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ORIGINAL PAPER

Quadrilateral and Hexagonal Maps Corresponding to the Subgroups $\Gamma_0(N)$ of the Modular Group

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Abstract

Let $N = 2^\alpha 3^\beta$. The normalizer $\Gamma_B(N)$ of $\Gamma_0(N)$ in $PSL(2, \mathbb{R})$ is the triangle group $(2, 4, \infty)$ for $\alpha = 1, 3, 5, 7$; $\beta = 0, 2$ and the triangle group $(2, 6, \infty)$ for $\alpha = 0, 2, 4, 6$; $\beta = 1, 3$. In this paper we examine relationship between the normalizer and the regular maps. We define a family of subgroups of the normalizer and then we study maps with quadrilateral and hexagonal faces using these subgroups and calculating the associated arithmetic structure.

Keywords Regular maps · Modular group · Normalizer

Mathematics Subject Classification 11G32 · 14H57 · 30F35

1 Introduction

The purpose of this paper is to relate the subgroups $\Gamma_0(N)$ of the modular group to regular maps on surfaces. In the theory of maps as developed by Jones and Singerman in [8], maps on surfaces are parametrized by subgroups of (2, m, n) triangle groups and the regular maps correspond precisely to the normal subgroups. In the papers of Akbas and Singerman [1, 2], they studied the signature of the normalizer of $\Gamma_0(N)$ and determined when these are triangle groups. For this reason, the normalizer of $\Gamma_0(N)$ could sometimes be used to study maps. Thus the normalizer $\Gamma_B(N)$ of $\Gamma_0(N)$ in $PSL(2, \mathbb{R})$ acquire significance. Despite the group

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structure of the normalizer $\Gamma_B(N)$ has been studied by many authors, its signature is still an open problem [1–4, 11, 12, 14].

In particular Akbaş and Singerman showed that the normalizer is a triangle group for exactly 26 values of N [2]. Our approach is mainly based on this result. In a previous work of the authors of the present paper, triangular maps corresponding to $\Gamma_0(N)$ is investigated [16]. In this study we will continue to investigate other types of regular maps, quadrilateral or hexagonal, corresponding to $\Gamma_0(N)$. In this manner the present paper could be thought to be the sequel of the previous paper.

We start by recalling some necessary background. A map on an orientable surface is a decomposition of the surface into simply-connected polygonal cells called faces. Thus a map considered to be have vertices and edges formed by the underlying graph and the faces formed by the polygonal cells. Here we will be interested in quadrilateral and hexagonal cells. Also we use the study of Ivrissimtzis and Singerman [6] as a guide. Their work is based on the fact that every triangular map is a quotient of the Farey map, in the sense that given a triangular map T on an orientable surface X, there is a subgroup M of Γ such that $\mathscr{U}$ (upper half plane with extended rationals $\hat{\mathbb{Q}}$) is conformally equivalent to X and $\mathscr{F}/M = T$, where $\mathscr{F}$ is the Farey map [15]. Also, it is known that any subgroup L of the modular group Γ gives rise to a triangular map $\mathscr{U}/L$ on the surface $\mathscr{U}/L$. Here we interested in the case where $L = \Gamma_0(N)$. In order to investigate the regular maps corresponding to $\Gamma_0(N)$, we first construct two universal maps $\mathscr{M}_4$ and $\mathscr{M}_6$ from the normalizer and then we give some results to understand the maps $\mathscr{M}_4(N_1)$ (quadrilateral case) and $\mathscr{M}_6(N_2)$ (hexagonal case) arithmetically.

On the other hand it is well-known that $PSL(2,\mathbb{R})$ acts on $\mathscr{U}$ and that if G is a discrete subgroup of PSL(2,$\mathbb{R}$) the quotient $S = \mathscr{U}/G$ is a Riemann surface [8]. As the points of S correspond to the orbits of G acting on $\mathscr{U}$ we will say that they form the orbit space, and we wish to define an action on S induced from the action of $PSL(2,\mathbb{R})$ on $\mathscr{U}$. However it is not necessary that an element $T \in PSL(2,\mathbb{R})$ induces a transformation of S. There might be two points of $\mathscr{U}$ in the same orbit of $\mathscr{U}/G$ such that T sends them in two different orbits of $\mathscr{U}/G$. So, we have to restrict the action to a subgroup of $PSL(2,\mathbb{R})$ that respects the G-orbits of $\mathscr{U}$. The following lemmas could be given using the facts in [13].

Lemma 1.1 *Let H_1 be a group acting on a set X and let $H_2 \lhd H_1$ a normal subgroup of H_1. The action of H_1 respects the H_2-orbits.*

Lemma 1.2 *Let H_1 be a group acting on a set X and let $H_2 \lhd H_1$. Let also $x \in X$, and let S_x be the stabilizer of x in H_1. Then, $S_{H_2(x)}$ the set-wise stabilizer in H_1 of the H_2-orbit of x, is*

$$S_{H_2(x)} = S_x H_2$$

We now give some information about on relations between subgroups of the normalizer in the next section.

2 The Structure of the Normalizer $\Gamma_B(N)$

As described in [5] , the normalizer of $\Gamma_0(N)$ consists of the transformations corresponding to the matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix} \tag{2.1}$$

where all symbols represent integers, h is the largest divisor of 24 for which $h^2 \mid N$, $e > 0$ is an exact divisor of N/h^2 and the determinant is e. (We say that e is an exact divisor of K if $e \mid K$ and $(e, K/e) = 1$). We now describe some subgroups of $\Gamma_B(N)$. The subgroup $\Gamma_C^0(N)$ consists of the transformations corresponding to the matrices

$$\begin{pmatrix} a & b/h \\ cN & d \end{pmatrix}, \quad ad - bcN/h = 1 \tag{2.2}$$

where h is as above. We can easily verify that

$$\Gamma_0(N/h) = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \Gamma_C^0(N) \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \tag{2.3}$$

that is $\Gamma_C^0(N)$ is a conjugate of $\Gamma_0(N/h)$ by $\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$. The subgroup $\Gamma_B^0(N)$ consists of the transformations in $\Gamma_B(N)$ corresponding to the matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}, \quad c^2N/eh^2 \equiv 0 \bmod h. \tag{2.4}$$

Thus we have the following relation

$$\Gamma_0(N) \le \Gamma_C^0(N) \le \Gamma_B^0(N) \le \Gamma_B(N). \tag{2.5}$$

The following lemma in [2] is a useful one to calculate the indices of subgroups in $\Gamma_B(N)$.

Lemma 2.1 $|\Gamma_B(N) : \Gamma_0(N)| = 2^\rho h^2 \tau$, *where* ρ *is the number of distinct prime factors of* N/h^2 *and*

$$\tau = \prod_{p \mid N}\left(1 + \frac{1}{p}\right) \Big/ \prod_{p \mid N/h^2}\left(1 + \frac{1}{p}\right).$$

Proposition 2.2 $|\Gamma_C^0(N) : \Gamma_0(N)| = h.$

Proof By using (2.3), we get $|\Gamma_C^0(N) : \Gamma_0(N)| = |\Gamma_0(N/h) : \Gamma_0(N)|$. It is known that the index of $\Gamma_0(N)$ in the modular group Γ is $N\prod_{p \mid N}(1 + \frac{1}{p})$. Thus we obtain

$$|\Gamma_C^0(N):\Gamma_0(N)| = |\Gamma_0(N/h):\Gamma_0(N)| = \frac{|\Gamma:\Gamma_0(N)|}{|\Gamma:\Gamma_0(N/h)|}$$
$$= \frac{N\prod_{p|N}(1+\frac{1}{p})}{N/h\prod_{p|N/h}(1+\frac{1}{p})} = h$$

since the prime divisors of N and N/h are the same. □

Corollary 2.3 $|\Gamma_B(N):\Gamma_C^0(N)| = 2^\rho h\tau$, *where* ρ *is the number of distinct prime factors of* N/h^2 *and*

$$\tau = \prod_{p|N}\left(1+\frac{1}{p}\right)/\prod_{p|N/h^2}\left(1+\frac{1}{p}\right).$$

We now give an important theorem of [2] which characterises the structure of $\Gamma_B(N)$.

Theorem 2.4 *Let* $N = 2^\alpha 3^\beta$ *and* $\alpha \le 8, \beta \le 3$. *Then* $\Gamma_B(N)$ *is a triangle group as follows*:

α	β	signature
$0,2,4,6$	$0,2$	$(0;2,3,\infty)$
$1,3,5,7$	$0,2$	$(0;2,4,\infty)$
$0,2,4,6$	$1,3$	$(0;2,6,\infty)$
8		$(0;2,\infty,\infty)$

Let $N = 2^\alpha 3^\beta$, by these two theorems, if $\beta = 0,2$ and $\alpha = 0,2,4,6$, then regular triangular maps correspond to normal subgroups of $\Gamma_B(N)$; if $\beta = 0,2$ and $\alpha = 1,3,5,7$, then regular quadrilateral maps correspond to normal subgroups of $\Gamma_B(N)$; if $\beta = 1,3$ and $\alpha = 0,2,4,6$, then regular hexagonal maps correspond to normal subgroups of $\Gamma_B(N)$.

Recently, Kattan and Singerman published a series of papers [9, 10] in which they relate q-gonal maps with the Hecke group H_q. The basic idea of the works of Kattan and Singerman is that triangular maps are quotients of the Farey map $\mathscr{F}$ which is the universal triangular map whose automorphism group is the modular group (see the Introduction). Triangular maps are parametrised by subgroups of the modular group which is the Hecke group H_3. Moreover, they construct the universal q-gonal maps basically by replacing the modular group with the Hecke group H_q. Hence q-gonal maps are now parametrised by subgroups of the Hecke groups H_q. Only Hecke groups where the elements are clearly known are when $q = 3,4,6$, and the cases correspond the our quadrilateral and hexagonal maps. Thus Kattan-Singerman's and our approaches are essentially the same. The reason behind this phenomenon could be explained by the following observation:

The groups $\Gamma_B(N_1)$ *and* $\Gamma_B(N_2)$ *studied in the present paper are isomorphic to the Hecke groups* H_4 *and* H_6*, respectively.*

One can easily verify the abovementioned statement by observing the following conjugations:

$$H_q = \begin{pmatrix} \sqrt{\frac{q}{2}h} & 0 \\ 0 & 1 \end{pmatrix} \Gamma_B(N_{\frac{q}{2}-1}) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{q}{2}h} \end{pmatrix}, \tag{2.6}$$

for $q = 4$ and 6.

Now we are ready to investigate regular quadrilateral and hexagonal maps corresponding to normal subgroups of $\Gamma_B(N)$.

3 Quadrilateral Maps

3.1 The Normalizer Map

From now on, unless otherwise stated explicitly, N_1 will denote an integer such that $N_1 = 2^{\alpha}3^{\beta}$ for $\beta = 0, 2$ and $\alpha = 1, 3, 5, 7$. For these values of N_1, we have the group $\Gamma_B(N_1)$ as the set of transformations corresponding to the matrices

I.

$$\begin{pmatrix} a & b/h \\ 2ch & d \end{pmatrix}, \quad ad - 2bc = 1,$$

II.

$$\begin{pmatrix} 2a & b/h \\ 2ch & 2d \end{pmatrix}, \quad 4ad - 2bc = 2or2ad - bc = 1,$$

where h is the largest divisor of 24 for which $h^2|N_1$.

Definition 3.1 The elements of $\Gamma_B(N_1)$ of type I will be called even elements and the elements of type II will be called odd elements.

Theorem 3.1 [2, 17] *The action of* $\Gamma_B(N_1)$ *on* $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ *is transitive.*

We now want to construct the universal map (the normalizer map) $\mathscr{M}_4$. Let us firstly construct the vertices. Since $\mathscr{M}_4$ is the image of the imaginary axis under $\Gamma_B(N_1)$, it is clear that 0 and ∞ are vertices of $\mathscr{M}_4$. We can write $0 = \frac{0}{1.h}$ and $\infty = \frac{1}{2.0.h}$. If we use an even element of $\Gamma_B(N_1)$, then the orbit of ∞ consists of the vertices $\dfrac{a}{2ch}$ with a is odd and $(a, c) = 1$; the orbit of 0 consists of the vertices $\dfrac{a}{ch}$ with c is odd and $(a, c) = 1$. In a similar way, one can use an odd element of $\Gamma_B(N_1)$, but the form of the resulting vertices are the same. The vertices in the orbit of ∞ are called even vertices and the other ones are called odd vertices.

We now construct the edges and the quadrilaterals of $\mathscr{M}_4$. The edges of $\mathscr{M}_4$ are the images of the edge connecting 0 and ∞ under $\Gamma_B(N_1)$. On the other hand, by definition, since 0 is an odd vertex and ∞ is an even vertex of $\mathscr{M}_4$, any edges of $\mathscr{M}_4$ simply connect an odd vertex and an even vertex. Thus the vertices $\dfrac{a}{2ch}$ and $\dfrac{b}{dh}$ are connected by an edge if and only if $ad - 2bc = \pm 1$. The quadrilaterals of $\mathscr{M}_4$ are the images of the basic quadrilateral with vertices $\infty, 0, \frac{1}{2h}, \frac{1}{h}$ under $\Gamma_B(N_1)$. Therefore the quadrilaterals of $\mathscr{M}_4$ are characterized by the vertices $\dfrac{a}{2ch}, \dfrac{b}{dh}, \dfrac{a+2b}{2(c+d)h}, \dfrac{a+b}{(2c+d)h}$.

In [15], the authors showed that any triangular map on a surface is the quotient of the universal triangular map by a subgroup of $\Gamma(2,\infty,3)$ and any regular triangular map is the quotient of the universal triangular map by a normal subgroup of $\Gamma(2,\infty,3)$. Here, we use $\Gamma_B(N_1)$ as the triangle group $\Gamma(2,\infty,4)$ and aim to study the maps $\mathscr{M}_4^0(N_1) = \mathscr{M}_4/\Gamma_0(N_1)$.

3.2 The Map $\mathscr{M}_4^0(N_1)$

Definition 3.2 The map $\mathscr{M}_4^0(N_1)$ is defined by the triangular map $\mathscr{M}_4/\Gamma_0(N_1)$.

We first want to determine the vertices of $\mathscr{M}_4^0(N_1)$. By definition, the vertices of the map are $\hat{\mathbb{Q}}/\Gamma_0(N_1)$. Also, since $\Gamma_0(N_1) \lhd \Gamma_B(N_1)$ and $\Gamma_B(N_1)$ acts transitively on $\hat{\mathbb{Q}}$, we have a transitive action of $\Gamma_B(N_1)$ on the set of vertices $\hat{\mathbb{Q}}/\Gamma_0(N_1)$ by $A \in \Gamma_B(N_1)$ acting as $A[x]_{\Gamma_0(N_1)} = [Ax]_{\Gamma_0(N_1)}$, where $[x]_{\Gamma_0(N_1)}$ denotes the $\Gamma_0(N_1)$-orbit of x.

Before giving a characterization, we give some useful lemmas.

Lemma 3.2 *Let G be a group acting transitively on a set X and let $H \lhd G$. Then, there exists a one-to-one correspondence between the cosets of $S_{[x_0]}$ in G and X/H, where $S_{[x_0]}$ is the stabiliser of $[x_0]_H$ in G.*

Proof Proof is straightforward by the orbit-stabiliser theorem. □

Lemma 3.3 *The stabiliser of $\infty = \frac{1}{0}$ in $\Gamma_B(N_1)$ is*

$$S_\infty = \left\{ \begin{pmatrix} 1 & u/h \\ 0 & 1 \end{pmatrix} : u \in \mathbb{Z} \right\}.$$

Proof Since the odd elements of $\Gamma_B(N_1)$ do not preserve the even vertices and also since ∞ is an even vertex [17], the odd elements of $\Gamma_B(N)$ could not stabilise ∞. Thus an even element of $\Gamma_B(N_1)$ stabilising ∞ we have

$$\begin{pmatrix} a & b/h \\ 2ch & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

giving

$$\begin{pmatrix} a \\ 2ch \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

that is, $a = 1$ and $c = 0$. Then the condition of the determinant $ad - 2bc = 1$ gives $d = 1$ and so the stabiliser of ∞ in $\Gamma_B(N_1)$ is contained in S_∞. The converse inclusion is straightforward because

$$\begin{pmatrix} 1 & u/h \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the proof is complete. □

Lemma 3.4 *Let* $S_{[\infty]}$ *be the set-wise stabiliser of* $[\infty]_{\Gamma_0(N_1)}$ *in* $\Gamma_B(N_1)$. *Then*

$$S_{[\infty]} = S_\infty \Gamma_0(N_1)$$

Proof Let A be an element of $S_{[\infty]}$. That means $A[\infty]_{\Gamma_0(N_1)} = [\infty]_{A\Gamma_0(N_1)} = [\infty]_{\Gamma_0(N_1)}$ giving, $[\infty]_{\Gamma_0(N_1)A} = [\infty]_{\Gamma_0(N_1)}$ because $\Gamma_0(N_1) \lhd \Gamma_B(N_1)$. Thus there exist $B, C \in \Gamma_0(N_1)$ such that $BA\infty = C\infty$ giving $C^{-1}BA\infty = \infty$, that is, $A \in S_\infty \Gamma_0(N_1)$.

For the converse inclusion, let $C \in S_\infty \Gamma_0(N_1)$. C is in the form $C = AB$ such that $A \in S_\infty$ and $B \in \Gamma_0(N_1)$. Thus it is obtained

$$C[\infty]_{\Gamma_0(N_1)} = AB[\infty]_{\Gamma_0(N_1)} = A[\infty]_{\Gamma_0(N_1)} = [\infty]_{\Gamma_0(N_1)}.$$

that is, C stabilises $[\infty]_{\Gamma_0(N_1)}$. The proof is completed. □

Lemma 3.5 $S_\infty \Gamma_0(N_1) = \Gamma_C^0(N_1)$.

Proof Since $N_1 = 2^\alpha 3^\beta$ for $\beta = 0, 2$ and $\alpha = 1, 3, 5, 7$, checking the definition of $\Gamma_C^0(N_1)$, we see that

$$\Gamma_C^0(N_1) = \left\{ \begin{pmatrix} a & b/h \\ 2ch^2 & d \end{pmatrix} : ad - 2bch = 1 \right\}.$$

It is clear that $S_\infty \leq \Gamma_C^0(N_1)$, and because $\Gamma_0(N_1) \leq \Gamma_C^0(N_1)$ also holds, we find that $S_\infty \Gamma_0(N_1) \leq \Gamma_C^0(N_1)$. For the converse inclusion we notice that if $A \in \Gamma_C^0(N_1)$ we can write

$$A = \begin{pmatrix} a & b/h \\ 2ch^2 & d \end{pmatrix} \text{with } a, b, c, d \in \mathbb{Z},$$

then we can verify that

$$\begin{pmatrix} 1 & ab/h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - 2abch & -2b^2c \\ 2ch^2 & d \end{pmatrix} = \begin{pmatrix} a & b/h \\ 2ch^2 & d \end{pmatrix} = A$$

where the matrix

$$\begin{pmatrix} a - 2abch & -2b^2c \\ 2ch^2 & d \end{pmatrix}$$

is an element of $\Gamma_0(N_1)$. So, we also have $\Gamma_C^0(N_1) \leq S_\infty \Gamma_0(N_1)$. This completes the proof. □

Theorem 3.6 *There exists a one-to-one correspondence between the left cosets of* $\Gamma_C^0(N_1)$ *in* $\Gamma_B(N_1)$ *and the vertices of* $\mathscr{M}_4^0(N_1)$.

Proof To find a bijection between the set of the vertices of $\mathscr{M}_4^0(N_1)$ and the set of the cosets of $\Gamma_C^0(N_1)$ in $\Gamma_B(N_1)$, we apply Lemma 3.2 with $G = \Gamma_B(N_1)$ acting transitively on $\hat{\mathbb{Q}}$, $H = \Gamma_0(N_1)$, $x_0 = \infty$, and we use Lemmas 3.4 and 3.5 which say that the stabiliser of the $\Gamma_0(N_1)$-orbit of ∞ is $\Gamma_C^0(N_1)$. □

The theorem means that the number of vertices is the index

$$|\Gamma_B(N_1) : \Gamma_C^0(N_1)| = 2^\rho h\tau.$$

Now, to find the vertices we first give the following propositions.

Proposition 3.7 *Let* $A = \begin{pmatrix} a_1 & b_1/h \\ 2c_1h & d_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & b_2/h \\ 2c_2h & d_2 \end{pmatrix}$ *be two even elements of* $\Gamma_B(N_1)$. *Then A and B determine the same left coset of* $\Gamma_C^0(N_1)$ *in* $\Gamma_B(N_1)$ *if and only if* $\begin{pmatrix} a_1 \\ c_1 \end{pmatrix} \equiv \pm \begin{pmatrix} a_2 \\ c_2 \end{pmatrix} \bmod h$.

Proof A straightforward calculation shows that $B^{-1}A \in \Gamma_C^0(N_1)$ if and only if $c_1a_2 - a_1c_2 \equiv 0 \bmod h$ giving $\begin{pmatrix} a_1 \\ c_1 \end{pmatrix} \equiv \pm \begin{pmatrix} a_2 \\ c_2 \end{pmatrix} \bmod h$. □

Proposition 3.8 *Let* $A = \begin{pmatrix} 2a_1 & b_1/h \\ 2c_1h & 2d_1 \end{pmatrix}, B = \begin{pmatrix} 2a_2 & b_2/h \\ 2c_2h & 2d_2 \end{pmatrix}$ *be two odd elements of* $\Gamma_B(N_1)$. *Then A and B determine the same left coset of* $\Gamma_C^0(N_1)$ *in* $\Gamma_B(N_1)$ *if and only if* $\begin{pmatrix} a_1 \\ c_1 \end{pmatrix} \equiv \pm \begin{pmatrix} a_2 \\ c_2 \end{pmatrix} \bmod h$.

Proof First we divide all the terms of the matrix $B^{-1}A$ so that the determinant is 1. Then similarly as the above proposition, we have $B^{-1}A \in \Gamma_C^0(N_1)$ if and only if $c_1a_2 - a_1c_2 \equiv 0 \bmod h$ giving $\begin{pmatrix} a_1 \\ c_1 \end{pmatrix} \equiv \pm \begin{pmatrix} a_2 \\ c_2 \end{pmatrix} \bmod h$. □

Proposition 3.9 *The cosets determined by the even elements and the odd elements of* $\Gamma_B(N_1)$ *are disjoint.*

Proof Let A and B be even and odd elements of $\Gamma_B(N_1)$, respectively. Then the proof follows from the fact that $B^{-1}A \notin \Gamma_C^0(N_1)$. □

Using Proposition 3.9, the vertices corresponding to these disjoint cosets are disjoint. We call cosets determined by the even elements as even vertices and other cosets as odd vertices. By the propositions above, we identify the left coset determined by $\begin{pmatrix} a & b/h \\ 2ch & d \end{pmatrix}$ with the row vector $(a, 2ch)$ and the left coset determined by $\begin{pmatrix} 2a & b/h \\ 2ch & 2d \end{pmatrix}$ with the row vector (a, ch). Hence we identify the vertices with the row vectors $(a, 2ch)$ (even vertex) and (a, ch) (odd vertex) such that $(a, c, h) = 1$. Since $(a, 2ch)$ and $(-a, 2(-c)h)$ determine the same left coset, we obtain the set of even vertices as

$$\{(a, 2ch)\ :\ a, c \in \mathbb{Z}_h,\ (a, c, h) = 1\}/\sim$$

where $(a, 2ch) \sim (h - a, 2(h - c)h)$. Similarly, since (a, ch) and $(-a, (-c)h)$ determine the same left coset, we obtain the set of odd vertices as

$$\{(a, ch)\ :\ a, c \in \mathbb{Z}_h,\ (a, c, h) = 1\}/\sim$$

where $(a, ch) \sim (h - a, (h - c)h)$.

We now investigate the darts, edges and faces of $\mathscr{M}_4^0(N_1)$.

It is known that a dart is a directed edge. Since $\Gamma_B(N_1)$ acts transitively on $\hat{\mathbb{Q}}$, it acts transitively on the darts of $\mathscr{M}_4$. Thus, $\Gamma_B(N_1)/\Gamma_0(N_1)$ acts transitively on the darts of $\mathscr{M}_4^0(N_1)$. By the results of [7], a map is regular if its automorphism group acts transitively on its darts which makes $\mathscr{M}_4^0(N_1)$ is a regular map.

Theorem 3.10 *There exists a one-to-one correspondence between the cosets of* $\Gamma_0(N_1)$ *in* $\Gamma_B(N_1)$ *and the darts of* $\mathscr{M}_4^0(N_1)$.

Proof To prove the theorem, we again apply Lemma 3.2. So, we first choose a dart and then find its stabiliser. Choose $[0]_{\Gamma_0(N_1)} \longrightarrow [\infty]_{\Gamma_0(N_1)}$, if $A \in \Gamma_B(N_1)$ stabilises the dart, firstly it is needed to stabilise $[\infty]_{\Gamma_0(N_1)}$. By Lemmas 3.4 and 3.5, we have $A \in \Gamma_C^0(N_1)$. On the other hand, A is needed to stabilise $[0]_{\Gamma_0(N_1)}$. It follows that $AB \in \Gamma_0(N_1)$ for any $B \in \Gamma_0(N_1)$. Thus, for arbitrary $B = \begin{pmatrix} x & y \\ zN_1 & t \end{pmatrix} \in \Gamma_0(N_1)$, where $N_1 = 2h^2$, we have

$$AB = \begin{pmatrix} a & b/h \\ 2ch^2 & d \end{pmatrix}\begin{pmatrix} x & y \\ 2h^2z & t \end{pmatrix} = \begin{pmatrix} ax + 2bzh & ay + bt/h \\ 2(cx + dz)h^2 & 2cyh^2 + dt \end{pmatrix}.$$

For the matrix in the right hand side of the above equation to be in $\Gamma_0(N_1)$, it is needed to be $h|b$ giving $A \in \Gamma_0(N_1)$. Conversely, for $T = \begin{pmatrix} a & b \\ cN_1 & d \end{pmatrix} \in \Gamma_0(N_1)$, it is clear that $T[\infty]_{\Gamma_0(N_1)}$ and $T[0]_{\Gamma_0(N_1)}$. Thus the stabilizer of the dart $[0]_{\Gamma_0(N_1)} \longrightarrow [\infty]_{\Gamma_0(N_1)}$ is $\Gamma_0(N_1)$. This completes the proof. □

By the theorem, the number of the darts is $|\Gamma_B(N_1) : \Gamma_0(N_1)|$.

Consequently, let (a, $2ch$) and (b, dh) are two vertices of $\mathscr{M}_4^0(N_1)$, then they are joined by an edge if $ad - 2bc \equiv \pm 1 \mod h$, where $a, b, c, d \in \mathbb{Z}_h$. By regularity, the number of the edges is $\frac{|\Gamma_B(N_1):\Gamma_0(N_1)|}{2}$; the number of the faces is $\frac{|\Gamma_B(N_1):\Gamma_0(N_1)|}{4}$ because the map is quadrilateral. The genus of $\mathscr{U}/\Gamma_0(N_1)$ can be found by the euler characteristic $2 - 2g = V - E + F$, where V is the number of vertices; E is the number of edges; F is the number of faces corresponding to $N_1 > 2$.

By the above results we can give the following corollary that formulates the genus of the maps $\mathscr{M}_4^0(N_1)$.

Corollary 3.11 *The genus $g_4(N_1)$ of the maps $\mathscr{M}_4^0(N_1)$ can be computed by the formula*

$$g_4(N_1) = 1 + \frac{1}{6}(h^2 - 4h) \prod_{p|N_1} \left(1 + \frac{1}{p}\right).$$

3.3 Examples

For $N_1 = 8$, we have $h = 2$. In this case, we obtain the map $\mathscr{M}_4^0(8)$ (see Fig. 1) which lies on $\mathscr{U}/\Gamma_0(8)$. This map has 4 vertices, 4 edges, and 2 face and genus 0. The vertices of $\mathscr{M}_4^0(8)$ are

$$(0, 2), (1, 0), (1, 2), (1, 4).$$

For $N_1 = 18$, we have $h = 3$. In this case, we obtain the map $\mathscr{M}_4^0(18)$ (see Fig. 2) which lies on $\mathscr{U}/\Gamma_0(18)$. This map has 8 vertices, 12 edges, and 6 faces and genus 0. The vertices of $\mathscr{M}_4^0(18)$ are

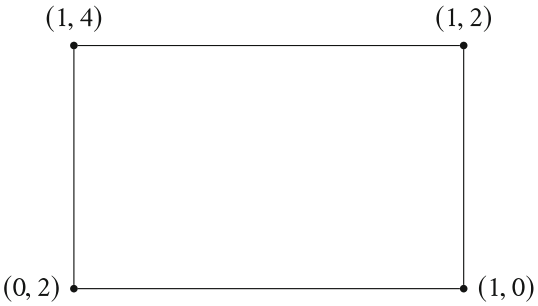

Fig. 1 $\mathscr{M}_4^0(8)$: quadrilateral

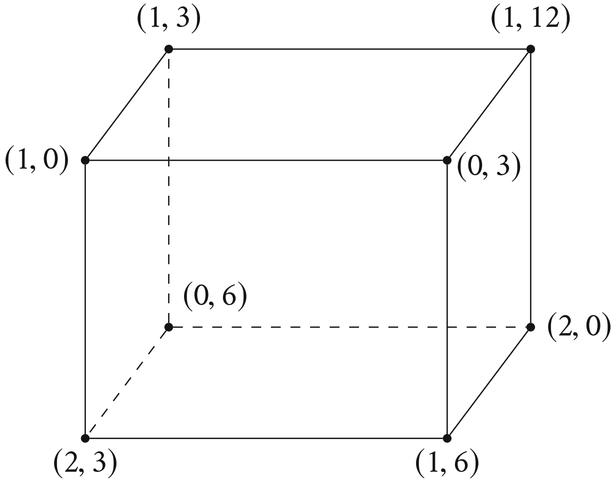

Fig. 2 $\mathscr{M}_4^0(18)$: cube

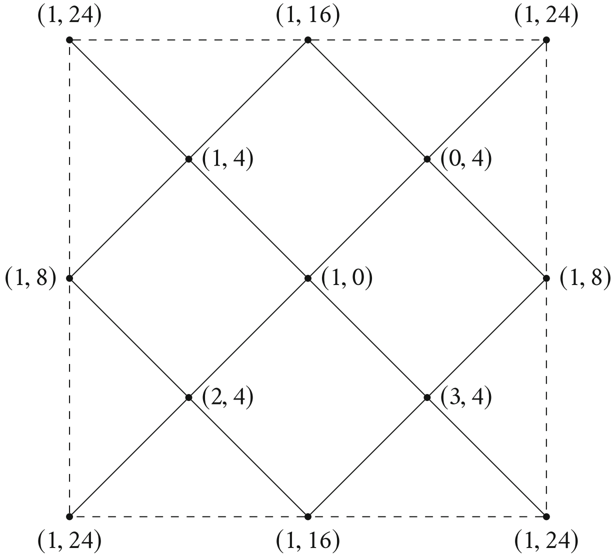

Fig. 3 $\mathscr{M}_4^0(32)$: $\{4,4\}_{2,2}$

$$(0,3),(1,0),(1,3),(0,6),(2,0),(1,6),(1,12),(2,3).$$

For $N_1 = 32$, we have $h = 4$. In this case, we obtain the map $\mathscr{M}_4^0(32)$ (see Fig. 3) which lies on $\mathscr{U}/\Gamma_0(32)$. This map has 8 vertices, 16 edges, and 8 faces and genus 1. The vertices of $\mathscr{M}_4^0(32)$ are

Table 1 The complete table of regular maps for N_1

	D	E	V	F	g	Regular map
$N_1 = 8$	8	4	4	2	0	Quadrilateral
$N_1 = 18$	24	12	8	6	0	Cube
$N_1 = 32$	32	16	8	8	1	$\{4,4\}$
$N_1 = 72$	96	48	16	24	5	$\{4,6\}$
$N_1 = 128$	128	64	16	32	9	$\{4,8\}$
$N_1 = 288$	384	192	32	96	33	$\{4,12\}$
$N_1 = 1152$	1536	768	64	384	161	$\{4,24\}$

$$(0,4),(1,0),(1,4),(1,8),(1,16),(1,24),(3,4),(2,4).$$

See Table 1 for complete table of quadrilateral maps.

4 Hexagonal Maps

4.1 The Normalizer Map

From now on, unless otherwise stated explicitly, N_2 will denote an integer such that $N_2 = 2^\alpha 3^\beta$ for $\beta = 1,3$ and $\alpha = 0,2,4,6$. For these values of N_2, we have the group $\Gamma_B(N_2)$ as the set of transformations corresponding to the matrices

I.

$$\begin{pmatrix} a & b/h \\ 3ch & d \end{pmatrix}, \quad ad - 3bc = 1,$$

II.

$$\begin{pmatrix} 3a & b/h \\ 3ch & 3d \end{pmatrix}, \quad 9ad - 3bc = 3or3ad - bc = 1,$$

where h is the largest divisor of 24 for which $h^2|N_2$.

Definition 4.1 The elements of $\Gamma_B(N_2)$ of type I again will be called even elements and the elements of type II will be called odd elements.

Theorem 4.1 [2, 17] *The action of* $\Gamma_B(N_1)$ *on* $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ *is transitive.*

The universal map $\mathscr{M}_6$ can be easily constructed similar to the that of $\mathscr{M}_4$. The even vertices of $\mathscr{M}_6$ are $(a,\ 3ch)$ with $(a,c) = 1$ and $3 \not| a$, and the odd vertices are $(a,\ ch)$ with $(a,c) = 1$ and $3 \not| c$. The vertices $\dfrac{a}{2ch}$ and $\dfrac{b}{dh}$ are connected by an edge if and only if $ad-3bc=\pm 1$. The hexagons of $\mathscr{M}_6$ are the images of the basic hexagon with vertices $\infty, 0, \frac{1}{3h}, \frac{1}{2h}, \frac{2}{3h}, \frac{1}{h}$ under $\Gamma_B(N_2)$. Therefore the hexagons of $\mathscr{M}_6$ are characterized by the vertices $\frac{a}{3ch}, \frac{b}{dh}, \frac{a+3b}{3(c+d)h}, \frac{a+2b}{(3c+2d)h}, \frac{2a+3b}{3(2c+d)h}, \frac{a+b}{(3c+d)h}$.

4.2 The Map $\mathscr{M}_6^0(N_2)$

Definition 4.2 The map $\mathscr{M}_6^0(N_2)$ is defined by the triangular map $\mathscr{M}_6/\Gamma_0(N_2)$.

Here all the results obtained for $\mathscr{M}_4^0(N_1)$ also hold for the map $\mathscr{M}_6^0(N_2)$. There are slight differences. Therefore we give only the differences in this section. We have the following lemmas.

Lemma 4.2 $S_{[\infty]}=S_\infty\Gamma_0(N_2)$.

Lemma 4.3 $S_\infty\Gamma_0(N_2)=\Gamma_C^0(N_2)$.

Here we give the following theorem without proof for the vertices.

Theorem 4.4 *There exists a one-to-one correspondence between the left cosets of* $\Gamma_C^0(N_2)$ *in* $\Gamma_B(N_2)$ *and the vertices of* $\mathscr{M}_6^0(N_2)$.

The theorem means that the number of vertices is the index

$$|\Gamma_B(N_2):\Gamma_C^0(N_2)|=2^\rho h\tau.$$

Thus we similarly obtain the set of even vertices as

$$\{(a,3ch)\ :\ a,c\in\mathbb{Z}_h,\ (a,c,h)=1\}/\sim$$

where $(a,3ch)\sim(h-a,3(h-c)h)$ and the odd vertices as

$$\{(a,ch)\ :\ a,c\in\mathbb{Z}_h,\ (a,c,h)=1\}/\sim$$

where $(a,ch)\sim(h-a,(h-c)h)$.

Again we give the following theorem without proof for the darts.

Theorem 4.5 *There exists a one-to-one correspondence between the cosets of* $\Gamma_0(N_2)$ *in* $\Gamma_B(N_2)$ *and the darts of* $\mathscr{M}_6^0(N_2)$.

By the theorem, the number of the darts is $|\Gamma_B(N_2):\Gamma_0(N_2)|$.

Now let $(a,\ 3ch)$ and $(b,\ dh)$ are two vertices of $\mathscr{M}_6^0(N_2)$, then they are joined by an edge if $ad-3bc\equiv\pm1 \bmod h$, where $a,b,c,d\in\mathbb{Z}_h$. By regularity, the number of the edges is $\frac{|\Gamma_B(N_2):\Gamma_0(N_2)|}{2}$; the number of the faces is $\frac{|\Gamma_B(N_2):\Gamma_0(N_2)|}{6}$ because the map is hexagonal.

Again, by the above results we can give the following corollary that formulates the genus of the maps $\mathscr{M}_6^0(N_2)$.

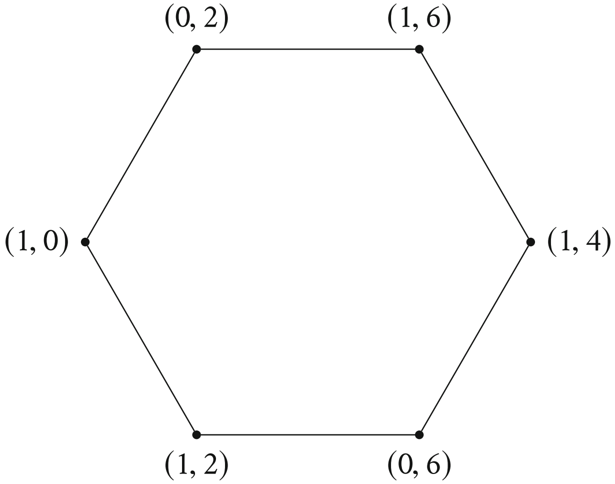

Fig. 4 $\mathscr{M}_6^0(12)$: hexagon

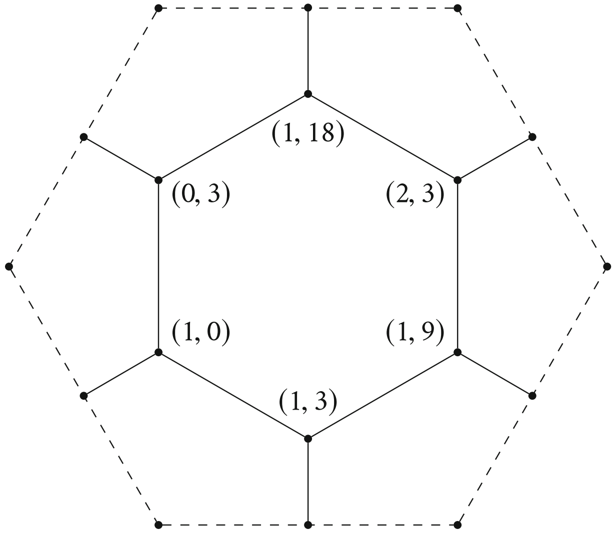

Fig. 5 $\mathscr{M}_6^0(27)$: $\{6, 3\}_6$

Corollary 4.6 *The genus* $g_6(N_2)$ *of the maps* $\mathscr{M}_6^0(N_2)$ *can be computed by the formula*

$$g_6(N_2) = 1 + \frac{1}{4}(h^2 - 3h) \prod_{p|N_2} \left(1 + \frac{1}{p}\right).$$

Table 2 The complete table of regular maps for N_2

	D	E	V	F	g	Regular map
$N_2 = 8$	12	6	6	2	0	Hexagon
$N_2 = 27$	18	9	6	3	1	$\{6, 3\}$
$N_2 = 48$	48	24	12	8	3	$\{6, 4\}$
$N_2 = 108$	108	54	18	18	10	$\{6, 6\}$
$N_2 = 192$	192	96	24	32	21	$\{6, 8\}$
$N_2 = 432$	432	216	36	72	55	$\{6, 12\}$
$N_2 = 1728$	1728	864	72	288	253	$\{6, 24\}$

4.3 Examples

For $N_2 = 12$, we have $h = 2$. In this case, we obtain the map $\mathscr{M}_6^0(12)$ (see Fig. 4) which lies on $\mathscr{U}/\Gamma_0(12)$. This map has 6 vertices, 6 edges, and 2 face and genus 0. The vertices of $\mathscr{M}_6^0(12)$ are

$$(0, 2), (1, 0), (1, 6), (1, 2), (0, 6), (1, 4).$$

For $N_2 = 27$, we have $h = 3$. In this case, we obtain the map $\mathscr{M}_6^0(27)$ (see Fig. 5) which lies on $\mathscr{U}/\Gamma_0(27)$. This map has 6 vertices, 9 edges, and 3 faces and genus 1. The vertices of $\mathscr{M}_6^0(27)$ are

$$(0, 3), (1, 0), (1, 9), (1, 3), (1, 18), (2, 3).$$

See Table 2 for complete table of hexagonal maps.

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Declarations

Conflict of interest The authors have not disclosed any competing interests.

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