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Identities for the Glasser transform and their applications

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Abstract. In the present paper the author shows that an iteration of the $\mathscr{L}_2$-transform by itself is a constant multiple of the Glasser transform. Using this iteration identity, a Parseval-Goldstein type theorem for the $\mathscr{L}_2$-transform and the Glasser transform is given. By making use of these results a number of new Parseval-Goldstein type identities are obtained for these and many other well-known integral transforms. The identities proven in this paper are shown to give rise to useful corollaries for evaluating infinite integrals of special functions. Some examples are also given as illustration of the results presented here.

Key words. Laplace transforms, $\mathscr{L}_2$-transforms, Glasser transforms, Fourier sine transforms, Fourier cosine transforms, Hankel transforms, $\mathscr{K}$-transforms, $\mathscr{E}_1$-transforms, $\mathscr{E}_{2,1}$-transforms, Parseval-Goldstein type theorems.

Glasser dönüşümü için özdeşlikler ve bunların uygulamaları

Özet. Bu makalede yazarlar $\mathscr{L}_2$-dönüşümünün kendisi ile iterasyonunun Glasser dönüşümünün sabit bir katı olduğunu gösterirler. Bu iterasyon özdeşliğini kullanarak, $\mathscr{L}_2$-dönüşümü ve Glasser dönüşümü için bir Parseval-Goldstein tip teorem verilmektedir. Bu sonuçların kullanılması ile bu ve bir çok iyi bilinen integral dönüşümü için yeni Parseval-Goldstein tip özdeşlikler elde edilmiştir. Bu makalede ispatlanan özdeşlikler özel fonksiyonların belirsiz integrallerini hesaplamak için kullanışlı sonuçların gösterilmesini sağlar. Burada sunulan sonuçlara örnek olarak bazı alıştırmalar verilmiştir.

Anahtar kelimeler. Laplace dönüşümü, $\mathscr{L}_2$-dönüşümü, Glasser dönüşümü, Fourier sinus dönüşümü, Fourier kosinüs dönüşümü, Hankel dönüşümü, $\mathscr{K}$-dönüşümü, $\mathscr{E}_1$-dönüşümü, $\mathscr{E}_{2,1}$-dönüşümü, Parseval-Goldstein tip teoremler.

1 Introduction

Over two decades ago, the author [1] introduced the $\mathscr{L}_2$-transform:

$$\mathscr{L}_2\big\{f(x);y\big\} = \int_0^\infty x\exp\big(-x^2y^2\big)f(x)dx \tag{1.1}$$

and presented its systematic account in [1, 2]. The $\mathscr{L}_2$-transform is related to the classical Laplace transform

$$\mathscr{L}\big\{f(x);y\big\} = \int_0^\infty \exp(-xy)f(x)dx \tag{1.2}$$

by means of the following relationships:

$$\mathscr{L}_2\left\{f(x);y\right\} = \frac{1}{2}\mathscr{L}\Big\{f\big(\sqrt{x}\big);y^2\Big\}, \tag{1.3}$$

$$\mathscr{L}\left\{f(x);y\right\} = 2\mathscr{L}_2\Big\{f\big(x^2\big);\sqrt{y}\Big\}. \tag{1.4}$$

Subsequently, various Parseval-Goldstein type identities were given in (for example) [3–8] for the $\mathscr{L}_2$-transform. Using this integral transform new solution techniques were obtained for the Bessel differential equation in [9] and the Hermite differential equation in [10]. There are numerous analogous results in the literature on various integral transforms (see, for instance [11–14]). Some of the results from [12, 15] are applied to generalized functions in [16].

Over three decades ago, Glasser [6] considered so-called the Glasser transform

$$\mathscr{G}\big\{f(x);y\big\} = \int_0^\infty \frac{f(x)}{\sqrt{x^2+y^2}}dx. \tag{1.5}$$

Glasser gave the following Parseval-Goldstein type theorem (cf. [6, p. 171, Eq. (4)])

$$\int_0^\infty f(x)\mathscr{G}\big\{g(y);x\big\}dx = \int_0^\infty g(x)\mathscr{G}\big\{f(y);x\big\}dx, \tag{1.6}$$

and evaluated a number of infinite integrals involving Bessel functions. Additional results about the Glasser transform can be found in [11, 17].

In this article, we consider several other integral transforms and potentially useful identities of these and the other integral transforms considered earlier. First of all, the Fourier sine transform and the Fourier cosine transform are defined as

$$\mathscr{F}_s\{f(x);y\} = \int_0^\infty \sin(xy)f(x)dx, \tag{1.7}$$

and

$$\mathscr{F}_c\{f(x);y\} = \int_0^\infty \cos(xy)f(x)dx, \tag{1.8}$$

respectively.

The Hankel transform is defined by

$$\mathscr{H}_\nu\{f(x);y\} = \int_0^\infty \sqrt{xy}J_\nu(xy)f(x)dx, \tag{1.9}$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν. Using the formula (cf. [18, p. 306, Eq. 32:13:10])

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x), \tag{1.10}$$

definition (1.7) of the Fourier sine transform, and the definition (1.9) of the Hankel transform, we obtain the familiar relationship

$$\mathscr{H}_{1/2}\{f(x);y\} = \sqrt{\frac{2}{\pi}}\mathscr{F}_S\{f(x);y\}. \tag{1.11}$$

Similarly, using the formula (cf. [18, p. 306, Eq. 32:13:11])

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos(x), \tag{1.12}$$

definition (1.8) of the Fourier cosine transform, and definition (1.9) of the Hankel transform, we obtain the relationship

$$\mathscr{H}_{-1/2}\{f(x);y\} = \sqrt{\frac{2}{\pi}}\mathscr{F}_C\{f(x);y\}. \tag{1.13}$$

The $\mathscr{K}$-transform is defined by

$$\mathscr{K}_\nu\{f(x);y\} = \int_0^\infty \sqrt{xy}K_\nu(xy)f(x)dx, \tag{1.14}$$

where K_ν is the Bessel function of the second kind of order ν. Using the formula (cf. [18, p. 239, Eq. 26:13:5])

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\exp(-x), \tag{1.15}$$

definition (1.2) of the Laplace transform, and the definition (1.14) of the $\mathscr{K}$-transform, we obtain the relationship

$$\mathscr{K}_{1/2}\{f(x);y\} = \sqrt{\frac{\pi}{2}}\mathscr{L}\{f(x);y\} \tag{1.16}$$

which incidentally holds true also when $\mathscr{K}_{1/2}$ is replaced by $\mathscr{K}_{-1/2}$.

In this article, we show that an iteration of the $\mathscr{L}_2$-transform by itself is a constant multiple of the Glasser transform defined by (1.5). Using this iteration identity, we establish a Parseval-Goldstein type theorem relating the $\mathscr{L}_2$-transform and the Glasser transform. The Parseval-Goldstein type theorem established here yields potentially new identities for the various integral transform introduced above. As applications of the resulting identities and theorems, some illustrative examples are also given.

2 The main theorem

In the following lemma, we give an iteration identity involving the $\mathscr{L}_2$-transform (1.1) and the Glasser transform (1.5).

Lemma 2.1. *The identity*

$$\mathscr{L}_2\Big\{\frac{1}{u}\mathscr{L}_2\Big\{f(x);u\Big\};y\Big\} = \frac{\sqrt{\pi}}{2}\mathscr{G}\big\{xf(x);y\big\} \tag{2.1}$$

holds true, provided that the integrals involved converge absolutely.

Proof. Using definition (1.1) of the $\mathscr{L}_2$-transform, we have

$$\mathscr{L}_2\Big\{\frac{1}{u}\mathscr{L}_2\big\{f(x);u\big\};y\Big\} = \int_0^\infty \exp\big(-y^2u^2\big)\left[\int_0^\infty x\exp\big(-x^2u^2\big)f(x)dx\right]du. \tag{2.2}$$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved, and then using definition (1.1) of the $\mathscr{L}_2$-transform once more, we find from (2.2) that

$$\begin{aligned}\mathscr{L}_2\Big\{\frac{1}{u}\mathscr{L}_2\big\{f(x);u\big\};y\Big\} &= \int_0^\infty xf(x)\left[\int_0^\infty \exp\Big[\big(-y^2+x^2\big)u^2\Big]du\right]dx\\ &= \int_0^\infty xf(x)\mathscr{L}_2\Big\{\frac{1}{u};\big(x^2+y^2\big)^{1/2}\Big\}dx.\end{aligned} \tag{2.3}$$

Furthermore, we have

$$\mathscr{L}_2\Big\{\frac{1}{u};\big(x^2+y^2\big)^{1/2}\Big\} = \frac{\sqrt{\pi}}{2}\big(x^2+y^2\big)^{-1/2}. \tag{2.4}$$

Now assertion (2.1) follows from (2.3), (2.4), and definition (1.5) of the Glasser transform. ∎

Lemma 2.1 yields some useful corollaries that will be required in our investigation.

Corollary 2.2. *We have (cf. [6, p. 171, (2)])*

$$\mathscr{G}\Big\{x^{\mu-1};y\Big\} = 2^{-\mu}\mathrm{B}\Big(\mu,\frac{1}{2}-\frac{\mu}{2}\Big)y^{\mu-1}, \quad 0<\Re(\mu)<1, \tag{2.5}$$

where $\mathrm{B}(x,y)$ *is the beta function defined by*

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x>0, y>0, \tag{2.6}$$

and it is related to the gamma function through

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \mathrm{B}(y,x). \tag{2.7}$$

Proof. We set

$$f(x) = x^{\mu-2}, \quad 0<\Re(\mu)<1 \tag{2.8}$$

in Lemma 2.1. Using relation (1.3) and the known formula [19, p. 137, Entry (1)], we find that

$$\mathscr{L}_2\Big\{x^{\mu-2};u\Big\} = \frac{1}{2}\mathscr{L}\Big\{x^{(\mu-2)/2};u^2\Big\} = \frac{1}{2}\Gamma\Big(\frac{\mu}{2}\Big)u^{-\mu}. \tag{2.9}$$

Multiplying equation (2.9) through by $1/u$ and then applying the $\mathscr{L}_2$-transform, we obtain

$$\mathscr{L}_2\Big\{\frac{1}{u}\mathscr{L}_2\Big\{x^{\mu-2};u\Big\};y\Big\} = \frac{1}{2}\Gamma\Big(\frac{\mu}{2}\Big)\mathscr{L}_2\Big\{u^{-\mu-1},y\Big\}. \tag{2.10}$$

Using relation (1.3) and formula [19, p. 137, Entry (1)] once more on the right hand side of (2.10), we deduce that

$$\mathscr{L}_2\Big\{\frac{1}{u}\mathscr{L}_2\Big\{x^{\mu-2};u\Big\};y\Big\} = \frac{1}{4}\Gamma\Big(\frac{\mu}{2}\Big)\Gamma\Big(\frac{1}{2}-\frac{\mu}{2}\Big)y^{\mu-1}. \tag{2.11}$$

Utilizing the well-known duplication formula for the gamma function (cf. [18, p. 414, Eq. (43:5:7)])

$$\Gamma(2\alpha) = \frac{4^{\alpha}}{2\sqrt{\pi}}\Gamma(\alpha)\Gamma\Big(\frac{1}{2}+\alpha\Big) \tag{2.12}$$

with $\mu = 2\alpha$, relationship (2.7) for the beta function on the right hand side of (2.11) and finally identity (2.1) of Lemma 2.1, we obtain the desired result (2.5). ■

Corollary 2.3. *We have (cf. [6, p. 171, (h)])*

$$\mathscr{G}\Big\{x^{\nu+1}J_{\nu}(zx);y\Big\} = \sqrt{\frac{2}{\pi z}}y^{\nu+\frac{1}{2}}K_{\nu+\frac{1}{2}}(zy). \tag{2.13}$$

Proof. We set

$$f(x) = x^{\nu}J_{\nu}(zx), \quad -1<\Re(\nu)<\frac{1}{2} \tag{2.14}$$

in Lemma 1. Using relation (1.3) and the known formula [19, p. 185, Entry (30)], we find that

$$\mathscr{L}_2\big\{x^\nu J_\nu(zx);u\big\} = \frac{1}{2}\mathscr{L}\Big\{x^{\nu/2}J_\nu\big(zx^{1/2}\big);u^2\Big\}$$
$$= \frac{1}{2}\Big(\frac{z}{2}\Big)^\nu u^{-2\nu-2}\exp\bigg(-\frac{z^2}{4u^2}\bigg). \tag{2.15}$$

Multiplying equation (2.15) through by $1/u$ and then applying the $\mathscr{L}_2$-transform, we obtain

$$\mathscr{L}_2\Big\{\frac{1}{u}\mathscr{L}_2\big\{x^\nu J_\nu(zx);u\big\};y\Big\} = \frac{1}{2}\Big(\frac{z}{2}\Big)^\nu \mathscr{L}_2\bigg\{u^{-2\nu-3}\exp\bigg(-\frac{z^2}{4u^2}\bigg);y\bigg\}. \tag{2.16}$$

We evaluate the $\mathscr{L}_2$-transform on the right hand side of (2.16) by using relation (1.3) and the known formula [19, p. 146, Entry (29)]:

$$\mathscr{L}_2\bigg\{u^{-2\nu-3}\exp\bigg(-\frac{z^2}{4u^2}\bigg);y\bigg\} = \frac{1}{2}\mathscr{L}\bigg\{u^{-(2\nu+3)/2}\exp\bigg(-\frac{z^2}{4u}\bigg);y^2\bigg\}$$
$$= \Big(\frac{2y}{z}\Big)^{\nu+\frac{1}{2}}\mathrm{K}_{\nu+\frac{1}{2}}(zy). \tag{2.17}$$

Now assertion (2.13) immediately follows upon substituting the result (2.17) into equation (2.16) and using identity (2.1) of Lemma 2.1. ∎

Corollary 2.4. *We have (cf. [6, p. 174, (g)])*

$$\mathscr{G}\big\{J_\nu(zx);y\big\} = I_{\nu/2}\Big(\frac{zy}{2}\Big)K_{\nu/2}\Big(\frac{zy}{2}\Big), \quad \Re(\nu) > -1. \tag{2.18}$$

Proof. We set

$$f(x) = \frac{1}{x}J_\nu(zx), \quad \Re(\nu) > -1 \tag{2.19}$$

in Lemma 2.1. Using relation (1.3) and the known formula [19, p. 185, Entry 29], we find that

$$\mathscr{L}_2\Big\{\frac{1}{x}J_\nu(zx);u\Big\} = \frac{\sqrt{\pi}}{2u}\exp\bigg(-\frac{z^2}{8u^2}\bigg)I_{\nu/2}\bigg(\frac{z^2}{8u^2}\bigg). \tag{2.20}$$

Multipliying equation (2.20) through by $1/u$ and then applying the $\mathscr{L}_2$-transform, we obtain

$$\mathscr{L}_2\bigg\{\frac{1}{u}\mathcal{L}_2\Big\{\frac{1}{x}J_\nu(zx);u\Big\};y\bigg\} = \frac{\sqrt{\pi}}{2}\mathscr{L}_2\bigg\{\frac{1}{u^2}\exp\bigg(-\frac{z^2}{8u^2}\bigg)I_{\nu/2}\bigg(\frac{z^2}{8u^2}\bigg);y\bigg\}. \tag{2.21}$$

Now assertion (2.18) immediately follows upon using the relation (1.3) once more and than utilizing the known formula [20, p. 325, Entry 10] and (2.1) of Lemma 2.1. ∎

Theorem 2.5. *If the conditions stated in Lemma 2.1 are satisfied, then the Parseval-Goldstein type relations*

$$\int_0^\infty \mathscr{L}_2\left\{f(x);y\right\}\mathscr{L}_2\left\{g(u);y\right\}dy = \frac{\sqrt{\pi}}{2}\int_0^\infty xf(x)\mathscr{G}\big\{ug(u);x\big\}dx \tag{2.22}$$
$$\int_0^\infty \mathscr{L}_2\left\{f(x);y\right\}\mathscr{L}_2\left\{g(u);y\right\}dy = \frac{\sqrt{\pi}}{2}\int_0^\infty ug(u)\mathscr{G}\big\{xf(x);u\big\}du \tag{2.23}$$

and

$$\int_0^\infty xf(x)\mathscr{G}\{ug(u);x\}dx = \int_0^\infty ug(u)\mathscr{G}\{xf(x);u\}du \tag{2.24}$$

hold true.

Proof. We only give the proof of (2.22), as the proof of (2.23) is similar. Identity (2.24) follows from the identities (2.22) and (2.23).

Using definition (1.1) of the $\mathscr{L}_2$-transform, we have

$$\begin{aligned}&\int_0^\infty \mathscr{L}_2\{f(x);y\}\mathscr{L}_2\{g(u);y\}dy\\ &\qquad = \int_0^\infty \mathscr{L}_2\{g(u);y\}\left[\int_0^\infty x\exp\left(-x^2y^2\right)f(x)dx\right]dy. \end{aligned}\tag{2.25}$$

Changing the order of integration (which is permissible by absolute convergence of the integrals involved) and using definition (1.1) of the $\mathscr{L}_2$-transform once again, we find from (2.25) that

$$\begin{aligned}&\int_0^\infty \mathscr{L}_2\{f(x);y\}\mathscr{L}_2\{g(u);y\}dy\\ &\qquad = \int_0^\infty xf(x)\left[\int_0^\infty \exp\left(-x^2y^2\right)\mathscr{L}_2\{g(u);y\}dy\right]dx\\ &\qquad = \int_0^\infty xf(x)\mathscr{L}_2\left\{\frac{1}{y}\mathscr{L}_2\{g(u);y\};x\right\}dx. \end{aligned}\tag{2.26}$$

Now assertion (2.22) easily follows from (2.26) and (2.1) of the Lemma 2.1. ■

Corollary 2.6. *If the integrals involved converge absolutely and* $0 < \Re(\mu) < 1$*, then we have*

$$\int_0^\infty y^{-\mu}\mathscr{L}_2\{f(x);y\}dy = \frac{1}{2}\Gamma\Big(\frac{1}{2}-\frac{\mu}{2}\Big)\int_0^\infty x^\mu f(x)dx, \tag{2.27}$$

$$\int_0^\infty y^{-\mu}\mathscr{L}_2\{f(x);y\}dy = \frac{\sqrt{\pi}}{\Gamma(\mu/2)}\int_0^\infty u^{\mu-1}\mathscr{G}\{xf(x);u\}du, \tag{2.28}$$

and

$$\int_0^\infty u^{\mu-1}\mathscr{G}\{xf(x);u\}du = \frac{1}{2}\mathrm{B}\Big(\frac{\mu}{2},-\frac{\mu}{2}+\frac{1}{2}\Big)\int_0^\infty x^\mu f(x)dx. \tag{2.29}$$

Proof. We start with the proof of assertion (2.27) by setting

$$g(u) = u^{\mu-2} \tag{2.30}$$

in Theorem 2.5. Utilizing the formulas (2.5), (2.9) and the identity (2.22) of the Theorem 2.5, we obtain that

$$\int_0^\infty y^{-\mu}\mathscr{L}_2\{f(x);y\}dy = \frac{\sqrt{\pi}}{2^\mu}\left[\Gamma\Big(\frac{\mu}{2}\Big)\right]^{-1}B\Big(\mu,\frac{1}{2}-\frac{\mu}{2}\Big)\int_0^\infty x^\mu f(x)dx. \tag{2.31}$$

Using the duplication formula (2.12) for the gamma function with $\mu = 2\alpha$ on the right hand side of (2.31) we deduce assertion (2.27).

Similarly, the proof of assertion (2.28) follows upon utilizing (2.30) and (2.9) into the identity (2.23) of Theorem 2.5.

Finally, assertion (2.29) easily follows from the identities (2.27), (2.28) and the relationship (2.7) between the beta function and the gamma function. ■

Corollary 2.7. *If the integrals involved converge absolutely and* $-1 < \Re(\nu) < 1/2$*, then we have*

$$\mathscr{L}_2\Big\{y^{2\nu-1}\mathscr{L}_2\Big\{f(x);\frac{1}{2y}\Big\};z\Big\} = 2^{-\nu-\frac{1}{2}}z^{-\nu-1}\mathscr{K}_{\nu+\frac{1}{2}}\big\{x^{\nu+1}f(x);z\big\}, \tag{2.32}$$

$$\mathscr{L}_2\Big\{y^{2\nu-1}\mathscr{L}_2\Big\{f(x);\frac{1}{2y}\Big\};z\Big\} = \frac{\sqrt{\pi}}{2^{\nu+1}}z^{-\nu-\frac{1}{2}}\mathscr{H}_\nu\Big\{u^{\nu+\frac{1}{2}}\mathscr{G}\big\{xf(x);u\big\};z\Big\}, \tag{2.33}$$

and

$$\mathscr{K}_{\nu+\frac{1}{2}}\big\{x^{\nu+1}f(x);z\big\} = \Big(\frac{\pi z}{2}\Big)^{1/2}\mathscr{H}_\nu\Big\{u^{\nu+\frac{1}{2}}\mathscr{G}\big\{xf(x);u\big\};z\Big\}, \tag{2.34}$$

where $\mathscr{H}_\nu\{f(x);y\}$ *and* $\mathscr{K}_\nu\{f(x);y\}$ *denote the Hankel transform and the* $\mathscr{K}$*-transform as defined by* (1.9) *and* (1.14)*, respectively.*

Proof. We put

$$g(u) = u^\nu J_\nu(zu) \tag{2.35}$$

in Theorem 2.5. Utilizing identity (2.13) of Corollary 2.3, equation (2.15) and the Parseval-Goldstein type relation (2.22) of Theorem 2.5, we obtain

$$\int_0^\infty \frac{1}{y^{2\nu+2}}\exp\Big(-\frac{z^2}{4y^2}\Big)\mathscr{L}_2\big\{f(x);y\big\}dy = \Big(\frac{2}{z}\Big)^{\nu+\frac{1}{2}}\int_0^\infty x^{\nu+\frac{3}{2}}\mathrm{K}_{\nu+\frac{1}{2}}(zx)f(x)dx. \tag{2.36}$$

Making a simple change of variable in the integral on the left hand side and using definition (1.14) of the $\mathscr{K}$-transform on the right hand side of (2.36) we obtain the desired identity (2.32).

Assertion (2.33) is obtained similarly using the Parseval-Goldstein relation (2.23) of Theorem 2.5. Assertion (2.34) immediately follows from the relations (2.22) and (2.23). ■

Remark 2.8. *If we let* $\nu = 0$ *in our Corollary 2.7 and than use the formula* (1.10) *and definition* (1.2) *of the Laplace transform, we obtain*

$$\mathscr{L}_2\Big\{\frac{1}{y}\mathscr{L}_2\Big\{f(x);\frac{1}{2y}\Big\};z\Big\} = \frac{\sqrt{\pi}}{2z}\mathscr{L}\big\{xf(x);z\big\} \tag{2.37}$$

$$\mathscr{L}_2\Big\{\frac{1}{y}\mathscr{L}_2\Big\{f(x);\frac{1}{2y}\Big\};z\Big\} = \frac{1}{2}\sqrt{\frac{\pi}{z}}\mathscr{H}_0\Big\{\sqrt{u}\mathscr{G}\big\{xf(x);u\big\};z\Big\} \tag{2.38}$$

and

$$\mathscr{L}\Big\{xf(x);z\Big\} = \sqrt{z}\mathscr{H}_0\Big\{\sqrt{u}\mathscr{G}\Big\{xf(x);u\Big\};z\Big\}. \tag{2.39}$$

Remark 2.9. *If we let* $\nu = -1/2$ *in our Corollary 2.7 and than use the formula* (1.13) *and definition* (1.8) *of the Fourier cosine transform, we obtain*

$$\mathscr{L}_2\Big\{\frac{1}{y^2}\mathscr{L}_2\Big\{f(x);\frac{1}{2y}\Big\};z\Big\} = \frac{1}{\sqrt{z}}\mathscr{K}_0\Big\{x^{1/2}f(x);z\Big\} \tag{2.40}$$

$$\mathscr{L}_2\Big\{\frac{1}{y^2}\mathscr{L}_2\Big\{f(x);\frac{1}{2y}\Big\};z\Big\} = \mathscr{F}_c\Big\{\mathscr{G}\Big\{xf(x);u\Big\};z\Big\} \tag{2.41}$$

and

$$\mathscr{K}_0\Big\{x^{1/2}f(x);z\Big\} = \sqrt{z}\mathscr{F}_c\Big\{\mathscr{G}\Big\{xf(x);u\Big\};z\Big\}. \tag{2.42}$$

Remark 2.10. *If we let* $\nu = 1/2$ *in our Corollary 2.7 and than use the formula* (1.11) *and definition* (1.7) *of the Fourier sine transform, we obtain*

$$\mathscr{L}_2\Big\{\mathscr{L}_2\Big\{f(x);\frac{1}{2y}\Big\};z\Big\} = \frac{1}{2z^{3/2}}\mathscr{K}_1\Big\{x^{3/2}f(x);z\Big\} \tag{2.43}$$

$$\mathscr{L}_2\Big\{\mathscr{L}_2\Big\{f(x);\frac{1}{2y}\Big\};z\Big\} = \frac{1}{2z}\mathscr{F}_s\Big\{u\mathscr{G}\Big\{xf(x);u\Big\};z\Big\} \tag{2.44}$$

and

$$\mathscr{K}_1\Big\{x^{3/2}f(x);z\Big\} = \sqrt{z}\mathscr{F}_s\Big\{u\mathscr{G}\Big\{xf(x);u\Big\};z\Big\}. \tag{2.45}$$

Corollary 2.11. *If the integrals involved converge absolutely, then we have*

$$\int_0^\infty \frac{1}{y}\exp\Big(-\frac{z^2}{8y^2}\Big)I_{\frac{\nu}{2}}\Big(-\frac{z^2}{8y^2}\Big)\mathscr{L}_2\Big\{f(x);y\Big\}dy = \int_0^\infty xf(x)I_{\frac{\nu}{2}}\Big(\frac{zx}{2}\Big)\mathrm{K}_{\frac{\nu}{2}}\Big(\frac{zx}{2}\Big)dx \tag{2.46}$$

$$\int_0^\infty \frac{1}{y}\exp\Big(-\frac{z^2}{8y^2}\Big)\mathrm{I}_{\frac{\nu}{2}}\Big(-\frac{z^2}{8y^2}\Big)\mathscr{L}_2\Big\{f(x);y\Big\}dy = z^{-1/2}\mathscr{H}_\nu\Big\{u^{-1/2}\mathscr{G}\Big\{xf(x);u\Big\};z\Big\} \tag{2.47}$$

and

$$\int_0^\infty xf(x)I_{\frac{\nu}{2}}\Big(\frac{zx}{2}\Big)K_{\frac{\nu}{2}}\Big(\frac{zx}{2}\Big)dx = z^{-1/2}\mathscr{H}_\nu\Big\{u^{-1/2}\mathscr{G}\Big\{xf(x);u\Big\};z\Big\}. \tag{2.48}$$

Proof. The proof of Corollary 2.11 is analogous to the previous Corollary 2.7. The assertions (2.46), (2.47), and (2.48) are obtained by putting

$$g(u) = \frac{\mathrm{J}_\nu(zu)}{u} \tag{2.49}$$

in Theorem 2.5 and by using the known formulas [6, p. 174, Entry (g)] and [19, p. 185, Entry (29)]. ∎

The following corollary contains an identity involving $\mathscr{L}_2$-transform, the Glasser transform, the $\mathscr{E}_1$-transform defined by

$$\mathscr{E}_1\big\{f(x);y\big\} = \int_0^\infty \exp\big(xy\big)E_1(xy)f(x)dx \tag{2.50}$$

introduced in [3, p. 1377, Eq. (1.1)], $\mathscr{E}_{2,1}$-transform defined by

$$\mathscr{E}_{2,1}\big\{f(x);y\big\} = \int_0^\infty x\exp\big(x^2y^2\big)E_1(x^2y^2)f(x)dx \tag{2.51}$$

introduced in [4, p. 1557, Eq. (1.1)] and the Widder transform defined by

$$\mathscr{P}\big\{f(x);y\big\} = \int_0^\infty \frac{xf(x)}{x^2+y^2}dx \tag{2.52}$$

introduced by Widder [21]. The function $\mathrm{E}_1(x)$ is the second member of a family of functions defined by

$$\mathrm{E}_n(x) = \int_1^\infty \frac{\exp(-xt)}{t^n} \qquad n = 0, 1, \ldots. \tag{2.53}$$

The functions $\mathrm{E}_n(x)$ were introduced by Schlömilch. The function $\mathrm{E}_1(x)$ in the definitions (2.50) and (2.51) of the $\mathscr{E}_{2,1}$ is related in a simple way to exponential integral function:

$$\mathrm{E}_1(x) = -\mathrm{Ei}(-x), \tag{2.54}$$

where the exponential integral function is defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^x \frac{\exp(t)}{t}dt. \tag{2.55}$$

Corollary 2.12. *If the integrals involved converge absolutely, then we have*

$$\mathscr{E}_{2,1}\left\{\frac{1}{y}\mathscr{L}_2\big\{f(x);y\big\};z\right\} = \sqrt{\pi}\mathscr{P}\Big\{\mathscr{G}\big\{xf(x);u\big\};z\Big\}. \tag{2.56}$$

Proof. Assertion (2.56) immediately follows upon putting

$$g(u) = \frac{1}{u^2+z^2} \tag{2.57}$$

identity (2.23) of Theorem 2.5 and by using the known formula [19, p. 185, Entry (29)]. ∎

3 Illustrative examples

An interesting illustration for identity (2.1) asserted by Lemma 2.1 is contained in the following example.

Example 3.1. *Suppose that* $|z| > |y|$. *Then*

$$\mathscr{L}_2\left\{\frac{1}{u}\exp\left(z^2u^2\right)E_1\left(z^2u^2\right);y\right\} = \sqrt{\pi}\frac{\pi - 2\arcsin(y/z)}{\sqrt{z^2-y^2}}, \tag{3.1}$$

$$\mathscr{L}\left\{\frac{1}{\sqrt{u}}\exp\left(zu\right)E_1\left(zu\right);y\right\} = \sqrt{\pi}\frac{\pi - 2\arcsin\left(\sqrt{y/z}\right)}{\sqrt{z-y}}, \tag{3.2}$$

$$\mathscr{E}_{2,1}\left\{\frac{1}{u}\exp\left(-y^2u^2\right);z\right\} = \sqrt{\pi}\frac{\pi - 2\arcsin(y/z)}{\sqrt{z^2-y^2}}, \tag{3.3}$$

and

$$\mathscr{E}_1\left\{\frac{1}{\sqrt{u}}\exp\left(-yu\right);z\right\} = \sqrt{\pi}\frac{\pi - 2\arcsin\left(\sqrt{y/z}\right)}{\sqrt{z-y}}\cdot \tag{3.4}$$

Proof. We put

$$f(x) = \frac{1}{x^2+z^2}\cdot \tag{3.5}$$

Using the known result [22, p. 10, Entry (47)] we find that

$$\mathscr{G}\left\{\frac{x}{x^2+z^2};y\right\} = \frac{\pi - 2\arcsin(y/z)}{2\sqrt{z^2-y^2}}\cdot \tag{3.6}$$

Using relationship (1.3) between the Laplace transform and the $\mathscr{L}_2$-transform and the known formula [20, p. 17, Entry (5)], we obtain

$$\mathscr{L}_2\left\{\frac{1}{x^2+z^2};u\right\} = \frac{1}{2}\mathscr{L}\left\{\frac{1}{x+z^2};u^2\right\} = \frac{1}{2}\exp\left(z^2x^2\right)E_1\left(z^2x^2\right)\cdot \tag{3.7}$$

Substituting the results (3.6) and (3.7) into identity (2.1) of Lemma 2.1, we obtain the asserted formula (3.1).

From relationship (1.3) we deduce assertion (3.2). The assertions (3.3) and (3.4) follow from the definitions (2.50) and (2.51) of the $\mathscr{E}_1$-transform and $\mathscr{E}_{2,1}$-transform, respectively. ■

The following illustration involves the error function defined by

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-x^2\right)dx, \tag{3.8}$$

and the Dawson integral defined by

$$\text{daw}(x) = \int_0^x \exp\left(t^2-x^2\right)dt. \tag{3.9}$$

The Dawson integral and the error function are related via the identity

$$\operatorname{daw}(x) = \frac{-i\sqrt{\pi}}{2} \exp\big(-x^2\big)\operatorname{Erf}(ix) \tag{3.10}$$

(cf. [18, p. 405, Eq. 42:0:1)]).

Example 3.2. *We show*

$$\mathscr{L}_2\Big\{\frac{1}{u^2}\exp\Big(-\frac{z^2}{4u^2}\Big)\operatorname{Erf}\Big(i\frac{z}{2u}\Big);y\Big\} = \frac{\pi i}{2}\big[\mathrm{I}_0(zy) - \mathbf{L}_0(zy)\big] \tag{3.11}$$

and

$$\mathscr{L}_2\Big\{\frac{1}{u^2}\operatorname{daw}\Big(\frac{z}{2u}\Big);y\Big\} = \frac{\pi^{3/2}}{4}\big[\mathrm{I}_0(zy) - \mathbf{L}_0(zy)\big], \tag{3.12}$$

where $\mathrm{I}_0(x)$ *denotes the modified Bessel function of the first kind of order zero and* $\mathrm{L}_0(x)$ *denotes the modified Struve function of order zero.*

Proof. We put

$$f(x) = \frac{\sin(zx)}{x}\cdot \tag{3.13}$$

Using relationship (1.3) and the known formula [19, p. 154, Entry (36)], we have

$$\begin{aligned}\mathscr{L}_2\big\{x^{-1}\sin(zx);u\big\} &= \frac{1}{2}\mathscr{L}\big\{x^{-1/2}\sin(zx^{1/2});u^2\big\}\\ &= -\frac{i\sqrt{\pi}}{2u}\exp\Big(-\frac{z^2}{4u^2}\Big)\operatorname{Erf}\Big(i\frac{z}{2u}\Big)\cdot\end{aligned} \tag{3.14}$$

Multiplying both sides of (3.14) by $1/u$ and then applying the $\mathscr{L}_2$-transform, we find that

$$\mathscr{L}_2\Big\{\frac{1}{u}\mathscr{L}_2\big\{x^{-1}\sin(zx);u\big\};y\Big\} = -\frac{i\sqrt{\pi}}{2}\mathscr{L}_2\Big\{\frac{1}{u^2}\exp\Big(-\frac{z^2}{4u^2}\Big)\operatorname{Erf}\Big(i\frac{z}{2u}\Big);y\Big\}\cdot \tag{3.15}$$

From the known formula [6, p. 174, Entry (a)] we have

$$\mathscr{G}\big\{\sin(zx);y\big\} = \frac{\pi}{2}\big[I_0(zy) - \mathbf{L}_0(zy)\big]\cdot \tag{3.16}$$

Substituting the formulas (3.15) and (3.16) into identity (2.1) of Lemma 2.1, we obtain the desired result (3.11).

From relationship (3.10) and formula (3.11) we deduce, assertion (3.12). ∎

Remark 3.1. *Using relationship* (1.3) *and setting* $z = 2a^{1/2}$ *we can restate the formulas* (3.11) *and* (3.12) *as*

$$\mathscr{L}\Big\{\frac{1}{u}\exp\Big(-\frac{a}{u}\Big)\operatorname{Erf}\Big(i\sqrt{\frac{a}{u}}\Big);y\Big\} = \pi i\big[\mathrm{I}_0\big(2\sqrt{ay}\big) - \mathbf{L}_0\big(2\sqrt{ay}\big)\big] \tag{3.17}$$

and

$$\mathscr{L}\left\{\frac{1}{u}\mathrm{daw}\Big(\sqrt{\frac{a}{u}}\Big);y\right\} = \pi\big[I_0\big(2\sqrt{ay}\big) - \mathbf{L}_0\big(2\sqrt{ay}\big)\big]. \tag{3.18}$$

Example 3.3. *Suppose that* $\Re(z) > 0$ *and* $\max\{0, -2\Re(\nu)\} < \Re(\mu) < 1$. *Then*

$$\int_0^\infty y^{-\mu-1}\exp\left(-\frac{z^2}{2y^2}\right)I_\nu\left(\frac{z^2}{2y^2}\right)dy = \frac{\Gamma\Big(\frac{1}{2}-\frac{\mu}{2}\Big)\Gamma\Big(\nu+\frac{\mu}{2}\Big)}{2\sqrt{\pi}z^\mu\Gamma\Big(\nu-\frac{\mu}{2}+1\Big)} \tag{3.19}$$

and

$$\int_0^\infty u^{\mu-1}I_\nu(zy)K_\nu(zy)dy = \frac{\Gamma\Big(\frac{\mu}{2}\Big)\Gamma\Big(\frac{1}{2}-\frac{\mu}{2}\Big)\Gamma\Big(\nu+\frac{\mu}{2}\Big)}{4\sqrt{\pi}z^\mu\Gamma\Big(\nu-\frac{\mu}{2}+1\Big)} \tag{3.20}$$

(cf. [17, p. 13, Eq. (4.8)]).

Proof. Putting

$$f(x) = \frac{\mathrm{J}_{2\nu}(2zx)}{x} \tag{3.21}$$

into identity (2.27) of Corollary 2.6, we obtain

$$\int_0^\infty y^{-\mu}\mathscr{L}_2\left\{\frac{\mathrm{J}_{2\nu}(2zx)}{x};y\right\}dy = \frac{1}{2}\Gamma\Big(\frac{1}{2}-\frac{\mu}{2}\Big)\int_0^\infty x^{\mu-1}J_{2\nu}(2zx)dx. \tag{3.22}$$

Utilizing the formulas (2.20) and [19, p.326, Entry (1)] together with (3.22) we obtain assertion (3.19).

Similarly, using formula (2.18) and [19, p.326, Entry (1)] together with (2.29) we deduce the second assertion (3.20) of our Example 3.3. ■

We conclude this investigation by remarking that many other infinite integrals can be evaluated in this manner by applying the above Lemma, the above Theorem, and their various corollaries and consequences considered here.

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