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Enhancing pre-service mathematics teachers' understanding of function ideas

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ABSTRACT

Function is a multifaceted and essential concept in mathematics that includes various complex ideas such as mapping, relations between variables, algebra, input-output, and rule. Pre-service mathematics teachers' (PMTs) should have a rich understanding of these ideas for their future teaching practices. However, research showed that PMTs had a limited and superficial understanding of function as students do. Thus, a learning module was designed for improving PMTs' conceptualizations of function ideas as they relate to their personal concept definitions and examples of functions. A design-based research study was conducted with 17 senior PMTs. The study results suggest an improvement in the way PMTs define functions and identify a variety of function ideas in definitions and examples. Even if the mapping idea became dominant among all ideas in PMTs' personal concept definitions and examples, they could identify the implicit function ideas and develop an understanding of connections between ideas. We also provide suggestions for the second cycle of the intervention.

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Personal concept definition; design-based research; function; function ideas

1. Introduction

The function concept is a central and unifying concept in school mathematics that encompasses a rich complexity of mathematical ideas, interpretations, and meanings (Ayalon & Wilkie, 2019). Studies have identified that this complexity stems from function definition (Jones, 2006; Malik, 1980). Also, the existence of multiple mathematical ideas in function definitions and examples contributes to this complexity (Ayalon et al., 2017; Watson et al., 2013). Students often have difficulty with tackling this complexity and so they could develop an understanding of functions that is too narrow (Akkoç & Tall, 2002; Kalchman & Koedinger, 2005). The discrepancies that often exist between students' conceptual understanding of functions and formal definition of the concept (Akkoç, 2006; Vinner & Dreyfus, 1989) contributes to this difficulty in students' understanding.

Difficulties with defining functions are the first factor that contributes to the complexity of the function concept. The formal definition of function expresses the idea of

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a correspondence between two sets satisfying the univalence characteristic, and this is the definition given in modern mathematics textbooks (Gök et al., 2019; Jones, 2006; Malik, 1980). This definition may be abstract and difficult for students to understand (Selden & Selden, 1992) if insufficient emphasis is put on the formal definition of function and its different sub-concepts (domain, range, determiners such as 'for every' and 'unique') (Markovits et al., 1986; Panaoura et al., 2017; Tabach & Nachlieli, 2015). Also, the predominance of the idea of mapping in this formal definition may result in students' understanding that all mappings are functions (Spyrou & Zagorianakos, 2010) or that a function must be a one-to-one mapping (Dubinsky & Wilson, 2013).

The existence of a rich variety of ideas is the second factor that contributes to the complexity (Doorman et al., 2012). Students encounter various formal and informal definitions in mathematics textbooks that express different ideas relating to functions (Ayalon et al., 2017; Cooney et al., 2010; Watson et al., 2013). For instance, the rule idea may be described in the following way in a calculus textbook: 'a rule that assigns a unique value $f(x)$ … .to each x' (Hass et al., 2017, p. 1). At the same time, the idea of input-output may be expressed as 'a relationship between input and output' in an algebra textbook function definition (Holliday et al., 2005, p. 43). Students may not realize that the function concept that they have learned in different ways at different grades is the same (Jones, 2006). This situation results in developing a limited understanding of functions.

The third factor that contributes to the complexity was the inconsistencies between students' concept images of functions and the formal function definition (Akkoç, 2006; Vinner & Dreyfus, 1989). For instance, the concept image that 'function is a rule, an algebraic formula, an equation' (Tall & Bakar, 1992; Clement, 2001) may result in a conceptualization of every algebraic expression as a function (Tall & Bakar, 1992; Bardini et al., 2014). This image can limit students' understanding of the arbitrary nature of functions that refers to 'functions do not have to be defined on any specific sets of objects.… do not have to be sets of numbers" (Even, 1993, p. 96). All of these potential difficulties experienced by students highlight the importance of teachers' instructional practices to support students' understanding of the connections across function definitions and ideas.

Studies suggested that pre-service mathematics teachers (PMTs) have a limited understanding of the concept of functions as do high school and calculus students. For instance, many mathematics teachers and PMTs ignored the univalence characteristic of functions or thought that each input always entails different outputs. Hence, they could not define the function correctly (Even, 1993; Hatisaru, 2020; Huang & Kulm, 2012; Steele et al., 2013; Zazkis & Marmur, 2018). Also, they often have limited concept images (e.g. overemphasis on the rule or single formula, only one graph could go through two points (for details see Dubinsky & Wilson, 2013) that would result in either an incorrect or limited conceptualization of function (Dede & Soybas, 2011; Elia & Spyrou, 2006; Hatisaru, 2020; Zazkis & Marmur, 2018). Teachers' limited understanding of the function or even correct conceptualizations of them may not reflect different ideas about function. One particular idea might be dominant in teachers' or PMTs' concept images, generated function examples and personal concept definitions (Hatisaru, 2020; Viirman et al., 2010; Vinner & Dreyfus, 1989).

The findings of the studies mentioned above show that PMTs and mathematics teachers have a limited understanding of functions that mostly emphasized one or two function ideas (e.g. mapping, input-output) rather than focusing on a web of rich connections across ideas (e.g. relationship between variables, mapping, covariation). This limited understanding points to the need for supporting PMTs to develop a comprehensive and mathematically correct understanding of various function ideas and definitions. Because the depth and quality of teachers' knowledge about functions influence the rigour and quality of their instructional practices, and high-quality practices can contribute to students' mathematics learning (Hatisaru & Erbas, 2017; Watson & Harel, 2013). For this reason, mathematics teachers must be able to know and evaluate the accuracy of the different definitions of functions (Leikin & Zazkis, 2010; Nyikahadzoyi, 2015; Steele et al., 2013). Also, since teachers' concept images also guide their instructional practices, it is expected that teachers should have a sound understanding of multiple function ideas that can properly represent the context in which a function is represented or discussed (Nyikahadzoyi, 2015).

Developing PMTs' comprehensive understanding of various function ideas is very important for various reasons. Firstly, they could support their students to recognize and interpret functions from different and novel contexts (Cooney et al., 2010; Steele et al., 2013). Secondly, they could provide learning opportunities for their students to foster an understanding of different definitions and their emphasized ideas functions (Ayalon & Wilkie, 2019). Thirdly, their students could have varied and rich example spaces of functions that include multiple function ideas as well as explain non-examples of functions (Ayalon & Wilkie, 2019). At last, such comprehensive understanding is critical for their students' understanding of further mathematics concepts or topics such as rate of change, calculus, and its applications (Cooney et al., 2010).

Although understanding various function ideas are critical for developing a comprehensive understanding of functions, the majority of studies only described PMTs' existing knowledge about formal functions definitions (e.g. Chesler, 2012), their concept images, and personal concept definitions (McCulloch et al., 2020; Wilson, 1994; Zazkis & Marmur, 2018), their fluency in identifying and producing examples and non-examples of functions (Zazkis & Marmur, 2018). These studies also documented PMTs' limited understanding of functions. PMTs' knowledge and conceptual understanding of functions should be supported so that they could create a learning environment that also would foster a conceptual understanding of function for their future students. Thus, intervention studies (e.g. Steele et al., 2013; Weber et al., 2015) were conducted on how to develop PMTs' pedagogical content knowledge or mathematical knowledge for teaching. Steele et al. (2013) reflected some features of interventions that support teachers' learnings. These are; focusing on central concepts for mathematics curriculum, engaging tasks that allow the repeated consideration of the definition, and provide opportunities to revisit and revise their content knowledge. Nevertheless, supporting PMTs' understanding of different function ideas as it relates to their personal concept definitions and example of functions remains a relatively unexplored area. Considering such a gap in the literature, a learning module was designed for improving PMTs' understanding of function ideas. Then, the development of secondary PMTs' understanding of function ideas as they engaged in the learning module was explored. Considering this aim, the study sought to answer the following research questions:

- How did PMTs' understanding of function ideas evolve as they engaged in the learning module?

- o Which ideas of function were emphasized in PMTs' personal concept definitions of function before and after the module?
- o Which ideas of function were emphasized in function examples generated by PMTs before and after the module?

1.1. Conceptual framework

1.1.1. The ideas of function

Studies have documented various function ideas that are essential for having a robust understanding of functions (Ayalon et al., 2017; Cooney et al., 2010; Watson et al., 2013). The first idea of *mapping* approaches the meaning of a function as a 'single-valued mapping from one set-domain of the function to another-range' (Ayalon et al., 2017, p. 8). This can be either in the form of a one-to-one or many-to-one mapping between sets (Watson et al., 2013). Vinner and Dreyfus (1989) called the same idea *correspondence*. They defined the function as 'any correspondence between two sets that assigns to every element in the first set exactly one element in the second set' (p. 359). Ayalon et al. (2017) stated that correspondence is an approach to understand functions. They asserted that a mapping and an input-output model are methods that emphasize correspondence. In this paper, we will use the term *mapping* instead of correspondence to distinguish mapping from an *input-output* idea, which is more general.

The second idea of *input-output* entails the thought of processing inputs to give some types of outputs (Watson et al., 2013). Also, it is related to another function idea called *algebra* which includes generating new values from given values using algebraic calculations (Ayalon et al., 2017) and using expressions to calculate dependent y-values for each given independent x-value (Watson et al., 2013). In the literature, different terms such as formula (Vinner & Dreyfus, 1989), algebraic calculation (Ayalon & Wilkie, 2019), and computational process (Sfard, 1991) have been used. In this study, we prefer to use an overarching term, *algebra*, to express the idea of function in a more general sense. Another function idea is *rule* which is closely related to the *algebra* idea. The rule idea ignores the arbitrariness feature while it takes the regularity between input-output into account (Vinner & Dreyfus, 1989). This regularity refers to either a numerical or a verbal relationship (e.g. each letter should be delivered to the mailbox) between inputs and outputs.

Early learning related to functions can revolve around the idea in which a function can be seen as a mechanism that represents a pointwise mapping between two variables and one that produces a single output for each input (Watson et al., 2013). This kind of understanding is dominantly addressed in current mathematics textbooks (Ayalon et al., 2017). However, the transition to a more comprehensive understanding of an abstract idea of a function can be achieved through *covariation*[1] (Cooney et al., 2010) and the *relations between variables* (Ayalon et al., 2017). The fourth idea of covariation involves two or more variables varying simultaneously. In this simultaneous change of variables, there is a hidden parameter (e.g. time) (Thompson & Carlson, 2017). Ayalon et al. (2017) defined the idea, *relations between variables*, as follows: 'the relational aspect of functions and also dependency of variables; in other words, it went beyond talking about individual values or collections of individual values' (p.11).

Understanding these various ideas is necessary for developing a comprehensive and multi-faceted conceptualization of functions (Ayalon & Wilkie, 2019). Mapping and input-output ideas can facilitate understanding of domain and range concepts and properties such as onto-ness and one-to-one-ness (McGowen, 2017; Selden & Selden, 1992). If a future teacher overly emphasized mapping and input-output ideas as they teach functions, this could limit 'students' conceptualization of functional relationship to correspondence approach (i.e. associating a unique dependent value with a independent value)' (Hatisaru, 2020, p. 94). In addition, this overemphasis on the mapping between discrete data may result in a limited understanding of functions as 'a set of discrete data points and anything that can be mapped is a function' (Spyrou & Zagorianakos, 2010; Watson & Harel, 2013; as cited in Hatisaru, 2020, p. 94).

The algebra idea provides the representation of the quantitative relationship between input-output or variables (Driscoll, 1999). Yet, the algebra idea does not include the arbitrariness of functions (Even, 1993). Overemphasis on the algebra idea can cause 'students to view functions only in terms of symbolic manipulations and procedural techniques' (Carlson et al., 2010, p. 115) and difficulties in understanding the constant function since they might believe the change in an independent variable always result in a change in a dependent variable (Eisenberg, 1991). On the other hand, the algebra idea has the potential for contributing to an understanding of the relationship between variables and covariation ideas by representing quantitative situations (Driscoll, 1999).

Developing a comprehensive understanding of the relationship between variables and covariation ideas is central to understanding functions. This understanding is important for the conceptualization of key concepts such as variables and rate of change (Carlson & Oehrtman, 2005; Thompson & Carlson, 2017). This conceptualization can be achieved by using different function definitions and real-life examples (Cooney et al., 2010).

Considering constraints (if overly emphasized) and affordances of these ideas discussed above, developing PMTs' understanding of the ideas would enable them to teach function comprehensively by going beyond or modifying what the textbooks or resources offered.

1.1.2. Concept definition and examples as they relate to function ideas

The function concept carries different mathematical ideas that have a connection with concept definition and example spaces. Concept definition is described as 'a form of words used to specify that concept' and is 'accepted by the mathematical community at large' (Tall & Vinner, 1981, p.152). Although a concept has a formal definition, the way individuals express the concept may differ from the formal definition that is called personal concept definition. Personal concept definition is defined as 'the form of words that the student uses for his own explanation of his (evoked) concept images' (Tall & Vinner, 1981, p. 152). Concept image is identified as 'total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes.' (Tall & Vinner, 1981, p. 152).

The connection between function ideas and the formal definition of functions was confronted in different definitions. For instance, Dirichlet's definition (1837; as cited in Boyer, 1968) underlies the idea of a relation between two variables called x and y, along with identifying this relationship as a rule that produces an input-output relation. Bourbaki's definition explicitly refers to a set definition as a functional relation between two sets (Malik, 1980), which later evolved to the definition of a set of ordered pairs.

Nevertheless, an individual may have a concept image regarding function without having a grasp of the formal mathematical definition (Vinner, 2002). Also, concept images do not necessarily reflect all mathematical ideas of function (Vinner, 1992). For instance, a concept image of function as 'a machine—that is, when numbers are entered, numbers are produced' (Clement, 2001, p. 747) emphasizes the input-output idea. However, this image of functions does not reflect the definitional properties of functions.

Concept images can be used to express a personal concept definition that can reflect different function ideas. Besides, these ideas may or may not be expressed in individuals' personal concept definitions (Ayalon et al., 2017; Tall & Vinner, 1981). For instance, a PMT can define functions as 'a formula'. Although this definition reflects the idea of algebra and rule, other personal concept definitions may not reflect any function idea. For instance, in Even's (1990) study, some teachers used the 'vertical line test' rule in their function definition. In such a definition, the rule is used to decide whether a given graph represents a function rather than explicitly emphasize function ideas.

Goldenberg and Mason's (2008) notion of example space for a concept is a useful construct to analyze the role of examples in exploring concept images. The example space is defined as the collections of examples generated by learners (Watson & Mason, 2005). The examples provided by the students are the mirror of their understanding of a mathematical concept (Zazkis & Leikin, 2007). The richness, diversity, and complexity of given examples is an indicator of comprehensive and multi-faceted conceptualization (Ayalon & Wilkie, 2019). In this study, example spaces are not just a list of examples. Example spaces reflect different function ideas and concept images of pre-service mathematics teachers. For example, a PMT who thinks function as an input-output machine can give an example of a washing machine that represents such conceptualization of function (Ayalon & Wilkie, 2019).

2. Methodology

Design-based research was utilized for this study. The aim of design-based research is not only to develop a theory that will guide instructional decisions but also generate a product to enhance learning (Cobb et al., 2003). In this study, design-based research was used to systematically examine learning forms of PMTs through employing a particular learning module within the context of function ideas.

In design studies, the first step is to define learning objectives to be gained by students at the end of the intervention (Steffe & Thompson, 2000). In the second step, researchers design a series of interventions to accomplish these learning objectives. These interventions include teaching and learning tasks (Confrey & Lachance, 2000). The researchers then test these interventions in the classroom environment (Cobb, 2000). This allows researchers to assess how well the intervention works. During implementation cycles, researchers test the interventions and then, iteratively revise, refine, and retest them (McKenney & Reeves, 2013). In this study, a continuous analysis was carried out by three researchers, and based on this analysis new instructional tasks and interventions were designed during the study. For instance, an examination of textbook tasks was designed to determine whether or not the *mapping* aspect is dominant in textbooks as it is dominant in PMTs' conceptions. Also, these analyses led to further revisions for upcoming cycles. For instance, researchers revised existing learning goals and tasks based on the data analysis for the next cycle of design-based research. These revisions and refinements will be elaborated in the discussion section and are critical for generating a better learning module section on function ideas and examples.

2.1. The module

In the wider context, multiple modules were designed for developing secondary PMTs' understanding of function, limit, and continuity. This study only focused on the first cycle of the learning module lasting five weeks for developing PMTs' understanding of function ideas and their connections. In the following sections, the details of the module, including its objectives, tasks, and implementation, are presented.

2.1.1. Participants

Participants were a convenience sample of PMTs studying in a teacher preparation programme at a state university in Istanbul, Turkey. The first cycle of the module was completed with 17 PMTs who enrolled in the Mathematics Teaching I and II courses. The PMTs' learning experiences concerning the concept of function in high school were as follows: (a) PMTs were introduced to functions as a special kind of a relation at 9th grade, (b) the national mathematics curriculum and mathematics textbooks at that time privileged ideas of *mapping*, *input-output*, *algebra,* and *rul*e, (c) there was no learning objective in this curriculum that focused on *relations between variables* or *covariation*. Before the module, the PMTs took sixteen mathematics courses such as calculus, linear algebra, and analytic geometry. Also, all participants took mathematics education courses including geometry teaching, algebra teaching, and mathematics curriculum.

PMTs' teaching experience showed that only PMT5 had private one-to-one tutoring for five years and whole-class teaching experience for one month. Five PMTs did not have any teaching experience. The rest of the PMTs ($n = 11$) only had one-to-one private teaching experience. Of these, eight of them had less than one year and three of them had one to three years of experience. All participants were numbered from 1 to 17, and pseudonyms (e.g. PMT1) were used to replace their names.

2.1.2. Goals of the module

The module focused on the concept of function, within the scope of its definitions, function ideas, and examples. Therefore, we describe the module within this context. The goals of the module are for PMTs to be able to: (1) define the concept of function mathematically, (2) develop a rich understanding of function which incorporates different function ideas, (3) analyze examples by using definitional properties of functions, (4) identify function ideas in various definitions, and (5) identify function ideas in various examples.

2.1.3. The tasks and the intervention

Five tasks were implemented over five weeks. The researcher-teacher (R-T) and PMTs met four hours per week. The rationale behind task selections, the aim of each task, and the implementation order are explained below.

2.1.3.1. Concept definition and concept images. The concept definition and concept images framework of Tall and Vinner (1981) was the central focus of the task. The aim was to enrich PMTs' concept images and help them define the function concept mathematically. Initially, the PMTs discussed the importance of the notion of concept images for teaching mathematics in general. Then, they evaluated different examples of students' work based on the framework. To do that, tangent was chosen as the content because of its potential to illustrate how students' concept images might conflict with the definition of a concept.

2.1.3.2. Examination of students' examples and definitions of a function. PMTs were asked to define the function concept and examine the accuracy of their personal concept definitions, and identified the emphasized function ideas. Second, researchers purposefully selected five function definitions written by students. Three of them emphasized the algebra idea while the other two emphasized the mapping idea. Only one definition was mathematically correct.

Since mathematics textbooks in Turkey often use a machine analogy, researchers selected three examples that described a function as a machine. In the implementation, the PMTs analyzed personal concept definitions and students' examples. They were asked to assess the mathematical accuracy of them. Then, they determined the underlying function ideas in personal concept definitions and examples.

2.1.3.3. Video-based tasks. Although in the first tasks the PMTs had an initial understanding of function definition and ideas, in the second task they had the opportunity to reflect on the ideas in students' personal concept definitions and examples selected from classroom teaching practices. Thus, they would have an opportunity to actively apply and recognize function ideas in real teaching practices (Kersting et al., 2010). Three introductory lesson videos on functions on YouTube were selected since they reflected different function ideas (see Appendix A for video descriptions). After watching videos, PMTs were asked to evaluate the mathematical accuracy of the function definition given in the videos, to evaluate emphasized function ideas, and to identify missing function ideas. Also, they were asked to evaluate and compare the mathematical appropriateness of introducing functions as in the videos.

2.1.3.4. Students' misconceptions about functions. The aim of the task was the elimination of PMTs' possible misconceptions related to function concept. In this task, examples of students' written works and lesson video clips that included misconceptions were analyzed. The PMTs discussed the underlying reasons behind the misconceptions and suggested possible ways to remediate them. For example, misconceptions regarding the definition of function were presented such as 'a general neglect of domain and range' (Markovits et al., 1986, p. 23) and 'functions are (or can always be represented as) equations or formulas" (Even, 1993, p. 114). were presented.

2.1.3.5. Examination of textbooks. In students' personal concept definitions and examples, and the lesson video clips that were previously examined, mapping, and input-output ideas were dominant. Then, the PMTs and researchers discussed the underlying reasons for this. One of the major conjectures was that mostly mapping and input-output ideas are presented in the textbooks. This revealed a need to examine the topic of function in mathematics textbooks to test our conjecture. Thus, this task was designed and the introductory sections in three different mathematics textbooks (see Appendix B) on functions were selected. Each textbook represented different function ideas: input-output, mapping, and relations between variables, respectively.

Before the class, we shared introductory sections of the textbooks with the PMTs. During the class, PMTs evaluated and compared the introductory sections in terms of (a) function ideas used or neglected, (b) mathematical accuracy of textbook examples and definitions, and (c) potential function idea that the textbook examples might evoke, which might also influence their personal concept definitions of functions.

2.2. Data collection and analysis

Pre- and post-tests, and video recordings of the intervention were used as data collection tools. Data analysis was an ongoing reflective process. Based on these reflections, revisions and refinements on the tasks were made and new implementation session strategies were discussed. At the end of the study, all data collected was analyzed.

2.3. Pre–post tests

Pre- and post-tests were administered before and after the module implementation. In both tests, three questions were asked to elicit the used function ideas in PMTs' concept definition and examples. Thus, the first question asked PMTs to define the function concept. Then, the second question asked the PMTs to give two examples of functions. The third question asked to give two daily-life examples for functions. The reason for asking two separate example types is to elicit different functions ideas bounded with context such as real-life function examples have the potential to reveal PMTs' understanding of the relations between variables idea (Cooney et al., 2010).

Content analysis was used to analyze the data obtained from the tests. Three researchers coded data independently. The inter-rater reliability was 86.2%. PMTs' personal concept definitions were analyzed in terms of function ideas emphasized and definitional properties. The categories regarding ideas were *mapping, algebra, input-output, rule, and relations between variables*. The definitional properties of function definitions are 'each element of a set' and 'a unique element of a set'. The generated examples were analyzed in terms of emphasized function ideas (*mapping*, *relations between variables*, *input-output*, *algebra*, *rule*, and unspecified) and accuracy (correct, partially correct, and incorrect). Appendix C shows the description of each category.

2.4. Videos of intervention

Video recording is an important data collection tool (Powell et al., 2003) since it enables researchers to view an intervention multiple times and to detect events and interactions that may not be recognized during the intervention. For these reasons, a camera was used to record each implementation session. The video data was analyzed through the analytical model (Powell et al., 2003). The critical events for the video analysis were instances in which the PMTs discussed function ideas, defined and identified a function, identified function ideas reflected in students' examples and instructional videos selected from YouTube, and pointed out a misconception regarding the function concept and related ideas (e.g. equations always represent a function). These critical events were identified by two researchers independently and were coded under the categories, 'focusing on definitional properties' and 'function ideas' (see Appendix C). The third researcher shared her ideas in cases of inconsistency.

3. Findings

3.1. Personal function definitions

This section will compare the ideas detected in pre and post-test personal concept definitions of functions. Then, to document improvement in PMTs' understanding of function ideas as it relates to personal concept definitions, excerpts from intervention will be reported. Table 1 shows the ideas and definitional properties ('each element of a set' and 'a unique element of a set') that were emphasized in PMTs' personal definitions.

As seen in Table 1, the use of both definitional properties and the idea of *mapping* became prominent from pre- to post-test. The *rule* idea in the pre-test definitions did not appear in the post-test. While in the pre-test only two PMTs out of 17 PMTs mentioned two properties in their definitions, 13 PMTs used both properties in the post-test. In line with this progress, the accuracy of PMTs' personal concept definitions had also increased from pre (n = 2) to post-test (n = 13). Eight PMTs gave partially correct definitions by specifying one property in the pre-test and one PMTs post-test. Also, the accurate use of two definitional properties in the definitions results in a shift in the language of the definition from colloquial to more formal. The majority of the PMTs (n = 10) used a more formal language rather than colloquial language in their personal concept definitions in the post-test. In this context, the definitions were closer to the formal function definition such as the following:

> **PMT14:** Let A and B [be] two sets. Relations that map every element in set A to just one element in set B are called functions. (post-test)

Table 1. The function ideas and definitional properties in PMTs' definitions.

Function ideas	Definitional properties**	Pre-test f*	Post-test f*
Mapping	Mention two and correct	2	12
	Mention one and partially correct	6	1
	No mention and incorrect	2	2
Rule	Mention two and correct	0	0
	Mention one and partially correct	0	0
	No mention and incorrect	4	0
Input-output	Mention two and correct	0	1
	Mention one and partially correct	3	3
	No mention and incorrect	0	1
Relations between variables	No mention and incorrect	2	1
Algebra	Mention one and partially correct	1	0
Total		20	21

* We coded 17 personal concept definitions. Some of the definitions (pre-test: n = 2; post-test: n = 4) included more than one function idea.
** Mention two: Use of two definitional properties, Mention one: Use of only one of the definitional properties, No mention: None of the definitional properties were used.

Although the PMT defined function as a special relation with two definitional properties, this definition could not be considered as a formal definition because PMT14 did not use notations such as $f: A \rightarrow B$, $f(x) = y$. None of the PMTs used formal notations in their definitions in both tests.

The findings regarding personal concept definitions will be further explored under two themes: (i) *mapping* idea and definitional properties and (ii) other ideas and definitional properties.

3.1.1. Mapping idea and definitional properties

When the personal concept definitions in which the idea of *mapping* is dominant are examined in detail, it could be seen that only two out of 10 definitions in the pre-test include two definitional properties of function ('each element of a set' and 'a unique element of a set'), while six of them include only one of these properties. One definition was mathematically incorrect. In 12 of the 15 definitions in the post-test with the *mapping* idea, these two definitional properties of function were present. The following definitions of PMT17 illustrate this finding.

Pre-Test: It is a relation that takes every element in the domain to the codomain.

Post-Test: Let's define a relation from set A to set B and A is the domain and B is the codomain. If it maps every element of set A to one element of set B and it maps exactly to one element, this relation is a function.

Although both definitions include the *mapping* idea, only the 'each element of a set' property was expressed in the pre-test definition, while two properties (each element of a set and a unique element of a set) were expressed in the post-test. Like PMT17, other PMTs ($n = 11$) who used mapping idea and also wrote mathematically correct definitions in the post-test. This result is an indication of the PMTs' grasp of the meanings of the essential mathematical properties of the notion of function.

The PMTs' progression in emphasizing definitional properties also revealed itself in the classroom interactions while implementing the *examination of textbooks* task:

PMT11: Function rule [refers to the essential properties of function],… using this idea, each child has a unique mother.

PMT6: Mothers should be given as biological mothers.

PMT11: Should not include a case of a stepmother.

PMT1: By filling these gaps, we can give this example… Giving a concrete example will be good for us as well as relating to the definition. For instance, we [refers to their high school learning experiences] did not consider the definition [of the function concept]

As seen in the excerpt above, the PMTs recognized that the children in the domain were assigned to the mothers in the range. They also specified that the mother should not be a stepmother for this example to be a function. This is an indication of an understanding of the definitional properties of functions.

3.1.2. Other ideas and definitional properties

This section examines how the definitions emphasizing ideas other than the *mapping* idea had changed from pre-test to post-test. In the instances of PMTs' definitions emphasizing the ideas other than mapping (*input-output, relations between variables, algebra, rule*), the definitional properties were not explicitly emphasized.

In the pre-test, four PMTs emphasized the *rule idea* in their definitions by ignoring arbitrariness of function which resulted in the presence of a mathematical misconception that every function has a rule or every rule is a function (Table 1). For instance,

> **PMT4:** Everything is a function that has a domain and a *rule*.

As a result of the intervention, four PMTs defined function correctly by abandoning the rule misconception in their post definitions. This progression may be explained by a classroom discussion on the arbitrariness of functions. During this discussion, arbitrary and non-arbitrary function examples were examined. All PMTs concluded that '[Function] also might be arbitrary and it does not necessarily have a rule' and some functions can have a rule such as in the example of 'oil liters are mapped to the price'.

Different from the pre-test, four PMTs used multiple ideas as they defined function correctly in the post-test. One example was as follows:

> **Pre-test:** The function is a process that allows us to transform any data as input. This data cannot have more than one value when it is processed.

> **Post-test:** A function defined from A to B, the relation that maps all elements in A to only one element in B. More than one element in A can be processed and eventually give us the same element from B, but there are no two different outcomes in B for an element in A.

PMT16's post-test definition reflected a tendency to the mapping idea but also preserved the input-output idea. Also, all definitional properties were apparent in her personal concept definition in the post-test.

The increase in the use of multiple ideas and identifying definitional properties implicitly embedded in the definitions emphasized ideas other than mapping could be attributed to the module intervention. For example, definitional properties in the example (oil amount-price relationship) that highlights the idea of *relations between variables* was an issue of discussion during the *examination of textbooks* task. PMT1 wanted to see 'each element of a set' and 'a unique element of a set' properties in this example. The R-T explained the mapping between domain (oil amount) and range (price). As a result of this interaction, the PMTs concluded that there was always a mapping even if the example explicitly highlighted the idea of *relations between variables*.

As PMTs progressed in the module implementation, they made sense of the definitional properties of functions. For instance, in the *examination of students' examples and definitions of a function task*, the PMTs examined a student definition: 'Let $A \to B$, [function] is formed by mapping each element of the domain (set A) to at least one element of the range (set B)'. All the PMTs indicated that this definition was not correct because of the expression 'at least one element'. Then, they corrected the definition. In another case, the PMTs discussed whether the circle graph or equation is a function. They concluded that it is not a function because there are two images for an x. Such discussions enabled the PMTs to conceptualize the definitional properties of function and correctly define functions.

Table 2. Ideas and accuracy of generated function examples.

		Function Examples		Daily Life Function Examples		Total	
Function Ideas	Accuracy*	Pre	Post	Pre	Post	Pre	Post
Mapping	C	0	5	0	14	0	19
	PC	3	2	1	0	4	2
	IC	0	0	1	0	1	0
Relations between variables	C	1	0	14	17	15	17
	PC	0	1	3	0	3	1
Input-output	C	0	0	3	3	3	3
	PC	0	0	1	0	1	0
Algebra	C	11	20	1	4	12	24
	PC	20	4	0	0	20	4
Rule	C	0	1	1	0	1	1
	PC	0	0	0	0	0	0
Unspecified	C	0	1	0	0	0	1
	PC	1	7	1	0	1	7
Does not specify a function		0	0	4	0	4	0
No response		1	0	6	1	7	1

* C: Correct, PC: Partially Correct, IC: Incorrect

3.1.3. Generated function examples

This section reports how emphasized ideas in the PMT's generated examples had changed from pre-test to post-test. We report the findings under two sections as (i) examples including multiple ideas and (ii) accuracy and clarity of examples. Table 2 shows the function ideas that were revealed in the generated function examples and their mathematical accuracy.

As shown in Table 2, when all the function examples were examined, it was seen that the idea of *algebra* was dominant in the pre-test (32 examples) and the post-test (28 examples). Only five of these examples were from daily life. In daily life examples, the *relations between variables* idea was dominant in both tests. While the frequency of examples reflecting *algebra* ideas decreased in the post-test, the frequency of *mapping* and *other* categories increased. The *mapping* idea was found in three function examples in the pre-test and seven examples in the post-test. In daily life examples, this idea was emphasized in two examples in the pre-test and 14 in the post-test. This shows that the *mapping* idea became prominent in two types of function examples after the implementation.

As seen in Table 2, the dominant idea in one pre-test example and eight post-test examples was not determined. One example in the pre-test and two examples in the post-test were example clusters (e.g. polynomial functions or trigonometry functions). In the post-test, seven examples were coded as *unspecified* (e.g. functions represented with graphics without any context or domain & range specified). Another noteworthy finding is that while PMTs ($n = 4$) gave examples that did not specify functions in the pre-test, in the post-test there was not an example of a non-function case. In addition, in the pre-test, there were seven instances of no response. Six out of seven of these instances were for the daily life example. In the post-test, there was the only one with no response for daily life examples. This indicates that the PMTs could generate more daily life examples after the intervention. The distribution of ideas in these examples varied including *mapping*, *relations between variables*, *algebra*, and *rule*.

3.1.4. Examples including multiple ideas

When the ideas existing in the examples from both tests were examined, it was found that the PMTs emphasized more than one idea in the post-test, especially for daily life examples. In the pre-test, only two PMTs emphasized two different ideas in a daily life example, while in the post-test this number increased to six. These six PMTs used *relations between variables* and *algebra* ideas in four examples and *relations between variables* and *input-output* ideas in two. One example is given below:

> **PMT13:** It is rendering a certain amount of wood placed in the factory machine into a certain amount of sawdust. (post-test)

Both *relations between variables* and *input-output* ideas were present in the above example. The PMT13 did not only give an example of a machine that renders the wood into sawdust but also expressed a quantitative relationship between inputs and outputs. This quantitative relationship was also an issue of discussion during the implementation.

> **R-T:** [R-T referred to an example in the textbook showing a function machine (oven) that represents inputs (ingredient) and outputs (bread) without quantification]. In such examples, transformation is always emphasized, but they don't refer to quantities. What should we revise/do here to make these examples more mathematically meaningful?
>
> **PMT5:** [We can express] algebraically [refers to the relations between variables.]
>
> **PMT16:** Like, with this amount [of ingredient], this amount of bread could be baked.

Based on the discussion of various examples similar to the one above, the PMTs concluded that just merely referring to ingredients and bread without any quantification was insufficient to conclude that this example was a function.

The development in PMTs' ability to use and identify more than one idea in the same example was also observed throughout the intervention. For example, in the *video-based task*, which is one of the first activities in the module, lesson videos that highlight different ideas of functions were watched and evaluated together. In the beginning, the PMTs tended to evaluate the video with a general perspective, including comments on the voice or diction of the teacher. In the mathematical discussions of the videos, it was observed that all PMTs could not determine correct function ideas presented in the videos. For example, PMT17 proposed that the teacher could have used dependent and independent variables for input and output for the example $f = \{(a,d), (b,e), (x,e), (z,f)\}$ in the video (see Appendix B). Also, the R-T asked the PMTs to determine the missing function ideas. The PMTs gave responses such as 'graphical representation' and 'constant function ideas are missing.' Since all the PMTs could not identify the different ideas presented in the videos, the R-T introduced the ideas.

As the implementation progressed, the PMTs started to conceptualize function ideas. In the last week of the implementation, all PMTs successfully identified both explicit and implicit ideas in the given examples (machine, mother–child, etc.). In addition, they concluded that a function is a coherent whole and different ideas of functions should be emphasized during instruction as seen in the interaction below.

> **PMT16:** Eventually you will give all [function ideas]. You'll give both mapping and oil-price examples [relations between variables idea].

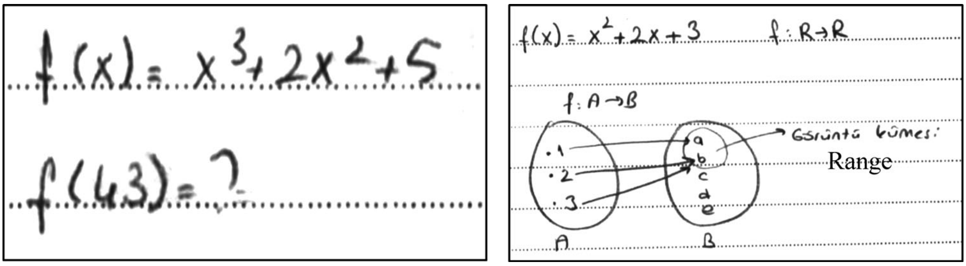

Figure 1. PMT16's pre-test and post-test function example.

PMT16: As PMT3 stated, there is also algebra idea. There is also mapping. We should give all the function ideas, not just mapping.

PMT5: Function is an integrated [coherent] whole. Mapping idea inherent in function … We can show dependent and independent variables by set mapping … for each liter of fuel, you write the price.

As seen in the interactions and examples above, the PMTs' awareness of function ideas and their ability to generate function examples that include these ideas had enhanced.

3.1.5. Accuracy and clarity of examples

After the module implementation, mathematical accuracy and clarity of the examples increased. The examples given by the PMTs became clearer by stating dependent and independent variables as measurable quantities and/or expressing definitional properties.

The examination of mathematical accuracy showed a remarkable increase in the examples that emphasized the idea of *algebra* (See Table 2). While only 12 out of 32 examples of the *algebra* idea were correct in the pre-test, 24 out of 28 examples were correct in the post-test. This increase might be explained by the decrease in partially correct examples from pre-test ($n = 20$) to post-test ($n = 4$). The strong emphasis on the importance of the domain and range and definitional properties in the implementation might have led to this increase. The following example shows how PMTs clearly defined the domain and range in the post-test compared to their pre-test examples (Figure 1).

Although the frequency of the *relations between variables* idea did not change after the module, all daily life examples given in the post-test were coded under the mathematically correct category. This increase was observed because PMTs clearly wrote dependent and independent variables in the post-test compared to the pre-test:

PMT12: Fuel consumed by a vehicle depending on the distance. (pre-test)

PMT12: Let's assume that a child is walking at a speed starting from a point and constantly accelerating. Depending on his acceleration, the distance he takes increases. This situation represents a function in which independent variable x is speed and dependent variable y is distance. (post-test)

Although PMT12 gave a function example that emphasized the *relations between variables* idea, she only mentioned two variables (distance and fuel consumed) without clarifying which one was dependent and which one was independent. In the post-test, she used the same context in her example. Different from the pre-test, however, she clearly stated the dependent (speed) and independent variables (distance). Also, she described the relations between the two variables. A similar progression was also observed in PMT3's responses:

> **Pre-test:** The amount of drug a patient needs to use is a function.
>
> **Post-test:** When the patient with asthma or any other illness receives medication, the concentration of the medication in the blood with respect to time is a function.

PMT3 did not specify the dependent variable (patient's need) clearly in the pre-test. However, in the post-test, she expressed the dependent variable (concentration) and independent variable (time) as measurable quantities. During the study, a clear statement of the variables was also discussed. In the example of filling water into a tank, the PMTs determined and clearly stated the possible dependent and independent variables respectively such as height-time, pressure-time, height-pressure, and density-time. This discussion also helped PMTs to elaborate on the *relations between variables* idea.

4. Discussion

In this study, a learning module that aimed to improve PMTs' understanding of function ideas was implemented. In line with the main research question, we examined how PMTs' understanding of function ideas evolved as they engaged in the learning module. We will discuss their development around our two sub-research questions, that is how they utilized multiple function ideas in their personal concept definitions and examples.

In relation to the first sub-research question, the results of this study regarding PMTs' personal concept definitions suggested that they failed to provide the correct mathematical definitions of functions before the intervention and showed an improvement in accuracy and depth as they engaged with the learning module throughout the implementation. These findings are also similar to the results of previous studies (e.g. Dede & Soybas, 2011; Huang & Kulm, 2012; Zazkis & Marmur, 2018). The initial state of the accuracy of PMTs' function definitions also supports the claim that teacher education courses might not provide PMTs with the opportunity to reflect on the accuracy of their mathematical understanding of the concepts (Jansen & Spitzer, 2009). This situation was very problematic since, without a solid and mathematically correct understanding of the definition of any mathematical concept, PMTs will not be able to teach these concepts effectively (Steele et al., 2013). As a result of the intervention in the study, the majority of the PMTs were able to use definitional properties in their personal concept definitions and they provided a correct mathematical definition. Studies in the related literature also found that similar interventions improved the accuracy of PMTs' definitions and their understanding of definitional properties of function (Even, 1993; McCulloch et al., 2020; Steele et al., 2013).

In response to the first sub-research question, this study highlights the improvement in the accuracy of PMTs' function definitions which prompted us to consider possible sources for why these senior PMTs could not define function correctly at the beginning of the study. Thus, we suggest the teacher training courses should prioritize mathematical definitions of the concepts.

In addition to the improvement in the accuracy of personal concept definitions, PMTs' examples, in line with the second sub-research question, also improved concerning mathematical clarity, the richness of examples of functions, and discarding non-examples. A qualitative analysis of post-test examples showed that PMTs attended different aspects of functions such as specifying dependent and independent variables. As a result, being able to produce a variety of examples that are mathematically correct and which also embody a variety of function ideas are important contributions of the intervention in helping PMTs develop a multi-faceted and comprehensive conceptualization of functions (Ayalon & Wilkie, 2019).

Another result regarding the second sub-research question is that the *algebra* idea became less dominant as the other function ideas became prominent in our participants' examples. The algebra idea was used in the function examples in the pre-test while the mapping idea was used in the post-test examples. There was no difference in the way function idea of relationship between variables was used. We think that the increase in the frequency of examples using the mapping idea might be related to the personal concept definitions which also used the mapping idea. From the perspective of our study, a personal concept definition could be either the personal reconstruction of a formal definition or a reflection of an individual's concept image (Akkoç & Tall, 2002). The correspondence idea forces students to abandon their need for an algorithm and instead focus more on the idea of mapping one set to another (Selden & Selden, 1992). The reason for the decrease in the frequency of algebraic examples is taught to be related to the notion of arbitrariness. As Freudenthal (1983) suggested, understanding arbitrariness is one of the essential features of a function (as cited in Even, 1990). In this study, tasks of the intervention focused on the arbitrariness feature of functions. After the intervention, all participants thought that a function could be arbitrary. Among the examples given in the post-test, arbitrary correspondences between two sets are remarkable. As Freudenthal (1983; as cited in Nyikahadzoyi, 2015) suggests examples that emphasize that arbitrariness should be among mathematics teachers' repertoire. In this respect, we can claim that our participants' understanding of function ideas improved after the intervention.

The frequency of examples that embody *relations between variables* did not change through the module. In particular, the idea of *relations between variables* became prominent in their daily-life examples of function after the intervention. As Cooney et al. (2010) suggested, the utilization of real-life examples of a function has the potential to reveal the complex relationship between variables idea of a function.

Overall, the study results regarding the first and second sub-questions suggested that the mapping idea became prominent in both PMT's personal concept definitions and examples. The result that the *mapping* idea became even more dominant after the intervention could be discussed both in cognitive and pedagogical terms. In cognitive terms, one of the possible underlying reasons for this result was that the mathematically correct definition of a function intrinsically emphasized *mapping* (Ayalon et al., 2017). We, as humans, are familiar with the idea of *mapping* (e.g. mapping pebbles to individuals). This idea is embedded in our experiences with discrete elements, which is also the case for discrete functions. The idea of a relation between variables, on the other hand, is embedded in experiences with continuous functions.

In pedagogical terms, we know from research that students in different countries follow different mathematics curricula that conceptualize functions in different ways and this greatly influences students' conception of functions (Ayalon et al., 2017; Ayalon & Wilkie, 2019). In Turkey, functions is introduced based on Dirichlet-Bourbaki definition and the mapping idea was prioritized in high-school and calculus courses. Thus, it is not surprising for the PMTs in this study to retain the mapping conceptualization of function despite the intervention on multiple ideas. Although their prior learning experiences seem still dominating PMTs' conceptualization of function, the results also suggested that they engaged with multiple ideas.

In relation to our main research question concerning how PMTs' understanding of function ideas evolved throughout the module, the results of the study suggested that the frequency of mathematically correct definitions and examples as well as the emphasis PMTs put on the mapping idea in their personal concept definitions and function examples increased. Although the mapping idea became prominent in the post-test definitions and examples, PMTs could identify various function ideas in the examined function examples and definitions during the intervention. Furthermore, they could identify the implicit ideas even when the definition and examples only explicitly showed a particular idea. They started to utilize multiple ideas in their definitions and examples.

Another important finding in relation to our main research question that they started to use multiple ideas and identify the connection between the ideas. This finding implies that the PMTs started to conceptualize different facets of the concept (Cooney et al., 2010) which results in a comprehensive understanding of function (Ayalon & Wilkie, 2019). Developing this understanding of the PMTs in this study could support them to create learning environments for fostering their students' understanding of these ideas and their connection to function concept. Since, the studies (Ayalon & Wilkie, 2019; Sfard, 1991) suggested that although students may have engaged in mathematics activities that introduce these ideas informally, the connection across ideas and their relations with function concept remain unexplored.

4.1. Implications for the second cycle of design-based research

Considering the overall results of the first cycle of this design-based research, the implications for designing a second cycle can be discussed under two themes. The first relates to potential revisions and refinements in the learning goals. The second relates to learning tasks. For instance, although during the implementation the connections between ideas in the definitions and examples were discussed, we would like to add an explicit goal of 'PMTs will be able to identify the connection between different function ideas'. Concerning this goal, a separate task fostering the understanding of connections across multiple ideas will be designed.

Another task suggestion could be giving PMTs opportunities to construct their personal concept definitions which emphasize function ideas other than the mapping idea and which includes definitional properties. Future mathematics teachers should have a grasp of various definitions highlighting different function ideas. A task could be designed based on the result that all PMTs could not define functions by using mathematical notation. To do that, we suggest discussing textbook definitions at the undergraduate level, which include a mathematical notation as well as high school textbooks' definitions or students' examples of the definition of a function, which are expressed in colloquial language.

5. Conclusion

We conclude our paper with discussing contributions that this study makes to the field of mathematics education and the study limitations. The results of the study have implications on how to prepare PMTs to have a deep understanding of function through engaging with a learning module situated in multiple function ideas. The PMTs' accuracy of function definitions and their knowledge of function ideas prior to this study suggested that the functions taught at university level courses remain insufficient for deep conceptualization of functions. One important contribution of this study is that the intervention supported the PMTs to acquire a rich and comprehensive understanding of function ideas that is essential for teaching conceptual mathematics. Designing this learning module on function ideas and definitions could set an example for the possible revisions in the current course design provided at the universities.

Another important contribution of this study was the PMTs' limited understanding of functions was challenged. Although the PMTs viewed functions predominantly as mapping, this study contributes to their awareness and conceptualization of a variety of function ideas and their relations. Thus, the design of the learning module has an important implication for university level course design on how PMTs' limited understanding of function could be challenged to reflect various function ideas in their definitions and examples.

One limitation of this study was that we only worked with a limited number of PMTs. Although this may limit the generalization of findings, it gave us the opportunity to deeply examine the changes in PMTs' understanding of function definitions and its ideas. However, we also acknowledged that, as McCulloch et al. (2020) suggested, conceptualization of a mathematical concept is often internal and researchers try to access those through examining evidence from PMTs' verbal and written articulations. Thus, it is hard to attend the details fully to the extent of changes in PMTs' function conceptions. The rich written and verbal articulations of PMTs' on function ideas during the study suggests the findings are worthy for further exploration.

Although, this study was limited to the ways PMTs define and give examples of functions that reflect the multiple function ideas. Beyond that, it would be worth investigating how their understanding of functions shapes their pedagogical choices. For practice, we suggest mathematics teacher educators implement the revised module and reflect on the implementation considering the findings of this study.

6. Declaration

All authors declare that this manuscript is original, has not been published before, and is not currently being considered for publication elsewhere. This study is part of a research project (project number EGT-K-091116-0515) funded by Marmara University Scientific Research Projects Commission. An earlier version of the study was presented in the 27th International Congress on Educational Sciences which was held in Antalya, Turkey. The authors have no conflicts of interest to declare that are relevant to the content of this article and have no relevant financial or non-financial interests to disclose.

Note

1. Our intervention does not involve the covariation idea of a function since the most recent mathematics curriculum that our participants will use when they enter the profession does not include this idea.

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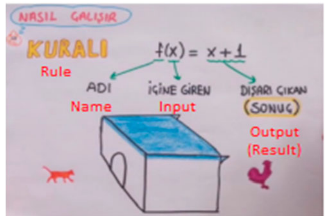
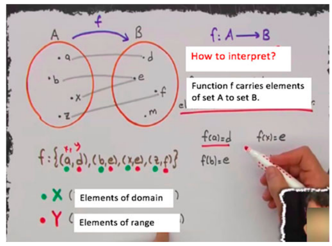
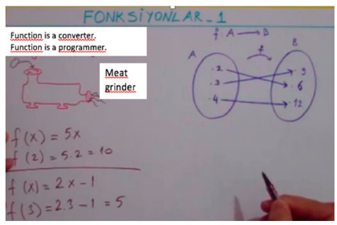
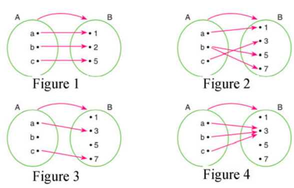
Appendix A

Details of videos

Video	Function Idea	Author and Link	Representation(s)	Example(s) in the Video	View Rate
1	Input-output Algebra	Tonguc Akademi, 2017 https://www.youtube.com/watch?v = AaGvrAjZE28	Algebraic		664000
2	Mapping	Hocalara Geldik, 2017 https://www.youtube.com/watch?v = 9_piroivxio	Set diagram List ordered-pairs Algebraic		1370000
3	Input-output Mapping Algebra	Şenol Hoca, 2017 https://www.youtube.com/watch?v = XhtwlQzOy1Q	Set diagram Algebraic		2900000

Appendix B

Details of textbooks

Textbook	Function Idea	Author(s)	Example(s) from textbook	Brief Description
1	Input-output	Bağrıaçık et al., 2010		In this textbook, three different pictures (factory, school, function machine) were given as a function examples. Students were asked to determine the similar-different aspects of the pictures are.
2	Mapping	Karakuyu & Bağcı, 2013		Mapping idea was presented in the second book in which four different set mappings were given. In these mappings, domain described as children, range described as mothers. Students were asked to examine whether it is a function or not.

Textbook	Function Idea	Author(s)	Example(s) from textbook	Brief Description
3	Relations between variables	MoNE, 2013		The textbook started with examples of functions from daily life (e.g. When driving a car, distance is a function of time). The function concept is explained by the relation between oil liter and its price.

Quantity (L)	Tutar (TL)
1	5
2	10
3	15
4	20
5	25
6	30
7	35

Appendix C

Analysis categories for personal concept definitions and function examples

Used for analysis of	Categories	Description of categories
The emphasized idea in personal concept definitions and function examples	Mapping	Any mapping between two sets that assigns to every element in the first set exactly one element in the second set.
	Algebra	Algebra idea of algebra includes generating new values from the given values using algebraic calculations (Ayalon et al., 2017)
	Input-output	Input-output idea entails the thought of processing inputs to give some type of outputs.
	Relations between variables	The relational aspect of functions and also dependency of variables; in other words, it went beyond talking about individual values or collections of individual values" (Ayalon et al., 2017, p.11).
	Rule	A function is a rule. A rule is expected to have some regularity, whereas a correspondence may be "arbitrary" (Vinner & Dreyfus, 1989)
The emphasized idea in personal concept definitions	Uncategorized	The emphasized function idea was not detected in this category. For example, PMTs defined function as a special type of relation.
The emphasized idea in function examples	Unspecified	The emphasized function idea was not detected in this category. For example, the functions in graph or list representation, and exemplar clusters were coded in this category. Exemplar clusters are the more specific cases of function, e.g. quadratics, trigonometric, exponentials (Authors, 2002).

Categories for accuracy of function examples

Categories	Description of categories
Correct	The domain and range of the function or the variables are accurately described.
Partially correct	The example represents a function but the details (domain, range, variables etc.) are not clearly included.
Incorrect	The domain and range of the function or the variables are specified but the example is not a function.

Details of critical events for the video analysis

Critical Events	Description of critical events
Focusing on definitional properties	The script(s) of PMTs' or R-T's about definitional properties of function (Mapping, algebra, input-output, relations between variables, rule) in the module implementation.
Function ideas	The script(s) of PMTs' or R-T's about function ideas (or concept images related to function ideas in the module implementation.