x}}_i^{(1)}(t) &= \bar{x}_i^{(2)}(t), \\ \dot{\bar{x}}_i^{(2)}(t) &= \bar{x}_i^{(3)}(t), \\ &\vdots \\ \dot{\bar{x}}_i^{(m-1)}(t) &= \bar{x}_i^{(m)}(t), \\ \dot{\bar{x}}_i^{(m)}(t) &= \sum_{q=1}^{m}\gamma_q \sum_{j=1}^{2n} \bar{a}_{ij}\left(\bar{x}_j^{(q)}(t) - \bar{x}_i^{(q)}(t)\right) \end{aligned} \tag{9}$$

for $i = 1,\dots,2n$ where $\bar{x}_i^{(q)}(t)$ $(q = 1,\dots,m)$ are the states at time t, $\bar{x}_i^{(q)}$ $(q = 2,\dots,m)$ denotes the $(q-1)$st derivative of $\bar{x}_i^{(1)}$.

The dynamics of system (9) are equivalent to

$$\begin{aligned} \dot{\bar{x}}_i^{(1)}(t) &= \bar{x}_i^{(2)}(t), \\ &\vdots \\ \dot{\bar{x}}_i^{(m-1)}(t) &= \bar{x}_i^{(m)}(t), \\ \dot{\bar{x}}_i^{(m)}(t) &= -\sum_{q=1}^{m}\gamma_q \sum_{j=1}^{2n} \bar{l}_{ij}\bar{x}_j^{(q)}(t), \quad i = 1,2,\dots,2n \end{aligned} \tag{10}$$

where $\bar{l}_{ij}$ is the element of the unsigned Laplacian matrix, $\bar{L}$, which is defined by

$$\bar{l}_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{2n} \bar{a}_{ik}, & \text{if } i = j \\ -\bar{a}_{ij}, & \text{otherwise.} \end{cases}$$

System (10) can be rewritten as in the following form:

$$\begin{bmatrix} \dot{\bar{x}}^{(1)}(t) \\ \dot{\bar{x}}^{(2)}(t) \\ \vdots \\ \dot{\bar{x}}^{(m)}(t) \end{bmatrix} = \bar{\mathcal{L}}_m \begin{bmatrix} \bar{x}^{(1)}(t) \\ \bar{x}^{(2)}(t) \\ \vdots \\ \bar{x}^{(m)}(t) \end{bmatrix}$$

where $\bar{x}^{(q)}(t) \in \mathbb{R}^{2n}, q = 1,2,\dots,m$ is a state vector with components $x_i^{(q)}(t), i = 1,2,\dots,2n$, at time t and

$$\bar{\mathcal{L}}_m = \begin{bmatrix} 0_{2n\times 2n} & I_{2n} & 0_{2n\times 2n} & \dots & 0_{2n\times 2n} \\ 0_{2n\times 2n} & 0_{2n\times 2n} & I_{2n} & \dots & 0_{2n\times 2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \dots & I_{2n} \\ -\gamma_1\bar{L} & -\gamma_2\bar{L} & -\gamma_3\bar{L} & \dots & -\gamma_m\bar{L} \end{bmatrix}; \tag{11}$$

$0_{2n\times 2n}$ is the $2n \times 2n$ matrix with all zero entries.

The eigenvalues of $\bar{\mathcal{L}}_m$ can be computed by solving the characteristic equation $\det(\bar{\mu}I_{2mn} - \bar{\mathcal{L}}_m) = 0$ where

$$\begin{aligned} \det(\bar{\mu}I_{2mn} - \bar{\mathcal{L}}_m) &= \det\left(\begin{bmatrix} \bar{\mu}I_{2n} & -I_{2n} & 0_{2n\times 2n} & \dots & 0_{2n\times 2n} & 0_{2n\times 2n} \\ 0_{2n\times 2n} & \bar{\mu}I_{2n} & -I_{2n} & \dots & 0_{2n\times 2n} & 0_{2n\times 2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \dots & \bar{\mu}I_{2n} & -I_{2n} \\ \gamma_1\bar{L} & \gamma_2\bar{L} & \gamma_3\bar{L} & \dots & \gamma_{m-1}\bar{L} & \bar{\mu}I_{2n} + \gamma_m\bar{L} \end{bmatrix}\right) \\ &= \det(\bar{\mu}^m I_{2n} + (\gamma_1 + \gamma_2\bar{\mu} + \dots + \gamma_m\bar{\mu}^{m-1})\bar{L}). \end{aligned}$$

The above expression can be rewritten with respect to the eigenvalues of the extended Laplacian matrix, $\bar{L}$, as

$$\det\left(\bar{\mu}I_{2mn} - \bar{\mathcal{L}}_m\right) = \prod_{i=1}^{n}\left(\bar{\mu}^m + \left(\gamma_1 + \gamma_2\bar{\mu} + \ldots + \gamma_m\bar{\mu}^{m-1}\right)\bar{\lambda}_i\right) = 0 \tag{12}$$

where $\bar{\lambda}_i$ is the i-th eigenvalue of $\bar{L}$. From (12), the eigenvalues of $\bar{\mathcal{L}}_m$ can be obtained from

$$\bar{\mu}^m + \gamma_m\bar{\lambda}_i\bar{\mu}^{m-1} + \ldots + \gamma_2\bar{\lambda}_i\bar{\mu} + \gamma_1\bar{\lambda}_i = 0 \tag{13}$$

for $i = 1, 2, \ldots, 2n$. Each eigenvalue of L corresponds to m eigenvalues of $\bar{\mathcal{L}}_m$.

The following result relates the trajectories of $\mathcal{S}_c$ described by a signed digraph to that of $\bar{\mathcal{S}}_c$ which is described by an unsigned digraph.

Theorem 1. The trajectory $x^{(q)}(t)$ of system $\mathcal{S}_c$ in (1) can be recovered from that of system $\bar{\mathcal{S}}_c$ in (9) as $x_i^{(q)}(t) = \bar{x}_i^{(q)}(t), i = 1, \ldots, n$ for all $t \geqslant 0$ by setting $\bar{x}_{i0}^{(q)} = x_{i0}^{(q)}$ and $\bar{x}_{i+n,0}^{(q)} = -x_{i0}^{(q)}$ for $i = 1, 2, \ldots, n$ and $q = 1, \ldots, m$.

Proof. The extended system $\bar{\mathcal{S}}_c$ can be rewritten as follows:

$$\begin{aligned}
\bar{S}_c: \quad \dot{\bar{x}}_i^{(1)}(t) &= \bar{x}_i^{(2)}(t),\\
&\vdots\\
\dot{\bar{x}}_i^{(m-1)}(t) &= \bar{x}_i^{(m)}(t),\\
\dot{\bar{x}}_i^{(m)}(t) &= \sum_{q=1}^{m}\gamma_q\left(\sum_{j=1}^{n}\bar{a}_{ij}\left(\bar{x}_j^{(q)}(t) - \bar{x}_i^{(q)}(t)\right) + \sum_{j=1}^{n}\bar{a}_{i,j+n}\left(\bar{x}_{j+n}^{(q)}(t) - \bar{x}_i^{(q)}(t)\right)\right)
\end{aligned} \tag{14a}$$

$$\begin{aligned}
\dot{\bar{x}}_{i+n}^{(1)}(t) &= \bar{x}_{i+n}^{(2)}(t),\\
&\vdots\\
\dot{\bar{x}}_{i+n}^{(m-1)}(t) &= \bar{x}_{i+n}^{(m)}(t),\\
\dot{\bar{x}}_{i+n}^{(m)}(t) &= \sum_{q=1}^{m}\gamma_q\left(\sum_{j=1}^{n}\bar{a}_{i+n,j+n}\left(\bar{x}_{j+n}^{(q)}(t) - \bar{x}_{i+n}^{(q)}(t)\right) + \sum_{j=1}^{n}\bar{a}_{i+n,j}\left(\bar{x}_j^{(q)}(t) - \bar{x}_{i+n}^{(q)}(t)\right)\right)
\end{aligned} \tag{14b}$$

for $i = 1, \ldots, n$. Let $\hat{x}_i^{(q)}(t) = \bar{x}_i^{(q)}(t) + \bar{x}_{i+n}^{(q)}(t)$ for $i = 1, \ldots, n$ and $q = 1, \ldots, m$. Then, we have

$$\begin{aligned}
\dot{\hat{x}}_i^{(1)}(t) &= \dot{\bar{x}}_i^{(1)}(t) + \dot{\bar{x}}_{i+n}^{(1)}(t)\\
&= \bar{x}_i^{(2)}(t) + \bar{x}_{i+n}^{(2)}(t)\\
&= \hat{x}_i^{(2)}(t)\\
\dot{\hat{x}}_i^{(2)}(t) &= \dot{\bar{x}}_i^{(2)}(t) + \dot{\bar{x}}_{i+n}^{(2)}(t)\\
&= \bar{x}_i^{(3)}(t) + \bar{x}_{i+n}^{(3)}(t)\\
&= \hat{x}_i^{(3)}(t)\\
&\vdots\\
\dot{\hat{x}}_i^{(m-1)}(t) &= \dot{\bar{x}}_i^{(m-1)}(t) + \dot{\bar{x}}_{i+n}^{(m-1)}(t)\\
&= \bar{x}_i^{(m)}(t) + \bar{x}_{i+n}^{(m)}(t)\\
&= \hat{x}_i^{(m)}(t)\\
\dot{\hat{x}}_i^{(m)}(t) &= \dot{\bar{x}}_i^{(m)}(t) + \dot{\bar{x}}_{i+n}^{(m)}(t)\\
&= \sum_{q=1}^{m}\gamma_q\left(\sum_{j=1}^{n}(\bar{a}_{ij} + \bar{a}_{i+n,j})\bar{x}_j^{(q)}(t) + (\bar{a}_{i,j+n} + \bar{a}_{i+n,j+n})\bar{x}_{j+n}^{(q)}(t)\right)\\
&\quad - \sum_{q=1}^{m}\gamma_q\left(\sum_{j=1}^{n}(\bar{a}_{ij} + \bar{a}_{i,j+n})\bar{x}_i^{(q)}(t) + (\bar{a}_{i+n,j} + \bar{a}_{i+n,j+n})\bar{x}_{i+n}^{(q)}(t)\right)
\end{aligned}$$

From Definition 3, we obtain

$$\bar{a}_{i+n,j} = \bar{a}_{i,j+n}, \quad \bar{a}_{ij} = \bar{a}_{i+n,j+n} \tag{15}$$

for $i = 1, \ldots, n$ and $|a_{ij}| = \bar{a}_{ij} + \bar{a}_{i+n,j}$. By using (15), we also have $\bar{a}_{i,j+n} + \bar{a}_{i+n,j+n} = \bar{a}_{ij} + \bar{a}_{i,j+n} = \bar{a}_{i+n,j} + \bar{a}_{i+n,j+n} = |a_{ij}|$. Then, we have

$$\begin{aligned}
\hat{S}_c: \quad \dot{\hat{x}}_i^{(1)}(t) &= \hat{x}_i^{(2)}(t)\\
\dot{\hat{x}}_i^{(2)}(t) &= \hat{x}_i^{(3)}(t)\\
&\vdots\\
\dot{\hat{x}}_i^{(m-1)}(t) &= \hat{x}_i^{(m)}(t)\\
\dot{\hat{x}}_i^{(m)}(t) &= \sum_{q=1}^{m}\gamma_q\left(\sum_{j=1}^{n}|a_{ij}|\left(\bar{x}_j^{(q)}(t) + \bar{x}_{j+n}^{(q)}(t)\right)\right) - \sum_{q=1}^{m}\gamma_q\left(\sum_{j=1}^{n}|a_{ij}|\left(\bar{x}_i^{(q)}(t) + \bar{x}_{i+n}^{(q)}(t)\right)\right)\\
&= \sum_{q=1}^{m}\gamma_q\sum_{j=1}^{n}|a_{ij}|\left(\hat{x}_j^{(q)}(t) - \hat{x}_i^{(q)}(t)\right)
\end{aligned} \tag{16}$$

The dynamics of the above system is equivalent to

$$\hat{\mathcal{S}}_c: \quad \dot{\hat{x}}(t) = \hat{\mathcal{L}}_m\hat{x}(t), \quad \hat{x}(0) = \hat{x}_0 \tag{17}$$

where $\hat{x}(t) = \left[\hat{x}^{(1)}(t)^T, \hat{x}^{(2)}(t)^T, \ldots, \hat{x}^{(m)}(t)^T\right]^T$ with $\hat{x}^{(q)}(t) = \left[\hat{x}_1^{(q)}(t)^T, \hat{x}_2^{(q)}(t)^T, \ldots, \hat{x}_n^{(q)}(t)^T\right]^T$ for $q = 1, \ldots, m$ and

$$\hat{\mathcal{L}}_m = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & \ldots & 0_{n\times n}\\ 0_{n\times n} & 0_{n\times n} & I_n & \ldots & 0_{n\times n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & I_n\\ -\gamma_1\hat{L} & -\gamma_2\hat{L} & -\gamma_3\hat{L} & \ldots & -\gamma_m\hat{L} \end{bmatrix} \tag{18}$$

where $\hat{L}$ is the unsigned Laplacian matrix whose elements are defined by

$$\hat{l}_{ij} = \begin{cases} \sum_{q=1,k\neq i}^{n}|a_{ik}|, & \text{if } i = j\\ -|a_{ij}|, & \text{otherwise.} \end{cases} \tag{19}$$

The solution of (17) is

$$\hat{x}(t) = e^{\hat{\mathcal{L}}_m t}\hat{x}_0 \tag{20}$$

Note that $\hat{x}_0 = [0, \ldots, 0]^T$ implies $\hat{x}(t) = [0, \ldots, 0]^T$ for all t, i.e., $\bar{x}_i^{(q)}(t) = -\bar{x}_{i+n}^{(q)}(t)$ for $i = 1, \ldots, n$ and $q = 1, \ldots, m$.

Substituting $\bar{x}_{j+n}^{(q)}(t) = -\bar{x}_j^{(q)}(t)$ into (14a), we obtain

$$\begin{aligned}
\dot{\bar{x}}_i^{(1)}(t) &= \bar{x}_i^{(2)}(t),\\
\dot{\bar{x}}_i^{(2)}(t) &= \bar{x}_i^{(3)}(t),\\
&\vdots\\
\dot{\bar{x}}_i^{(m-1)}(t) &= \bar{x}_i^{(m)}(t),\\
\dot{\bar{x}}_i^{(m)}(t) &= \sum_{q=1}^{m}\gamma_q\left(\sum_{j=1}^{n}(\bar{a}_{ij} - \bar{a}_{i,j+n})\bar{x}_j^{(q)}(t) - \sum_{j=1}^{n}(\bar{a}_{ij} + \bar{a}_{i,j+n})\bar{x}_i^{(q)}(t)\right)
\end{aligned}$$

for $i = 1, 2, \ldots, n$.

From Definition 3, we get

$$\begin{aligned}
\dot{\bar{x}}_i^{(1)}(t) &= \bar{x}_i^{(2)}(t),\\
\dot{\bar{x}}_i^{(2)}(t) &= \bar{x}_i^{(3)}(t),\\
&\vdots\\
\dot{\bar{x}}_i^{(m-1)}(t) &= \bar{x}_i^{(m)}(t),\\
\dot{\bar{x}}_i^{(m)}(t) &= \sum_{q=1}^{m}\gamma_q\sum_{j=1}^{n}a_{ij}\left(\bar{x}_j^{(q)}(t) - \operatorname{sgn}(a_{ij})x_i^{(q)}(t)\right)
\end{aligned} \tag{21}$$

for $i = 1, 2, \ldots, n$. Note that (21) is equal to (1). By setting $\bar{x}_i^{(q)}(0) = x_{i0}^{(q)}$ and $\bar{x}_{i+n}^{(q)}(0) = -x_{i0}^{(q)}$ for $i = 1, 2, \ldots, n$ and $q = 1, \ldots, m$ the trajectory $x^{(q)}(t)$ can be recovered. This completes the proof.

While Theorem 1 focuses on the relation of the trajectories of $\mathcal{S}_c$ and $\bar{\mathcal{S}}_c$, it does not address the relationship between their stability properties. By using the relation of L and $\bar{L}$, below we first study the stability relation for first-order dynamics and then extend the results to higher-order systems.

Theorem 2. For $m = 1$, the extended system $\bar{\mathcal{S}}_c$ in (9) is stable if and only if the system $\mathcal{S}_c$ in (1) is stable.

Proof. ($\Rightarrow$) Suppose $\bar{\mathcal{S}}_c$ is stable. From (8), the Laplacian system matrix, $\bar{L}$, for $\bar{\mathcal{S}}_c$ is expressed as

$$\bar{L} = \frac{1}{2}\begin{bmatrix} \hat{L}+L & \hat{L}-L \\ \hat{L}-L & \hat{L}+L \end{bmatrix}$$

where L and $\hat{L}$ are both Laplacian matrices defined as in (3) and (19), respectively. By using the similarity transformation matrix

$$T = \begin{bmatrix} I_n & 0_{n\times n} \\ I_n & I_n \end{bmatrix} \tag{22}$$

we have

$$T\bar{L}T^{-1} = \begin{bmatrix} L & \frac{1}{2}\left(\hat{L}-L\right) \\ 0_{n\times n} & \hat{L} \end{bmatrix} \tag{23}$$

from which we conclude that the eigenvalues of $\bar{L}$ are the combined eigenvalues of L and $\hat{L}$. Since $\bar{\mathcal{S}}_c$ is stable and L in (3) is the system matrix for $\mathcal{S}_c$, we conclude that $\mathcal{S}_c$ is stable.
($\Leftarrow$) Suppose $\mathcal{S}_c$ is stable, i.e., L is a Laplacian matrix. Note from (23) that $\bar{L}$ has eigenvalues of L and $\hat{L}$. As $\hat{L}$ is a Laplacian matrix by definition in (19) and $\mathcal{S}_c$ is stable, it follows that $\bar{\mathcal{S}}_c$ is stable.

Theorem 3. For $m \geqslant 2$, if the extended system $\bar{\mathcal{S}}_c$ in (9) is stable, then the system $\mathcal{S}_c$ in (1) is stable.

Proof. Suppose $\bar{\mathcal{S}}_c$ is stable. From (23), recall that the eigenvalues of $\bar{L}$ for $\bar{\mathcal{S}}_c$ are the union of the eigenvalues of L and $\hat{L}$. By using this result, the eigenvalues of $\bar{\mathcal{L}}_m$ can be obtained with respect to the eigenvalues of L and $\hat{L}$ as

$$\bar{\mu}^m + \gamma_m\lambda_i\bar{\mu}^{m-1} + \ldots + \gamma_2\lambda_i\bar{\mu} + \gamma_1\lambda_i = 0, \quad i = 1,2,\ldots,n \tag{24a}$$

$$\bar{\mu}^m + \gamma_m\hat{\lambda}_i\bar{\mu}^{m-1} + \ldots + \gamma_2\hat{\lambda}_i\bar{\mu} + \gamma_1\hat{\lambda}_i = 0, \quad i = 1,2,\ldots,n \tag{24b}$$

where $\bar{\mu}$ is an eigenvalue of $\bar{\mathcal{L}}_m$, λ_i and $\hat{\lambda}_i$ are the eigenvalues of L and $\hat{L}$, respectively. Note that (24a) and (24b) together are the characteristic polynomial of the system $\bar{\mathcal{S}}_c$ and only (24a) is the characteristic polynomial of the system $\mathcal{S}_c$. This implies that the characteristic polynomial of $\bar{\mathcal{S}}_c$ includes that of $\mathcal{S}_c$. Given that $\bar{\mathcal{S}}_c$ is stable, so is $\mathcal{S}_c$.

From Theorem 3, we note that the stability of $\bar{\mathcal{S}}_c$ implies the stability of $\mathcal{S}_c$ for $m \geqslant 2$. On the other hand, a stable $\mathcal{S}_c$ does not necessarily imply a stable $\bar{\mathcal{S}}_c$ as illustrated in the following example.

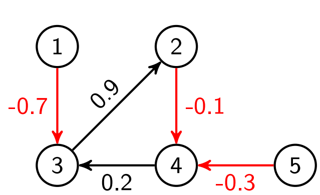

Fig. 3. Signed digraph in Example 2.

Example 2. Consider the signed network in Fig. 3.

For the given network, the signed Laplacian matrix, L, and its unsigned version, $\hat{L}$, are obtained as follows:

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & -0.9 & 0 & 0 \\ 0.7 & 0 & 0.9 & -0.2 & 0 \\ 0 & 0.1 & 0 & 0.4 & 0.3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\hat{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & -0.9 & 0 & 0 \\ -0.7 & 0 & 0.9 & -0.2 & 0 \\ 0 & -0.1 & 0 & 0.4 & -0.3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues of $-L$ and $-\hat{L}$ are calculated as $\{-1.0646, -0.6, -0.5354, 0, 0\}$ and $\{-0.9289 \pm j0.1773, -0.3422, 0, 0\}$, respectively. Note that the spectra of $-L$ and $-\hat{L}$ are on the OLHP (open left half plane) except zero eigenvalues. From Theorem 2, the eigenvalues of $-\bar{L}$ is the union of the eigenvalues of $-L$ and $-\hat{L}$, i.e., $\{-1.0646, -0.6, -0.5354, 0, 0\} \bigcup \{-0.9289 \pm j0.1773, -0.3422, 0, 0\}$. In the light of this fact, the following results are obtained:

- For $m = 1$, system matrices for $\mathcal{S}_c$ and $\bar{\mathcal{S}}_c$ are $-L$ and $-\bar{L}$, respectively. Therefore, both systems are stable.
- For $m = 2$ and $\gamma_1 = 10, \gamma_2 = 0.5$, system matrices for $\mathcal{S}_c$ and $\bar{\mathcal{S}}_c$ are $\mathcal{L}_m$ and $\bar{\mathcal{L}}_m$, respectively. All nonzero eigenvalues of $\mathcal{L}_m$ have negative real parts. In the corresponding extended system matrix $\bar{\mathcal{L}}_m$, there is a pair of complex conjugate eigenvalues whose real parts are positive $(0.0548 \pm j3.0971)$. In this case, $\mathcal{S}_c$ is stable whereas $\bar{\mathcal{S}}_c$ is unstable.

In the case of structurally balanced systems, the stability of $\mathcal{S}_c$ and $\bar{\mathcal{S}}_c$ are equivalent regardless of the degree of the dynamics as delineated in the next result.

Corollary 1: Consider a structurally balanced system $\mathcal{S}_c$ in (1). For $m \geqslant 1$, the extended system $\bar{\mathcal{S}}_c$ in (9) is stable if and only if the system $\mathcal{S}_c$ is stable.

Proof. Given a structurally balanced system $\mathcal{S}_c$ in (1), its signed Laplacian matrix, L, in (3) can be expressed as

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

where the elements of $L_{11} \in \mathbb{R}^{n_1\times n_1}$ and $L_{22} \in \mathbb{R}^{n_2\times n_2}$ contains positive edges and the elements of $L_{12} \in \mathbb{R}^{n_1\times n_2}$ and $L_{21} \in \mathbb{R}^{n_2\times n_1}$ contain negative edges. Therefore, one can obtain $\hat{L}$ in (19) by using the similarity transformation as

$$\hat{L} = \begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} -I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & I_{n_2} \end{bmatrix}\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} -I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & I_{n_2} \end{bmatrix}$$

This implies that L and $\hat{L}$ have the same eigenvalues. From (23), $\bar{L}$ also has the same eigenvalues with L. In the structurally balanced case, the stabilities of $\bar{\mathcal{S}}_c$ in (9) and $\mathcal{S}_c$ in (1) are equivalent for $m \geqslant 1$.

For any given higher-order system with negative links, its extended higher-order system with positive links can be obtained by using Definition 3. In this case, the following result in [21] will be useful in determining the number of clusters for any given higher-order system with antagonistic interactions.

Lemma 1. [21] The number of clusters for the m-th order system $\mathcal{S}_c$ in (1) associated with an unsigned digraph is equal to $l_p + l_s$ if and only if the roots of (6) are on the OLHP except ml_p zero eigenvalues where l_p and l_s are the number of primary and secondary layer subgraphs, respectively.

Note that Lemma 1 is applicable for networks whose communication links are positive weighted. By using Definition 3, a signed digraph can be augmented to its extended digraph whose edges are positive. In this context, the following result is obtained to determine the clusters for a general higher-order multi-agent system that has no restrictive structure properties.

Theorem 4. The number of clusters for the m-th order system $\mathcal{S}_c$ in (1) associated with an signed digraph is equal to $\bar{l}_p + \bar{l}_s - l'$ if the roots of (6) are on the OLHP where $\bar{l}_p$ and $\bar{l}_s$ are the number of primary and secondary layer subgraphs of the extended system $\bar{\mathcal{S}}_c$ in (9), respectively; and l' is the number of subgraphs whose node set consists of only a subset of $\bar{\mathcal{V}} \setminus \mathcal{V} = \{v'_1, \dots, v'_n\}$.

Proof. From Definition 3, it follows that the extended graph $\bar{\mathcal{G}}$ contains non-negative edge weights. Hence, Lemma 1 is applicable. For a given system $\mathcal{S}_c$ with n agents, its extended representation $\bar{\mathcal{S}}_c$ with $2n$ agents is obtained based on Definition 3. By using Theorem 1 in [21], we express by $\bar{n}_{p,i}$ and $\bar{n}_{s,j}$, the number of nodes in the i-th primary and j-th secondary layer subgraphs of the extended system, respectively for $i = 1, \dots, \bar{l}_p$ and $j = 1, \dots, \bar{l}_s$. Let $\bar{n}_p = \sum_i \bar{n}_{p,i}$ and $\bar{n}_s = \sum_j \bar{n}_{s,j}$ be the total number of nodes in the primary and secondary layer subgraphs, respectively and $\bar{n}_p + \bar{n}_s = \bar{n} = 2n$ holds. For positive γ_i $(i = 1, \dots, m)$, $\bar{\mathcal{L}}_m$ has exactly $m\bar{l}_p$ zero eigenvalues. From [21], there are $\bar{l}_p$ right eigenvectors associated with zero eigenvalue. Therefore, $\bar{\mathcal{L}}_m$ cannot be diagonal. The $\bar{l}_p$ right eigenvectors and $(m-1)\bar{l}_p$ generalized right eigenvectors associated with zero eigenvalue can be obtained as follows for $i = 1, \dots, \bar{l}_p$

$$
\begin{aligned}
\xi_{i,1} &= \begin{bmatrix} \kappa_i^T & -\bar{L}_s^{-1}\bar{L}_{sp}\kappa_i^T & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & \cdots & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T \end{bmatrix}^T \\
&\quad \text{are right eigenvectors} \\
\xi_{i,2} &= \begin{bmatrix} 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & \kappa_i^T & -\bar{L}_s^{-1}\bar{L}_{sp}\kappa_i^T & \cdots & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T \end{bmatrix}^T \\
&\ \vdots \\
\xi_{i,m-1} &= \begin{bmatrix} 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & \cdots & \kappa_i^T & -\bar{L}_s^{-1}\bar{L}_{sp}\kappa_i^T & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T \end{bmatrix}^T \\
\xi_{i,m} &= \begin{bmatrix} 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & \cdots & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & \kappa_i^T & -\bar{L}_s^{-1}\bar{L}_{sp}\kappa_i^T \end{bmatrix}^T \\
&\quad \text{are generalized right eigenvectors}
\end{aligned}
$$

where $\bar{L}_p$, $\bar{L}_s$ and $\bar{L}_{sp}$ are defined in [21]. The $(m-1)\bar{l}_p$ generalized left eigenvectors and $\bar{l}_p$ left eigenvectors associated with zero eigenvalue can be obtained as follows for $i = 1, \dots, \bar{l}_p$

$$
\begin{aligned}
\eta_{i,1} &= \begin{bmatrix} \varsigma_i^T & 0_{\bar{n}_s}^T & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & \cdots & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T \end{bmatrix}^T \\
\eta_{i,2} &= \begin{bmatrix} 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & \varsigma_i^T & 0_{\bar{n}_s}^T & \cdots & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T \end{bmatrix}^T \\
&\ \vdots \\
\eta_{i,m-1} &= \begin{bmatrix} 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & \cdots & \varsigma_i^T & 0_{\bar{n}_s}^T & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T \end{bmatrix}^T \\
&\quad \text{are generalized left eigenvectors} \\
\eta_{i,m} &= \begin{bmatrix} 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & \cdots & 0_{\bar{n}_p}^T & 0_{\bar{n}_s}^T & \varsigma_i^T & 0_{\bar{n}_s}^T \end{bmatrix}^T \\
&\quad \text{are left eigenvectors}
\end{aligned}
$$

where $\varsigma_i^T \kappa_i = 1$ and their properties are given in [21]. Therefore, $\bar{\mathcal{L}}_m$ has exactly a zero eigenvalue of algebraic multiplicity $m\bar{l}_p$ and all the other eigenvalues have negative real parts. Then there exists a nonsingular matrix $P \in \mathbb{R}^{m\bar{n} \times m\bar{n}}$ such that $P^{-1}\bar{\mathcal{L}}_m P = J$, where J is the Jordan canonical form associated with $\bar{\mathcal{L}}_m$. One can obtain the following:

$$
\bar{\mathcal{L}}_m = PJP^{-1} = \left[\xi_{1,1}, \dots, \xi_{1,m}, \xi_{2,1} \dots, \xi_{\bar{l}_p,1}, \xi_{m\bar{l}_p+1}, \dots, \xi_{m\bar{n}}\right]
\begin{bmatrix} J_0 & & & \\ & \ddots & & \\ & & J_0 & \\ & & & \tilde{J} \end{bmatrix}
\begin{bmatrix} \eta_{1,1}^T \\ \vdots \\ \eta_{1,m}^T \\ \vdots \\ \eta_{\bar{l}_p,1} \\ \vdots \\ \eta_{\bar{l}_p,m} \\ \eta_{m\bar{l}_p+1} \\ \vdots \\ \eta_{m\bar{n}} \end{bmatrix}
\text{ with } J_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{m \times m}
$$

where J_0 are the blocks associated with zero eigenvalue, $\tilde{J}$ is the upper diagonal Jordan block matrix with the nonzero eigenvalues and ξ_j and η_j, $j = m\bar{l}_p + 1, \dots, m\bar{n}$ are the rest of the right and left eigenvectors of $\bar{\mathcal{L}}_m$, respectively. One has

$$
e^{\bar{\mathcal{L}}_m t} = e^{P(Jt)P^{-1}} = P \begin{bmatrix} e^{J_0 t} & & & \\ & \ddots & & \\ & & e^{J_0 t} & \\ & & & e^{\tilde{J}t} \end{bmatrix} P^{-1}
$$

$$
\text{where } e^{J_0 t} = \begin{bmatrix} 1 & \frac{t}{1!} & \frac{t^2}{2!} & \cdots & \frac{t^{(m-1)}}{(m-1)!} \\ 0 & 1 & \frac{t}{1!} & \cdots & \frac{t^{(m-2)}}{(m-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \text{ and } \lim_{t\to\infty} e^{\tilde{J}t} \to 0_{m(\bar{n}-\bar{l}_p) \times m(\bar{n}-\bar{l}_p)}.
$$

Let $y(t)$ be defined as

$$
y(t) = \left(\begin{bmatrix}
I_{\bar{n}_p} & tI_{\bar{n}_p} & \cdots & \frac{t^{(m-1)}}{(m-1)!} I_{\bar{n}_p} \\
0_{\bar{n}_p \times \bar{n}_p} & I_{\bar{n}_p} & \cdots & \frac{t^{(m-2)}}{(m-2)!} I_{\bar{n}_p} \\
\vdots & \vdots & \ddots & \vdots \\
0_{\bar{n}_p \times \bar{n}_p} & 0_{\bar{n}_p \times \bar{n}_p} & \cdots & I_{\bar{n}_p} \\
-\bar{L}_s^{-1}\bar{L}_{sp} & -t\bar{L}_s^{-1}\bar{L}_{sp} & \cdots & -\frac{t^{(m-1)}}{(m-1)!}\bar{L}_s^{-1}\bar{L}_{sp} \\
0_{\bar{n}_s \times \bar{n}_p} & -\bar{L}_s^{-1}\bar{L}_{sp} & \cdots & -\frac{t^{(m-2)}}{(m-2)!}\bar{L}_s^{-1}\bar{L}_{sp} \\
\vdots & \vdots & \ddots & \vdots \\
0_{\bar{n}_s \times \bar{n}_p} & 0_{\bar{n}_s \times \bar{n}_p} & \cdots & -\bar{L}_s^{-1}\bar{L}_{sp}
\end{bmatrix} \left(I_m \otimes \boldsymbol{\kappa}\boldsymbol{\varsigma}^T\right) \right) \begin{bmatrix} \bar{x}_p^{(1)}(0) \\ \vdots \\ \bar{x}_p^{(m)}(0) \end{bmatrix}
$$

where $\bar{x}_p^{(q)}(t)$, $\bar{x}_s^{(q)}(t)$, $q = 1, \dots, m$ are the state vectors for the primary and secondary layer subgraphs; and $\boldsymbol{\kappa} = \left[\kappa_1, \dots, \kappa_{\bar{l}_p}\right]$, $\boldsymbol{\varsigma} = \left[\varsigma_1, \dots, \varsigma_{\bar{l}_p}\right]$ are $\bar{n}_p \times \bar{l}_p$ matrices containing the right and left eigenvectors associated with zero eigenvalues of $\bar{L}_p$, respectively. $\lim_{t\to\infty}\|\bar{x}(t) - y(t)\| = \lim_{t\to\infty}\|e^{\bar{\mathcal{L}}_m t}\bar{x}(0) - y(t)\|$ can be computed as

$$
\begin{aligned}
\lim_{t\to\infty}\|\bar{x}(t)-y(t)\| &= \lim_{t\to\infty}\left\| \begin{bmatrix} \bar{x}_p^{(1)}(t) \\ \vdots \\ \bar{x}_p^{(m)}(t) \\ \bar{x}_s^{(1)}(t) \\ \vdots \\ \bar{x}_s^{(m)}(t) \end{bmatrix} - y(t) \right\| = \lim_{t\to\infty}\left\| e^{\bar{\mathcal{L}}_m t} \begin{bmatrix} \bar{x}_p^{(1)}(0) \\ \vdots \\ \bar{x}_p^{(m)}(0) \\ \bar{x}_s^{(1)}(0) \\ \vdots \\ \bar{x}_s^{(m)}(0) \end{bmatrix} - y(t) \right\| \\
&= \lim_{t\to\infty}\left\| \begin{bmatrix}
\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & t\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & \cdots & \frac{t^{(m-1)}}{(m-1)!}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & 0_{\bar{n}_p\times m\bar{n}_s} \\
0_{\bar{n}_p\times\bar{n}_p} & \boldsymbol{\kappa}\boldsymbol{\varsigma}^T & \cdots & \frac{t^{(m-2)}}{(m-2)!}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & 0_{\bar{n}_p\times m\bar{n}_s} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{\bar{n}_p\times\bar{n}_p} & 0_{\bar{n}_p\times\bar{n}_p} & \cdots & \boldsymbol{\kappa}\boldsymbol{\varsigma}^T & 0_{\bar{n}_p\times m\bar{n}_s} \\
-\bar{L}_s^{-1}\bar{L}_{sp}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & -t\bar{L}_s^{-1}\bar{L}_{sp}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & \cdots & -\frac{t^{(m-1)}}{(m-1)!}\bar{L}_s^{-1}\bar{L}_{sp}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & 0_{\bar{n}_p\times m\bar{n}_s} \\
0_{\bar{n}_s\times\bar{n}_p} & -\bar{L}_s^{-1}\bar{L}_{sp}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & \cdots & -\frac{t^{(m-2)}}{(m-2)!}\bar{L}_s^{-1}\bar{L}_{sp}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & 0_{\bar{n}_p\times m\bar{n}_s} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{\bar{n}_s\times\bar{n}_p} & 0_{\bar{n}_s\times\bar{n}_p} & \cdots & -\bar{L}_s^{-1}\bar{L}_{sp}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & 0_{\bar{n}_p\times m\bar{n}_s}
\end{bmatrix} \begin{bmatrix} \bar{x}_p^{(1)}(0) \\ \vdots \\ \bar{x}_p^{(m)}(0) \\ \bar{x}_s^{(1)}(0) \\ \vdots \\ \bar{x}_s^{(m)}(0) \end{bmatrix} - y(t) \right\| \\
&= \lim_{t\to\infty}\left\| \begin{bmatrix}
\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & t\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & \cdots & \frac{t^{(m-1)}}{(m-1)!}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T \\
0_{\bar{n}_p\times\bar{n}_p} & \boldsymbol{\kappa}\boldsymbol{\varsigma}^T & \cdots & \frac{t^{(m-2)}}{(m-2)!}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T \\
\vdots & \vdots & \ddots & \vdots \\
0_{\bar{n}_p\times\bar{n}_p} & 0_{\bar{n}_p\times\bar{n}_p} & \cdots & \boldsymbol{\kappa}\boldsymbol{\varsigma}^T \\
-\bar{L}_s^{-1}\bar{L}_{sp}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & -t\bar{L}_s^{-1}\bar{L}_{sp}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & \cdots & -\frac{t^{(m-1)}}{(m-1)!}\bar{L}_s^{-1}\bar{L}_{sp}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T \\
0_{\bar{n}_s\times\bar{n}_p} & -\bar{L}_s^{-1}\bar{L}_{sp}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T & \cdots & -\frac{t^{(m-2)}}{(m-2)!}\bar{L}_s^{-1}\bar{L}_{sp}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T \\
\vdots & \vdots & \ddots & \vdots \\
0_{\bar{n}_s\times\bar{n}_p} & 0_{\bar{n}_s\times\bar{n}_p} & \cdots & -\bar{L}_s^{-1}\bar{L}_{sp}\boldsymbol{\kappa}\boldsymbol{\varsigma}^T
\end{bmatrix} \begin{bmatrix} \bar{x}_p^{(1)}(0) \\ \vdots \\ \bar{x}_p^{(m)}(0) \end{bmatrix} - y(t) \right\| \\
&= \lim_{t\to\infty}\left\| \left(\begin{bmatrix}
I_{\bar{n}_p} & tI_{\bar{n}_p} & \cdots & \frac{t^{(m-1)}}{(m-1)!}I_{\bar{n}_p} \\
0_{\bar{n}_p\times\bar{n}_p} & I_{\bar{n}_p} & \cdots & \frac{t^{(m-2)}}{(m-2)!}I_{\bar{n}_p} \\
\vdots & \vdots & \ddots & \vdots \\
0_{\bar{n}_p\times\bar{n}_p} & 0_{\bar{n}_p\times\bar{n}_p} & \cdots & I_{\bar{n}_p} \\
-\bar{L}_s^{-1}\bar{L}_{sp} & -t\bar{L}_s^{-1}\bar{L}_{sp} & \cdots & -\frac{t^{(m-1)}}{(m-1)!}\bar{L}_s^{-1}\bar{L}_{sp} \\
0_{\bar{n}_s\times\bar{n}_p} & -\bar{L}_s^{-1}\bar{L}_{sp} & \cdots & -\frac{t^{(m-2)}}{(m-2)!}\bar{L}_s^{-1}\bar{L}_{sp} \\
\vdots & \vdots & \ddots & \vdots \\
0_{\bar{n}_s\times\bar{n}_p} & 0_{\bar{n}_s\times\bar{n}_p} & \cdots & -\bar{L}_s^{-1}\bar{L}_{sp}
\end{bmatrix} \left(I_m \otimes \boldsymbol{\kappa}\boldsymbol{\varsigma}^T\right) \right) \begin{bmatrix} \bar{x}_p^{(1)}(0) \\ \vdots \\ \bar{x}_p^{(m)}(0) \end{bmatrix} - y(t) \right\| = 0
\end{aligned}
$$

From [22], the system $\bar{\mathcal{S}}_c$ that is extended representation of the system $\mathcal{S}_c$ converges to $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$ clusters. Since l' subgraphs in $\bar{l}_p + \bar{l}_s$ subgraphs do not contain any node of the system $\mathcal{S}_c$, the number of clusters for system $\mathcal{S}_c$ is found as

$$\mathcal{K} = \bar{l}_p + \bar{l}_s - l'.$$

Remark 3. Theorem 4 delineates the condition to determine the number of clusters for any digraph with positively or negatively weighted edges from its extended digraph. If the original system $\mathcal{S}_c$ is completely given by non-negative weights with l_p primary and l_s secondary layer subgraphs, then the corresponding extended system $\bar{\mathcal{S}}_c$ has $\bar{l}_p = 2l_p$ primary and $\bar{l}_s = 2l_s$ secondary layer subgraphs with $l' = l_p + l_s$. From Theorem 4, the number of clusters for $\mathcal{S}_c$ is $\mathcal{K} = \bar{l}_p + \bar{l}_s - l' = l_p + l_s$ which agrees with the result in [20].

Remark 4. It can also be shown that the existing results in the literature on the special class of structurally balanced signed digraphs can be recovered from Theorem 4. Systems with structurally balanced digraphs always converge to 2 clusters [13–17]. Extended digraph of a structurally balanced has 2 independent subgraphs each of which contains a spanning tree. Therefore, it can be found that $\bar{l}_p = 2$ primary and $\bar{l}_s = 0$ secondary layer subgraphs for the extended system. Since each primary layer subgraph always has at least one member of the original system, it can be obtained as $l' = 0$. Consequently, a structurally balanced system attains $\bar{l}_p + \bar{l}_s - l' = 2$ equilibria that agrees with the existing results in the literature.

The above result brings a novel contribution to the study of the cluster consensus problem for higher-order multi-agent systems. In the literature, there is no result on determining the number of clusters to which a given higher-order system with negative links converges.

Given a multi-agent system with a signed digraph, the application of Theorem 4 can be summarized as in Algorithm 1.

Algorithm 1:Algorithm to calculate the number of clusters in a given signed digraph

---

Given a multi-agent system of n agents with signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{v_1, \ldots, v_n\}$

Step 1: Construct the multi-agent system of $2n$ agents with $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$ where $\mathcal{V} = \{v_1, \ldots, v_n, v\prime_1, \ldots, v\prime_n\}, \mathcal{E}$ and $\mathcal{A}$ satisfy Definition 3.

Step 2: Determine the primary layer subgraphs $\bar{\mathcal{G}}_{p,i}$ for $i = 1, \ldots, \bar{l}_p$, in $\bar{\mathcal{G}}$ by using the detection algorithm in [20].

Step 3: Determine the secondary layer subgraphs $\bar{\mathcal{G}}_{s,i}$ for $i = 1, \ldots, \bar{l}_s$, in $\bar{\mathcal{G}}$ by using the detection algorithm in [20].

Step 4: Determine the subgraphs among $\bar{\mathcal{G}}_{p,i}$ and $\bar{\mathcal{G}}_{s,j}$ for $i = 1, \ldots, \bar{l}_p,\ j = 1, \ldots, \bar{l}_s$ whose node set consists of only a subset of $\bar{\mathcal{V}} \setminus \mathcal{V} = \{v'_1, \ldots, v'_n\}$. Let the total number of such subgraphs be l'

Step 5: Calculate the number of clusters in $\mathcal{G}$ as $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$.

Given a signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, the time complexity of Algorithm 1 can be obtained as follows:

- **Step 1:** The extended unsigned digraph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$ is constructed based on Definition 3 where the node set $\bar{\mathcal{V}}$ is duplicated that requires $2n$ operations and the edge set $\bar{\mathcal{E}}$ is extended based on the connections that requires $4n^2 - 2n$ operations. Hence, the time complexity of Step 1 is $O(n^2)$.
- **Step 2:** The time complexity of the primary layer subgraph detection algorithm is studied in [20] and found to be $O(n^4)$.
- **Step 3:** In [20], the time complexity of the secondary layer subgraph detection algorithm is found to be $O(n^6)$.
- **Step 4:** Searching n elements $(v'_1, \ldots, v'_n)$ in set of $2n$ elements in the worst case requires at most $2n^2$ comparisons. Therefore, the time complexity of Step 4 is $O(n^2)$.
- **Step 5:** As this step involves a scalar calculation, its time complexity is $O(1)$.

Based on the above discussion, Algorithm 1 is polynomial time and its time complexity is given by $O(n^6)$.

3.1. Controller parameter selection

In this section, conditions on the choice of controller parameters $\left(\gamma_q\right)$ are investigated to achieve cluster consensus for systems with any order of dynamics. In deriving the general stability condition for m-th order systems and their extended dynamics, consider the following characteristic polynomials with complex coefficients

$$f(\mu) = \mu^m + \gamma_m \lambda_i \mu^{m-1} + \ldots + \gamma_2 \lambda_i \mu + \gamma_1 \lambda_i \tag{25a}$$

$$\hat{f}(\mu) = \mu^m + \gamma_m \hat{\lambda}_i \mu^{m-1} + \ldots + \gamma_2 \hat{\lambda}_i \mu + \gamma_1 \hat{\lambda}_i \tag{25b}$$

that should have its roots in the open left half of the complex plane. Associated with (25a) or (25b), define Υ as

$$\Upsilon = \begin{bmatrix} \alpha'_1 & -\alpha''_2 & -\alpha'_3 & \alpha''_4 & \alpha'_5 & \cdots & (-1)^{m+1}\alpha'_{2m-1} \\ 1 & -\alpha''_1 & -\alpha'_2 & \alpha''_3 & \alpha'_4 & \cdots & (-1)^{m+1}\alpha'_{2m-2} \\ 0 & \alpha'_1 & -\alpha''_2 & -\alpha'_3 & \alpha''_4 & \cdots & (-1)^{m+1}\alpha'_{2m-2} \\ 0 & 1 & -\alpha''_1 & -\alpha'_2 & \alpha''_3 & \cdots & (-1)^{m+1}\alpha'_{2m-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{(2m-1)\times(2m-1)} \tag{26}$$

where $\alpha'_r = Re(\gamma_{m-r+1}\lambda_i)$ or $\alpha'_r = Re(\gamma_{m-r+1}\hat{\lambda}_i)$, $\alpha''_r = Im(\gamma_{m-r+1}\lambda_i)$ or $\alpha''_r = Im(\gamma_{m-r+1}\hat{\lambda}_i)$ and $\alpha_r = 0$ for $r > m$. Let $\Upsilon_{ii}, i = 1, 2, \ldots, 2m-1$ denote the leading principal minors of Υ. From [21], it follows that the roots of (25a) and (25b) are on the OLHP, i.e., the m-th order systems are stable if and only if all members of the sequence of $1, \Upsilon_{11}, \Upsilon_{33}, \ldots, \Upsilon_{2m-1,2m-1}$ are positive. These conditions can be simplified for second and third-order systems as follows.

For second-order systems:

- $\gamma_2 > 0$,
- $\frac{\gamma_2^2}{\gamma_1} > \max_i \frac{Im^2(\lambda_i)}{Re(\lambda_i)|\lambda_i|^2}$.

(λ_i is any non-zero eigenvalue of L or $\hat{L}$)

For third-order systems:

- $\gamma_3 > 0$,
- $\gamma_3^2\gamma_2 Re(\lambda_i)|\lambda_i|^2 - \gamma_3\gamma_1 Re^2(\lambda_i) - \gamma_2^2 Im^2(\lambda_i) > 0$,
- $\gamma_2^2\gamma_3^2 Re(\lambda_i)|\lambda_i|^4 - 2\gamma_1\gamma_2\gamma_3 Re^2(\lambda_i)|\lambda_i|^2 - \gamma_2^3 Im^2(\lambda_i)|\lambda_i|^2 + \gamma_1^2 Re^3(\lambda_i) > 0$.

(λ_i is any non-zero eigenvalue of L or $\hat{L}$)

Under the conditions given above for second and third-order systems, the extended representation of any signed network is stable from [21]. In this case, the number of clusters for the extended system can be calculated as $\bar{l}_p + \bar{l}_s$ based on Theorem 4 where $\bar{l}_p$ and $\bar{l}_s$ are the number of primary and secondary layer subgraphs of the extended system, respectively. Since l' is the number of subgraphs containing only nodes from $\{v'_1, \ldots, v'_n\}$, it can be concluded that the number of clusters for a signed network is equal to $\bar{l}_p + \bar{l}_s - l'$ provided that the following respective conditions are satisfied.

4. Cluster consensus in signed discrete-time networks

Consider a multi-agent system consisting of n agents with m-th order integrator dynamics in discrete time.

$$\begin{aligned} \mathcal{S}_d: \quad x_i^{(1)}(k+1) &= x_i^{(1)}(k) + x_i^{(2)}(k) \\ x_i^{(2)}(k+1) &= x_i^{(2)}(k) + x_i^{(3)}(k) \\ &\vdots \\ x_i^{(m-1)}(k+1) &= x_i^{(m-1)}(k) + x_i^{(m)}(k) \\ x_i^{(m)}(k+1) &= x_i^{(m)}(k) + u_i(k) \end{aligned} \tag{27}$$

with the initial condition $x_i^{(q)}(0) = x_{i0}^{(q)}$ where $x_i^{(q)}(k)$ ($i = 1, \ldots, n$ and $q = 1, \ldots, m$), are the states of agent i at time step k. We introduce the control input of agent i as follows

$$u_i(k) = \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} a_{ij}\left(x_j^{(q)}(k) - \mathrm{sgn}(a_{ij})x_i^{(q)}(k)\right) \tag{28}$$

where a_{ij} are the elements of the adjacency matrix and $\gamma_q > 0$ ($q = 1, \ldots, m$) are the controller parameters to be designed.

System $\mathcal{S}_d$ will be investigated under first and higher-order ($m \geqslant 2$) dynamics separately. In this context, system $\mathcal{S}_d$ can be expressed for first-order dynamics as

$$x(k+1) = Wx(k) \tag{29}$$

where $x(k) = [x_1(k), \ldots, x_n(k)]^T$ is the state vector of the first-order system at time step k and $W = [w_{ij}]$ is the system matrix. The following conditions on the system matrix are assumed to be satisfied.

Assumption 1.

(i) $w_{ii} > 0$ for all $i = 1, \ldots, n$,
(ii) $w_{ij} \neq 0$ for all $(v_j, v_i) \in \mathcal{E}$,
(iii) $w_{ij} = 0$ for all $(v_j, v_i) \notin \mathcal{E}$,
(iv) $\sum_{j=1}^{n} |w_{ij}| = 1$ for all $i = 1, \ldots, n$

Assumption 1(i) ensures that each agent uses its local information by positive weighting. Assumptions 1(ii) and (iii) guarantee that the weighting coefficients are nonzero if there is a direct connection between the agents and zero if there is none. The absolute values of the weighting coefficients used by an agent sum up to 1 as stated in Assumption 1(iv). Assumption 1 assures that the matrix $\mathbb{W} = [|w_{ij}|] \in \mathbb{R}^{n\times n}$ is a row-stochastic matrix.

For $m \geqslant 2$, the state dynamics given in (27) and (28) can be expressed in matrix form as

$$x(k+1) = \Gamma_m x(k) \tag{30}$$

where the system matrix Γ_m is defined as

$$\Gamma_m = \begin{bmatrix} I_n & I_n & 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & I_n & I_n & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} & I_n & I_n \\ -\gamma_1 L & -\gamma_2 L & \cdots & \cdots & -\gamma_{m-1}L & I_n - \gamma_m L \end{bmatrix}_{mn\times mn}. \tag{31}$$

Given $\mathcal{S}_d$ in (27) evolving over $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, the extended system $\bar{\mathcal{S}}_d$ is defined as

$$\begin{aligned} \bar{\mathcal{S}}_d: \quad \bar{x}_i^{(1)}(k+1) &= \bar{x}_i^{(1)}(k) + \bar{x}_i^{(2)}(k) \\ \bar{x}_i^{(2)}(k+1) &= \bar{x}_i^{(2)}(k) + \bar{x}_i^{(3)}(k) \\ &\vdots \\ \bar{x}_i^{(m-1)}(k+1) &= \bar{x}_i^{(m-1)}(k) + \bar{x}_i^{(m)}(k) \\ \bar{x}_i^{(m)}(k+1) &= \bar{x}_i^{(m)}(k) + \bar{u}_i(k) \end{aligned} \tag{32}$$

with the initial condition $\bar{x}_i^{(q)}(0) = \bar{x}_{i0}^{(q)}$ for $i = 1, \ldots, 2n$ and $q = 1, \ldots, m$. The control input is given as

$$\bar{u}_i(k) = \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{2n} \bar{a}_{ij}\left(\bar{x}_j^{(q)}(k) - \bar{x}_i^{(q)}(k)\right) \tag{33}$$

where $\bar{a}_{ij}$ are the non-negative weighting coefficients. The state dynamics given in (32) and (33) can be expressed in matrix form as

$$\bar{x}(k+1) = \bar{\Gamma}_m \bar{x}(k) \tag{34}$$

Here the system matrix $\bar{\Gamma}_m$ is defined as

$$\bar{\Gamma}_m = \begin{bmatrix} I_n & I_n & 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & I_n & I_n & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} & I_n & I_n \\ -\gamma_1\bar{L} & -\gamma_2\bar{L} & \cdots & \cdots & -\gamma_{m-1}\bar{L} & I_n-\gamma_m\bar{L} \end{bmatrix}_{2mn\times 2mn}. \tag{35}$$

and $\bar{x}(k) = \left[\bar{x}^{(1)}(k)^T, \bar{x}^{(2)}(k)^T, \ldots, \bar{x}^{(m)}(k)^T\right]^T \in \mathbb{R}^{2mn}$ with $\bar{x}^{(q)}(k) = \left[\bar{x}_1^{(q)}(k)^T, \ldots, \bar{x}_{2n}^{(q)}(k)^T\right]^T \in \mathbb{R}^{2n}$ $(q = 1,\ldots,m)$ and $\bar{L}$ is the unsigned Laplacian matrix of the extended graph $\bar{\mathcal{G}}$.

The state trajectory of the original system $\mathcal{S}_d$ can be recovered from that of the extended system $\bar{\mathcal{S}}_d$ as delineated by the following result:

Theorem 5. The trajectory $x(k) = \left[x_1(k)^T, x_2(k)^T, \ldots, x_n(k)^T\right]^T$ with $x_i(k) = \left[x_i^{(1)}(k), \ldots, x_i^{(m)}(k)\right]^T$ of system $\mathcal{S}_d$ in (27) can be recovered from that of system $\bar{\mathcal{S}}_d$ in (32) as $x_i^{(q)}(k) = \bar{x}_i^{(q)}(k)$, $i = 1,\ldots,n$ and $q = 1,\ldots,m$ for all $k \geqslant 0$ by setting $\bar{x}_{i0}^{(q)} = x_{i0}^{(q)}$ and $\bar{x}_{i+n,0}^{(q)} = -x_{i0}^{(q)}$ for $i = 1,\ldots,n$ and $q = 1,\ldots,m$.

Proof. The input of the extended system $\bar{\mathcal{S}}_d$ can be rewritten as follows

$$\bar{u}_i(k) = \sum_{q=1}^m \gamma_q \sum_{j=1}^n \bar{a}_{ij}\left(\bar{x}_j^{(q)}(k) - \bar{x}_i^{(q)}(k)\right) + \sum_{j=1}^n \bar{a}_{i,j+n}\left(\bar{x}_{j+n}^{(q)}(k) - \bar{x}_i^{(q)}(k)\right) \tag{36}$$

$$\begin{aligned} \bar{u}_{i+n}(k) &= \sum_{q=1}^m \gamma_q \sum_{j=1}^n \bar{a}_{i+n,j+n}\left(\bar{x}_{j+n}^{(q)}(k) - \bar{x}_{i+n}^{(q)}(k)\right) \\ &\quad + \sum_{j=1}^n \bar{a}_{i+n,j}\left(\bar{x}_j^{(q)}(k) - \bar{x}_{i+n}^{(q)}(k)\right) \end{aligned} \tag{37}$$

for $i = 1,\ldots,n$. Let $\hat{x}_i^{(q)}(k) = \bar{x}_i^{(q)}(k) + \bar{x}_{i+n}^{(q)}(k)$ for $i = 1,\ldots,n$ and $q = 1,\ldots,m$. Then we have

$$\begin{aligned} \hat{x}_i^{(q)}(k+1) &= \bar{x}_i^{(q)}(k+1) + \bar{x}_{i+n}^{(q)}(k+1) \\ &= \bar{x}_i^{(q)}(k) + \bar{x}_i^{(q+1)}(k) + \bar{x}_{i+n}^{(q)}(k) + \bar{x}_{i+n}^{(q+1)}(k) \\ &= \hat{x}_i^{(q)}(k) + \hat{x}_i^{(q+1)}(k) \end{aligned} \tag{38}$$

for $q = 1,\ldots,m-1$ and

$$\begin{aligned} \hat{x}_i^{(m)}(k+1) &= \bar{x}_i^{(m)}(k+1) + \bar{x}_{i+n}^{(m)}(k+1) \\ &= \bar{x}_i^{(m)}(k) + \bar{u}_i(k) + \bar{x}_{i+n}^{(m)}(k) + \bar{u}_{i+n}(k) \\ &= \hat{x}_i^{(m)}(k) + \sum_{q=1}^m \gamma_q \left(\sum_{j=1}^n \bar{a}_{ij}\left(\bar{x}_j^{(q)}(k) - \bar{x}_i^{(q)}(k)\right) + \sum_{j=1}^n \bar{a}_{i,j+n}\left(\bar{x}_{j+n}^{(q)}(k) - \bar{x}_i^{(q)}(k)\right)\right. \\ &\quad \left. + \sum_{j=1}^n \bar{a}_{i+n,j+n}\left(\bar{x}_{j+n}^{(q)}(k) - \bar{x}_{i+n}^{(q)}(k)\right) + \sum_{j=1}^n \bar{a}_{i+n,j}\left(\bar{x}_j^{(q)}(k) - \bar{x}_{i+n}^{(q)}(k)\right)\right) \end{aligned}$$

From Definition 3, we obtain

$$\bar{a}_{i+n,j} = \bar{a}_{i,j+n}, \quad \bar{a}_{ij} = \bar{a}_{i+n,j+n} \tag{39}$$

for $i = 1,\ldots,n$ and $|a_{ij}| = \bar{a}_{ij} + \bar{a}_{i+n,j}$. By using (39), we also have $\bar{a}_{i,j+n} + \bar{a}_{i+n,j+n} = \bar{a}_{ij} + \bar{a}_{i,j+n} = \bar{a}_{i+n,j} + \bar{a}_{i+n,j+n} = |a_{ij}|$. Then we have

$$\begin{aligned} \hat{x}_i^{(m)}(k+1) &= \bar{x}_i^{(m)}(k) \\ &\quad + \sum_{q=1}^m \gamma_q \left(\sum_{j=1}^n |a_{ij}|\bar{x}_j^{(q)}(k) + |a_{ij}|\bar{x}_{j+n}^{(q)}(k) - |a_{ij}|\bar{x}_i^{(q)}(k) - |a_{ij}|\bar{x}_{i+n}^{(q)}(k)\right) \\ &= \bar{x}_i^{(m)}(k) + \sum_{q=1}^m \gamma_q \sum_{j=1}^n |a_{ij}|\left(\hat{x}_j^{(q)}(k) - \hat{x}_i^{(q)}(k)\right) \end{aligned} \tag{40}$$

The dynamics of the system represented by (38) and (40) is equivalent to

$$\begin{aligned} \hat{\mathcal{S}}_d: \quad \hat{x}_i^{(1)}(k+1) &= \hat{x}_i^{(1)}(k) + \hat{x}_i^{(2)}(k) \\ \hat{x}_i^{(2)}(k+1) &= \hat{x}_i^{(2)}(k) + \hat{x}_i^{(3)}(k) \\ &\vdots \\ \hat{x}_i^{(m-1)}(k+1) &= \hat{x}_i^{(m-1)}(k) + \hat{x}_i^{(m)}(k) \\ \hat{x}_i^{(m)}(k+1) &= \bar{x}_i^{(m)}(k) + \sum_{q=1}^m \gamma_q \sum_{j=1}^n |a_{ij}|\left(\hat{x}_j^{(q)}(k) - \hat{x}_i^{(q)}(k)\right) \end{aligned} \tag{41}$$

which can be expressed in matrix form as

$$\hat{\mathcal{S}}_d: \quad \hat{x}(k+1) = \hat{\Gamma}_m \hat{x}(k), \quad \hat{x}(0) = \hat{x}_0 \tag{42}$$

where $\hat{x}(k) = \left[\hat{x}^{(1)}(k)^T, \hat{x}^{(2)}(k)^T, \ldots, \hat{x}^{(m)}(k)^T\right]^T \in \mathbb{R}^{mn}$ with $\hat{x}^{(q)}(k) = \left[\hat{x}_1^{(q)}(k)^T, \ldots, \hat{x}_n^{(q)}(k)^T\right]^T \in \mathbb{R}^n$ $(q = 1,\ldots,m)$ and

$$\hat{\Gamma}_m = \begin{bmatrix} I_n & I_n & 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & I_n & I_n & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} & I_n & I_n \\ -\gamma_1\hat{L} & -\gamma_2\hat{L} & \cdots & \cdots & -\gamma_{m-1}\hat{L} & I_n-\gamma_m\hat{L} \end{bmatrix}_{mn\times mn}. \tag{43}$$

where $\hat{x}_0$ is the initial condition. Since the solution of (42) is $\hat{x}(k) = \hat{\Gamma}_m^k \hat{x}_0$, the initial condition $\hat{x}_0 = [0,\ldots,0]^T$ implies $\hat{x}(k) = [0,\ldots,0]^T$ for all k, i.e., $\bar{x}_i^{(q)}(k) = -\bar{x}_{i+n}^{(q)}(k)$ for $i = 1,\ldots,n$ and $q = 1,\ldots,m$.

Substituting $\bar{x}_{j+n}^{(q)}(k) = -\bar{x}_j^{(q)}(k)$ into (36), we obtain

$$\bar{u}_i(k) = \sum_{q=1}^m \gamma_q \left(\sum_{j=1}^n \bar{a}_{ij}\left(\bar{x}_j^{(q)}(k) - \bar{x}_i^{(q)}(k)\right) + \sum_{j=1}^n \bar{a}_{i,j+n}\left(-\bar{x}_j^{(q)}(k) - \bar{x}_i^{(q)}(k)\right)\right) \tag{44}$$

$$= \sum_{q=1}^m \gamma_q \left(\sum_{j=1}^n (\bar{a}_{ij} - \bar{a}_{i,j+n})\bar{x}_j^{(q)}(k) - (\bar{a}_{ij} + \bar{a}_{i,j+n})\bar{x}_i^{(q)}(k)\right) \tag{45}$$

for $i = 1,\ldots,n$. From Definition 3, we get

$$\begin{aligned} \bar{u}_i(k) &= \sum_{q=1}^m \gamma_q \left(\sum_{j=1}^n \left(\frac{|a_{ij}| + a_{ij}}{2} - \frac{|a_{ij}| - a_{ij}}{2}\right)\bar{x}_j^{(q)}(k) - \left(\frac{|a_{ij}| + a_{ij}}{2} + \frac{|a_{ij}| - a_{ij}}{2}\right)\bar{x}_i^{(q)}(k)\right) \\ &= \sum_{q=1}^m \gamma_q \sum_{j=1}^n a_{ij}\left(x_j^{(q)}(k) - \mathrm{sgn}(a_{ij})x_i^{(q)}(k)\right) \end{aligned} \tag{46}$$

for $i = 1,\ldots,n$. Note that (46) is equal to (28). By setting $\bar{x}_i^{(q)}(0) = x_{i0}^{(q)}$ and $\bar{x}_{i+n}^{(q)}(0) = -x_{i0}^{(q)}$ for $i = 1,\ldots,n$ and $q = 1,\ldots,m$, the trajectory $x(k)$ can be recovered. This completes the proof.

While Theorem 5 focuses on the relation of the trajectories of $\mathcal{S}_d$ and $\bar{\mathcal{S}}_d$, it does not address the relationship between their stability properties.

Theorem 6. For $m = 1$, the extended system $\bar{\mathcal{S}}_d$ in (27) is stable if and only if the system $\mathcal{S}_d$ in (27) is stable.

Proof. ($\Rightarrow$) Suppose $\bar{\mathcal{S}}_d$ is stable. From (8), the system matrix $\bar{W}$ of the extended system $\bar{\mathcal{S}}_d$ is expressed as

$$\bar{W} = \frac{1}{2}\begin{bmatrix} \mathbb{W} + W & \mathbb{W} - W \\ \mathbb{W} - W & \mathbb{W} + W \end{bmatrix}$$

where $\mathbb{W} = [|w_{ij}|] \in \mathbb{R}^{n\times n}$. W and $\bar{W}$ are both system matrices satisfying Assumption 1. By using the similarity transformation matrix defined as in (22), we have

$$T\bar{L}T^{-1} = \begin{bmatrix} W & \frac{1}{2}(\mathbb{W} - W) \\ 0_{n\times n} & \mathbb{W} \end{bmatrix} \tag{47}$$

from which we conclude that the eigenvalues of $\bar{W}$ are the combined eigenvalues of W and $\mathbb{W}$. Since $\bar{\mathcal{S}}_d$ is stable and W is the system matrix for $\mathcal{S}_d$, we conclude that $\mathcal{S}_d$ is stable.
($\Leftarrow$) Suppose $\mathcal{S}_d$ is stable, i.e., W is a system matrix satisfying Assumption 1. Note from (47) that $\bar{W}$ has eigenvalues of W and $\mathbb{W}$. As $\mathbb{W}$ is a system matrix satisfying Assumption 1 and $\mathcal{S}_d$ is stable, it follows that $\bar{\mathcal{S}}_d$ is stable.

The following Theorem states the stability relationship between the original and the extended system.

Theorem 7. For $m \geqslant 2$, if the extended system $\bar{\mathcal{S}}_d$ in (32) is stable, then the system $\mathcal{S}_d$ in (27) is stable.

Proof. In order to prove the theorem, we show that the spectrum of $\bar{\Gamma}_m$ is the union of the spectra of Γ_m and $\hat{\Gamma}_m$.
The eigenvalues of Γ_m are the roots of the polynomial

$$f(\lambda) = \prod_{i=1}^{n}(\lambda-1)^m + \left(\sum_{d=1}^{m}\gamma_d(\lambda-1)^{d-1}\right)\mu_i$$

where μ_i are the eigenvalues of L. The eigenvalues of $\hat{\Gamma}_m$ are the roots of the polynomial

$$g(\lambda) = \prod_{i=1}^{n}(\lambda-1)^m + \left(\sum_{d=1}^{m}\gamma_d(\lambda-1)^{d-1}\right)\hat{\mu}_i$$

where $\hat{\mu}_i$ are the eigenvalues of $\hat{L}$. The eigenvalues of $\bar{\Gamma}_m$ are the roots of the polynomial

$$h(\lambda) = \prod_{i=1}^{n}(\lambda-1)^m + \left(\sum_{d=1}^{m}\gamma_d(\lambda-1)^{d-1}\right)\bar{\mu}_i$$

where $\bar{\mu}_i$ are the eigenvalues of $\bar{L}$. Recall from (23) that the spectrum of $\bar{L}$ is the union of the spectra of L and $\hat{L}$. Therefore, for all λ such that $f(\lambda) = 0$ or $g(\lambda) = 0$, we have $h(\lambda) = 0$ which concludes the proof.

Remark 5. From Theorem 7, we note that the stability of $\bar{\mathcal{S}}_d$ implies the stability of $\mathcal{S}_d$ for $m \geqslant 2$. On the other hand, a stable $\mathcal{S}_d$ does not necessarily imply a stable $\bar{\mathcal{S}}_d$.

In the case of structurally balanced systems, the stability of $\mathcal{S}_d$ and $\bar{\mathcal{S}}_d$ are equivalent regardless of the degree of the dynamics as delineated in the next result.

Corollary 2:Consider a structurally balanced system $\mathcal{S}_d$ in (27). For $m \geqslant 1$, the extended system $\bar{\mathcal{S}}_d$ in (32) is stable if and only if the system $\mathcal{S}_d$ is stable.

Proof. Suppose that the system $\mathcal{S}_d$ in (27) is structurally balanced. From Corollary 1, the signed Laplacian matrix L and the unsigned Laplacian matrix $\hat{L}$ have the same eigenvalues. From (23), $\bar{L}$ also has the same eigenvalues as L. Since the eigenvalues of $\bar{\Gamma}_m$ and Γ_m are functions of the eigenvalues of $\bar{L}$ and L, we conclude that the stabilities of $\bar{\mathcal{S}}_d$ in (32) and $\mathcal{S}_d$ in (27) are equivalent for structurally balanced graphs.

The stability conditions for cluster consensus of (27) evolving over an unsigned digraph are stated in the following lemma.

Lemma 2. [23] Suppose that $\gamma_1 \neq 0$ and the underlying unsigned digraph of the unsigned network consists of l_p primary and l_s secondary layer subgraphs. Then the system (27) achieves $\mathcal{K} = l_p + l_s$ cluster consensus if and only if the roots of the equation

$$\sum_{q=0}^{m-1} 2^q(\sigma-1)^{m-q}\gamma_{q+1}\mu_i + 2^m \tag{48}$$

are located in the left half of the complex plane where μ_i are the nonzero eigenvalues of L and the multiplicity of the eigenvalue 0 of L is ml_p.

The above lemma is only valid for unsigned multi-agent system. By using Definition 3, this lemma can be generalized for signed networks without any structural restriction. Therefore, the following theorem is obtained:

Theorem 8. The system $\mathcal{S}_d$ achieves $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$ cluster consensus if the roots of the equation

$$\sum_{q=0}^{m-1} 2^q(\sigma-1)^{m-q}\gamma_{q+1}\bar{\mu}_i + 2^m \tag{49}$$

are located in the left half of the complex plane where $\bar{l}_p$ and $\bar{l}_s$ are the number of primary and secondary layer subgraphs of the extended system $\bar{\mathcal{S}}_d$, respectively; l' is the number of subgraphs whose node set consists of only a subset of $\bar{\mathcal{V}} \setminus \mathcal{V} = \{v'_1, \ldots, v'_n\}$, $\bar{\mu}_i$ are the nonzero eigenvalues of $\bar{L}$ and the multiplicity of the eigenvalue 0 of $\bar{L}$ is ml_p.

Proof. From Lemma 2, the total number of clusters of extended system $\bar{\mathcal{S}}_d$ is $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$. Finally, the number of clusters for system $\mathcal{S}_d$ is found as

$$\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$$

since l' clusters have no member from the original system $\mathcal{S}_d$.

Remark 6. Lemma 2 states the stability conditions for cluster consensus of the extended system. Note that when the extended system achieves $\bar{\mathcal{K}}$-cluster consensus, the original system achieves $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$ cluster consensus where l' is the number of subgraphs whose node set consists of only a subset of $\bar{\mathcal{V}} \setminus \mathcal{V} = \{v'_1, \ldots, v'_n\}$.

4.1. Controller Parameter Selection

The choice of controller parameters γ_i plays a crucial role for the stability of the system (27). In order to ensure stability, the controller parameters should be chosen such that the roots of (49) are located in the left half of the complex plane and the conditions obtained in Section 4 can be applied. By using the results in [22,24], these conditions can be simplified for second and third-order systems.

For second-order systems:

- $\gamma_1 = (\epsilon_l/\delta)^2$,
- $\gamma_2 = \epsilon_l$.

($\delta = \sqrt{\frac{2}{r_l}}$ and ϵ_l are two positive numbers with $0 < r_l \leqslant \min Re(\mu_i), 0 < \epsilon_l \leqslant \min \frac{Re(\mu_i)}{|\mu_i|^2}$, and μ_i are the nonzero eigenvalues of $\bar{L}$)

For third-order systems:

- $\gamma_1 = \dfrac{\alpha\epsilon_{Re} - \epsilon_{Re} - 2\Delta_{Re} - 0.5\Delta_{Im}}{(\alpha - 1.5)(\alpha - 1)(\alpha^2 - 3\alpha + 2.5)}$,
- $\gamma_2 = \alpha\gamma_1$,
- $\gamma_3 = \dfrac{(\Delta_{Im} + 4\Delta_{Re})(\alpha^3 - 4.5\alpha^2 + 6.5\alpha - 3.25) + \epsilon_{Re}(\alpha - 1)^2}{(\alpha - 1.5)(\alpha - 1)(\alpha^2 - 3\alpha + 2.5)}$.

($\Delta_{Re}, \epsilon_{Re}, \Delta_{Im}$ and α are positive parameters such that $\Delta_{Re} \geqslant \max_i \frac{Re(\mu_i)}{|\mu_i|^2}, 0 < \epsilon_{Re} \leqslant \min_i \frac{Re(\mu_i)}{|\mu_i|^2}, \Delta_{Im} \geqslant \max_i \frac{Im(\mu_i)}{|\mu_i|^2}, \alpha = \alpha_0 \frac{2\epsilon_{Re}+4\Delta_{Re}+\Delta_{Im}}{2\epsilon_{Re}}$ with $\alpha_0 > 1$ hold where μ_i are the nonzero eigenvalues of $\bar{L}$)

Under the conditions given above for second and third-order systems, the extended representation of any signed network is stable from [22,24]. In this case, the number of clusters for the extended system can be calculated as $\bar{l}_p + \bar{l}_s$ based on Theorem 4 where $\bar{l}_p$ and $\bar{l}_s$ are the number of primary and secondary layer subgraphs of the extended system, respectively. Since l' is the number of subgraphs containing only nodes from $\{v'_1, \ldots, v'_n\}$, it can be concluded that the number of clusters for a signed network is equal to $\bar{l}_p + \bar{l}_s - l'$ provided that the following respective conditions are satisfied.

5. Simulation

Consider the network in Fig. 4 where the red arrows represent negative information flows and the black arrows represent positive information flows. *Note that the network is not structurally balanced and has no spanning tree. As such, the existing results in [13–17] are not applicable.*

The system matrix is generated as

$$L = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 4 & 5 & -1 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix}$$

The corresponding extended system matrix can be obtained as follows:

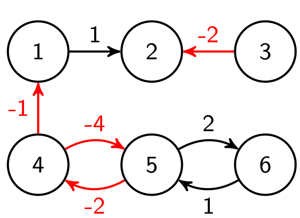

Fig. 4. A signed digraph not containing a spanning tree.

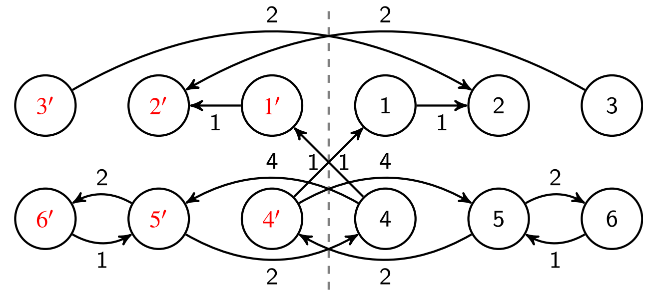

Fig. 5. Extended digraph for the network in Fig. 4.

$$\bar{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 5 & -1 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix}$$

The eigenvalues of $\bar{L}$ are $7, 7, 3, 3, 2, 2, 1, 1, 0, 0, 0, 0$. In the case where the network has second-order dynamics with $\gamma_1 = \gamma_2 = 1$, the stability conditions are satisfied. The signed digraph given in Fig. 4 can be extended into the digraph in Fig. 5. Notice that all connections between nodes of the extended network are positively weighted in Fig. 5. The extended digraph can also be redrawn as shown in Fig. 6 depicting also primary and secondary layer subgraphs. In this context, primary and secondary layer subgraphs of the extended network can be extracted as shown in Fig. 6 by using the primary and secondary layer detection algorithms from [20]. Subgraphs are labeled as $\mathcal{G}_{p,1}, \mathcal{G}_{p,2}, \mathcal{G}_{p,3}, \mathcal{G}_{p,4}$ for the primary layer subgraphs and $\mathcal{G}_{s,1}, \mathcal{G}_{s,2}$ for the secondary layer subgraphs. The members of the clusters in the extended digraph are determined as $\mathcal{V}_{p,1} = \{v_3\}$, $\mathcal{V}_{p,2} = \{v_4, v'_1, v'_5, v'_6\}, \mathcal{V}_{p,3} = \{v_1, v_5, v_6, v'_4\}, \mathcal{V}_{p,4} = \{v'_3\}, \mathcal{V}_{s,1} = \{v_2\}$ and $\mathcal{V}_{s,2} = \{v'_2\}$. The number of primary and secondary layer subgraphs of the extended network are $\bar{l}_p = 4$ and $\bar{l}_s = 2$, respectively. The number of subgraphs whose node set consists of only a subset of $\bar{\mathcal{V}} \setminus \mathcal{V} = \{v'_1, \ldots, v'_n\}$ is $l' = 2$ ($\mathcal{V}_{p,4}$ and $\mathcal{V}_{s,2}$). From Theorem 4, the original system in Fig. 4 reaches $\bar{l}_p + \bar{l}_s - l' = 4$ clusters.

For the signed digraph, shown in Fig. 4, with $x(0) = x_0 = [0.8,\ 0.25,\ 0.95,\ 0.35,\ 0.2,\ 0.25]^T$, the second-order simulation result is depicted in Fig. 7. The system converges to 4 clusters which are $\{v_1, v_5, v_6\}, \{v_2\}, \{v_3\}$ and $\{v_4\}$.

To recover the states of original nodes from its extended representation, initial states of the extended system are set as $\bar{x}(0) = \left[x_0^T,\ -x_0^T\right]^T$. With this setting, the related second-order simulation result is given in Fig. 8. Note that the trajectories of the original nodes are the same in both figures.

From this numerical example, the following comparative results can be obtained:

- The main result in [13] allows to determine whether the network converges to bipartite consensus or not. If given network is strongly connected, digon-sign symmetric and structurally balanced, it can be said that bipartite consensus is achieved.

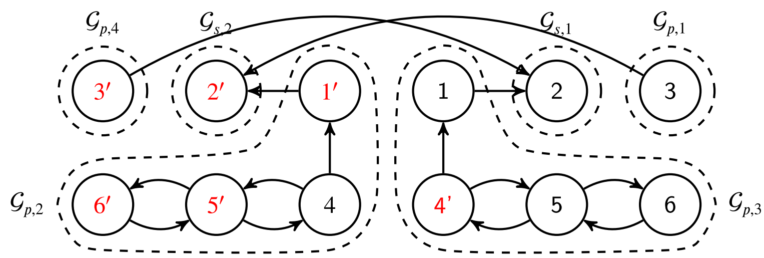

Fig. 6. Redrawing of the extended network in Fig. 5.

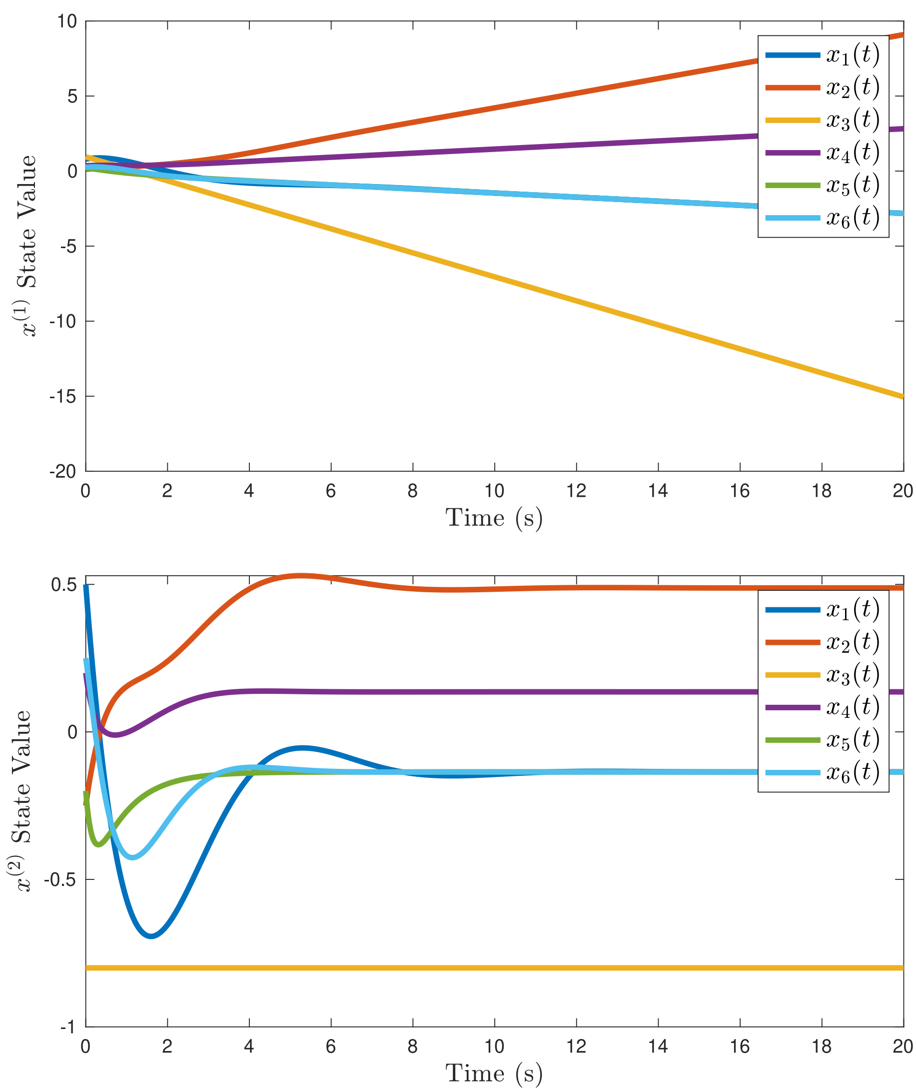

Fig. 7. State evolution of nodes associated with signed digraph in Fig. 4.

The studies [15–17] also focus on networks that have special graph structures such as structurally balanced, spanning tree. However, there are no results for networks that do not containing special graph structures. When structurally balanced systems are considered numerically, its extended representations have always only 2 primary layer subgraphs, i.e., $\bar{l}_p = 2, \ \bar{l}_s = 0$. Since the primary layer subgraphs always contain a member of the structurally balanced signed network, l' can be calculated as 0. Hence, one can conclude that a structurally balanced signed network always converges to $\bar{l}_p + \bar{l}_s - l' = 2$ clusters in perspective of numerical analysis.

- In [14], results are obtained on networks having a spanning tree. In addition to the related result in [13], if given network has a spanning tree, interval bipartite consensus is attained. However, the number of clusters and their members cannot be determined from this work.

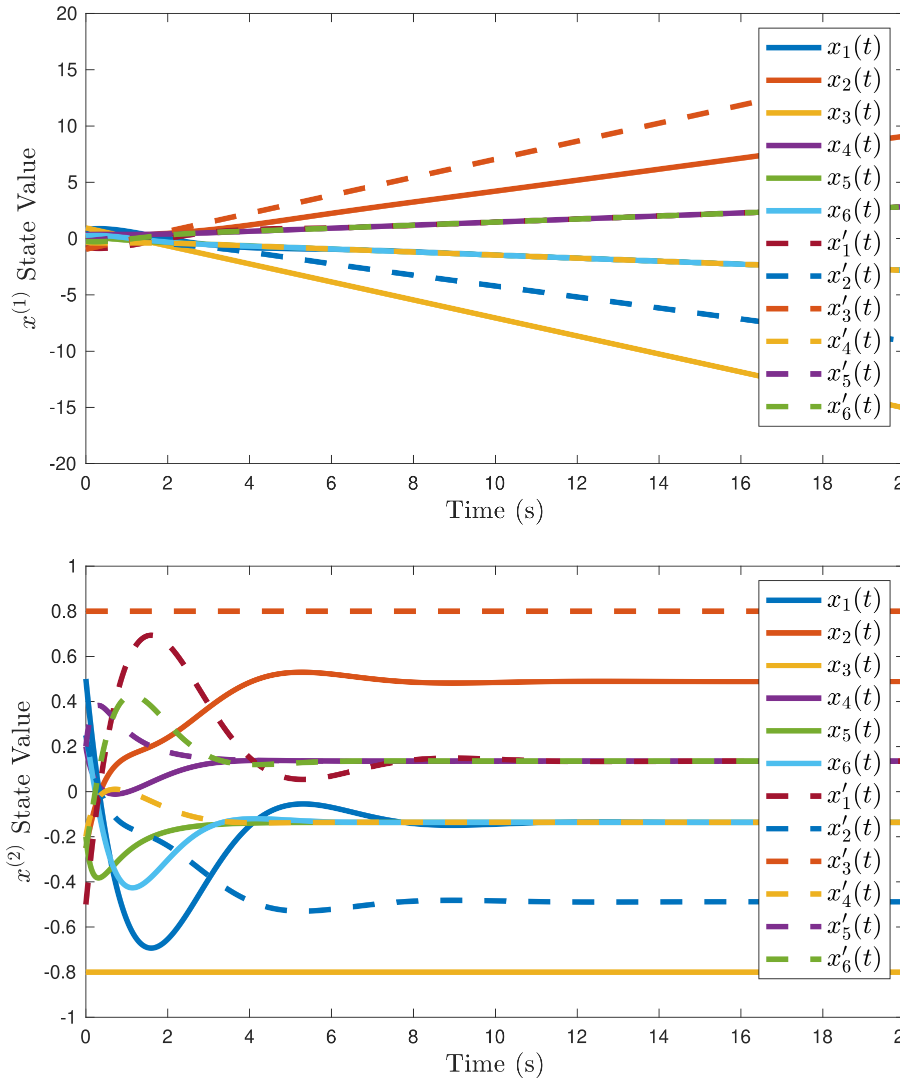

Fig. 8. State evolution of nodes associated with extended digraph in Fig. 5.

- The network in Fig. 4 is not structurally balanced and has no spanning tree. All papers mentioned above and others in the literature are insufficient to define how the given network behaves. In this paper, we determine the number of clusters and their members for a given network without any restrictive structural properties by using primary/secondary layer subgraph concepts and lifting approach. In numerical manner, $\bar{l}_p, \bar{l}_s$ and l' can be calculated as any value under primary and secondary layer subgraph conditions.

6. Conclusion

This paper has discussed the cluster agreement problem for a higher-order network of agents evolving over any given signed digraph $\mathcal{G}$. We have introduced an extended graph representation to analyze the convergence properties of any given signed digraph. This extended digraph structure has enabled us to compute the number of clusters and the agents in each cluster for a multi-agent network without any restriction on the digraph structure. Systematic controller parameter selection methods are also provided for both continuous-time and discrete-time networks. Furthermore, the application of the extended digraph for nonlinear networks is currently under investigation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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