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On Modules Satisfying S-Noetherian Spectrum Condition

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Abstract
Let R be a commutative ring having nonzero identity and M be a unital R-module. Assume that $S \subseteq R$ is a multiplicatively closed subset of R. Then, M satisfies S-Noetherian spectrum condition if for each submodule N of M, there exist $s \in S$ and a finitely generated submodule $F \subseteq N$ such that $sN \subseteq \mathrm{rad}_M(F)$, where $\mathrm{rad}_M(F)$ is the prime radical of F in the sense (McCasland and Moore in Commun Algebra 19(5):1327–1341, 1991). Besides giving many properties and characterizations of S-Noetherian spectrum condition, we prove an analogous result to Cohen's theorem for modules satisfying S-Noetherian spectrum condition. Moreover, we characterize modules having Noetherian spectrum in terms of modules satisfying the S-Noetherian spectrum condition.

Keywords Noetherian modules · S-Noetherian modules · Noetherian spectrum · S-Noetherian spectrum condition

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1 Introduction

In this paper, all rings under consideration are commutative with nonzero identity and all modules are nonzero unital. Let R always denote such a ring and M denote such an R-module. In algebra, various types of ascending chain conditions on rings and modules have been widely studied by many authors. See, for example, [4,5,9,14]. Recall that a commutative ring R has Noetherian spectrum if it satisfies the ascending chain condition (ACC) on radical ideals, that is, every increasing sequence $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ of radical ideals of R stops [18]. Note that a ring R has Noetherian spectrum if and only if for each ideal I of R, there exist $a_1, a_2, \ldots, a_n \in I$ such that $\sqrt{I} = \sqrt{Ra_1 + Ra_2 + \cdots + Ra_n}$ [18, Proposition 2.1], where $\sqrt{I}$ denotes the radical of an ideal I of R. Until this time, Noetherian spectrum condition on rings and modules has drawn attention. See, for example, [7,10,12,19,20]. Recently, Ahmed [1] have generalized the concept of Noetherian spectrum over rings to the concept of S-Noetherian spectrum. Let $S \subseteq R$ be a multiplicatively closed subset of R. An ideal I of R is called radically S-finite if there exist $s \in S$ and $a_1, a_2, \ldots, a_n \in I$ such that $sI \subseteq \sqrt{Ra_1 + Ra_2 + \cdots + Ra_n}$. In particular, R is said to satisfy the S-Noetherian spectrum condition if its each ideal is radically S-finite. It is clear that a ring R with Noetherian spectrum satisfies the S-Noetherian spectrum condition and the converse is true if $S \subseteq u(R)$, where $u(R)$ denotes the set of all units in R. The author, in [1], transferred many properties of rings having Noetherian spectrum to rings satisfying S-Noetherian spectrum condition. In particular, he showed that a ring R satisfies the S-Noetherian spectrum condition if and only if its each radical ideal is radically S-finite if and only if its each prime ideal is radically S-finite [1, Theorem 2.2]. Our aim in this paper is to extend the S-Noetherian spectrum condition on rings to modules and examine the properties of modules satisfying the S-Noetherian spectrum condition.

For the sake of completeness, we begin with some definitions and notations which will be followed throughout the paper. Let M be an R-module. For any submodule N of M, $(N : M) = \{r \in R : rM \subseteq N\}$ denotes the residual of N by M. In particular, we use $ann(M)$ to denote the $(0 : M)$. If $ann(M) = 0$, then M is called a faithful module [21]. Recall from [6] that an R-module M is said to be a multiplication module if each submodule K of M has the form $K = IM$ for some ideal I of R. The notion of prime submodule has a significant role in determining the structure of many classes of modules. A proper submodule P of M is called a prime submodule if for $a \in R, m \in M$ and whenever $am \in P$, then either $a \in (P : M)$ or $m \in P$ [13]. Similar to the concept of radical of an ideal in a commutative ring, for any submodule N of M, the prime radical $\mathrm{rad}_M(N)$ of N is defined to be the intersection of all prime submodules P of M containing N [17] . If there is no such a prime submodule, we say $\mathrm{rad}_M(N) = M$. Note that in multiplication (or finitely generated) modules, for each proper submodule N of M, there exists always a prime submodule containing N. D. Rush, in [20], extended the concept of Noetherian spectrum in rings to modules. Let N be a submodule of M. We denote the set of all prime submodules containing N by $V(N)$. N is said to be radically finite if there exists a finitely generated submodule F of N such that $V(N) = V(F)$, or equivalently, $\mathrm{rad}_M(N) = \mathrm{rad}_M(F)$ [20]. In particular, an R-module M has Noetherian spectrum if its all submodules are radically finite. Now, we introduce the concept of S-Noetherian spectrum condition. Let M be an R-

module and $S \subseteq R$ be a multiplicatively closed subset. We define a submodule N of M as a radically S-finite if there exist $s \in S$ and a finitely generated submodule F of N such that $sN \subseteq \text{rad}_M(F)$. Also, an R-module M is said to satisfy the S-Noetherian spectrum condition if each submodule N of M is radically S-finite. Note that an R-module M with Noetherian spectrum satisfies the S-Noetherian spectrum condition and the converse is also true if $S \subseteq u(R)$. In this article, we extend many properties of modules having Noetherian spectrum to modules satisfying the S-Noetherian spectrum condition. Among the other results, we examine the S-Noetherian spectrum condition under epimorphism, in factor modules, in Cartesian product of modules, and in trivial extension (idealization) of modules (see Proposition 2.10, Theorem 2.14 and Theorem 2.20). Also, we prove that if M satisfies the S-Noetherian spectrum condition, then the quotient module $S^{-1}M$ has Noetherian spectrum (see Proposition 2.4). We give a characterization of multiplication modules satisfying the S-Noetherian spectrum condition in terms of rings satisfying the S-Noetherian spectrum condition (see Theorem 2.7). Recall the following celebrated theorem (Cohen's theorem): a ring R is Noetherian if and only if every prime ideal of R is finitely generated [8, Theorem 3.6]. In several papers, this result was extended to finitely generated modules (see [11,13]). We prove an analogous result to Cohen's theorem for modules satisfying the S-Noetherian spectrum condition (see Theorem 2.6). Furthermore, we use the S-Noetherian spectrum condition to characterize modules having Noetherian spectrum (see Theorem 2.9). Recall that a submodule N of M is said to be radical submodule if $\text{rad}_M(N) = N$ [17]. In Theorem 2.17, we prove that a finitely generated multiplication R-module M satisfies the S-Noetherian spectrum condition if and only if each radical submodule is radically S-finite if and only if its each prime submodule is radically S-finite if and only if every nonempty family of radical submodules has an S-maximal element. Finally, we investigate the S-Noetherian spectrum condition and Noetherian spectrum condition in trivial extension $R \propto M$ of an R-module M (see Theorem 2.20 and Corollary 2.21).

2 Modules Satisfying S-Noetherian Spectrum Condition

Definition 2.1 Let M be an R-module and $S \subseteq R$ be a multiplicatively closed subset. A submodule N of M is called radically S-finite if there exist $s \in S$ and a finitely generated submodule F of N such that $sN \subseteq \text{rad}_M(F)$. In particular, an R-module M is said to satisfy the S-Noetherian spectrum condition if each submodule of M is radically S-finite.

Example 2.2 (i) Every radically finite submodule is radically S-finite. The converse also holds if $S \subseteq u(R)$.

(ii) Every module having Noetherian spectrum satisfies the S-Noetherian spectrum condition for each multiplicatively closed subset $S \subseteq R$.

(iii) Assume that $S \subseteq u(R)$. An R-module has Noetherian spectrum if and only if it satisfies the S-Noetherian spectrum condition.

Example 2.3 Suppose that M is an R-module and S is a multiplicatively closed subset of R such that $S \cap \text{ann}(M) \neq \emptyset$. Then, M trivially satisfies the S-Noetherian spectrum condition.

Let S be a multiplicatively closed subset of R and M be an R-module. The quotient module $S^{-1}M$ of M is an $S^{-1}R$ -module. Recall from [8], for any multiplicatively closed subset $S \subseteq R$, $S^{\star} = \left\{a \in R : \frac{a}{1} \text{ is a unit of } S^{-1}R\right\}$ denotes the saturation of S. Also, a multiplicatively closed subset $S \subseteq R$ is saturated if $S = S^{\star}$. Note that $S^{\star}$ is always a saturated set of R containing S. Here, we denote the set $\{r \in R : rM = M\}$ by $U_M(R)$. It is clear that $U_M(R)$ is a saturated multiplicatively closed subset containing $u(R)$.

Proposition 2.4 *Let M be an R-module and $S \subseteq R$ be a multiplicatively closed subset. The following statements are satisfied*:

(i) *If $S_1 \subseteq S_2$ are multiplicatively closed subsets of R and M satisfies the S_1-Noetherian spectrum condition, then M also satisfies the S_2-Noetherian spectrum condition.*

(ii) *Suppose that $S^{\star}$ is the saturation of S. M satisfies the S-Noetherian spectrum condition if and only if M satisfies the $S^{\star}$-Noetherian spectrum condition.*

(iii) *Let M be a multiplication R-module and $S \subseteq U_M(R)$ be a multiplicatively closed subset of R. M has Noetherian spectrum if and only if M satisfies the S-Noetherian spectrum condition.*

(iv) *Let M satisfy the S-Noetherian spectrum condition with $ann(M) \cap S = \emptyset$. Then, $S^{-1}R$-module $S^{-1}M$ has Noetherian spectrum.*

Proof (i): It is clear.

(ii): Suppose that M satisfies the S-Noetherian spectrum condition. Since $S \subseteq S^{\star}$, by (i), M satisfies the $S^{\star}$-Noetherian spectrum condition. For the converse, assume that M satisfies the $S^{\star}$-Noetherian spectrum condition. Let N be a submodule of M, then there exist $s \in S^{\star}$ and a finitely generated submodule F of N such that $sN \subseteq \text{rad}_M(F)$. Since $\frac{s}{1}$ is a unit of $S^{-1}R$, $\frac{s}{1}\frac{a}{t} = 1$ for some $a \in R$, $t \in S$. Then, we have $usa = ut$ for some $u \in S$. Put $s' = ut \in S$. This implies that $s'N = (ua)sN \subseteq sN \subseteq \text{rad}_M(F)$. Hence, M satisfies the S-Noetherian spectrum condition.

(iii): The "only if part" follows from (i) by taking $S_1 = \{1\} \subseteq S$. For the "if part", let N be a submodule of M. As M satisfies the S-Noetherian spectrum condition, there exists an $s \in S$ and a finitely generated submodule F of N such that $sN \subseteq \text{rad}_M(F)$. Since M is a multiplication module and $sM = M$, we have

$$\begin{aligned} N &= (N : M)M = (N : M)sM \\ &= s(N : M)M = sN \subseteq \text{rad}_M(F). \end{aligned}$$

Hence, M has Noetherian spectrum.

(iv): It is sufficient to point out that $S^{-1}\,(\text{rad}_M(F)) \subseteq \text{rad}_{S^{-1}M}(S^{-1}F)$, where F is a submodule of M. Also if F is a finitely generated submodule of M, then $S^{-1}F$ is also a finitely generated submodule of $S^{-1}M$. □

Proposition 2.5 *Let $\mathcal{A}$ be a family of submodules of M which are not radically S-finite. If $\mathcal{A} \neq \emptyset$, then $\mathcal{A}$ contains a maximal element and any such maximal element is a prime submodule.*

Proof Assume that $\mathcal{A} \neq \emptyset$. Then $(\mathcal{A}, \subseteq)$ is a partially ordered set. Let $\{N_i\}_{i\in\Delta}$ be an increasing chain in $\mathcal{A}$ and $N = \bigcup_{i\in\Delta} N_i$. Then N is not radically S-finite. If not, there exist an $s \in S$ and a finitely generated submodule F of N such that $sN \subseteq \mathrm{rad}_M(F)$. Since F is finitely generated, then there exists a $k \in \Delta$ such that $F \subseteq N_k$, then $sN_k \subseteq sN \subseteq \mathrm{rad}_M(F)$. This is a contradiction. Hence, N is an upper bound of $\{N_i\}_{i\in\Delta}$ in $\mathcal{A}$, so by Zorn's lemma, $\mathcal{A}$ has a maximal element P. Now, we will show that P is a prime submodule. Suppose to the contrary that there exist an $x \in M - P$ and an $a \in R$ such that $ax \in P$ with $a \notin (P : M)$. By the maximality of P, $P + aM$ and $(P :_M a)$ are radically S-finite. So there exist $s, t \in S$ and finitely generated submodules H and F_2 of $P + aM$ and $(P :_M a)$, respectively, such that $s(P + aM) \subseteq \mathrm{rad}_M(H)$ and $t(P :_M a) \subseteq \mathrm{rad}_M(F_2)$. By writing the finitely many generators h_i of H as $h_i = x_i + am_i$, where $x_i \in P$, $m_i \in M$ and letting F_1 the submodule generated by x_i, we have $F_1 \subseteq P$,

$$V(F_2) \subseteq V(t(P :_M a)) \text{ and } V(F_1 + aM) \subseteq V(s(P + aM)).$$

Note that $F_1 + aF_2 \subseteq P$ and $F_1 + aF_2$ is finitely generated. Now, we want to show that $stP \subseteq \mathrm{rad}_M(F_1 + aF_2)$ by showing that $V(F_1 + aF_2) \subseteq V(stP)$, so $stP \subseteq \mathrm{rad}_M(stP) \subseteq \mathrm{rad}_M(F_1 + aF_2)$. Let $P^* \in V(F_1 + aF_2)$, then $F_1 + aF_2 \subseteq P^*$. Thus, we have $aF_2 \subseteq P^*$, P^* is prime, this implies that $F_2 \subseteq P^*$ or $aM \subseteq P^*$. If $F_2 \subseteq P^*$, we get $stP \subseteq tP \subseteq t(P :_M a) \subseteq P^*$. If $aM \subseteq P^*$, then $F_1 + aM \subseteq P^*$ and so $stP \subseteq sP \subseteq s(P + aM) \subseteq P^*$. Hence, $P^* \in V(stP)$. Thus, P is radically S-finite which is a contradiction. Hence, P is a prime submodule. □

The following theorem is the counterpart of Cohen's theorem for modules satisfying the S-Noetherian spectrum condition.

Theorem 2.6 *Let M be an R-module and $S \subseteq R$ be a multiplicatively closed subset of R. The following statements are equivalent.*

(i) *M satisfies the S-Noetherian spectrum condition.*
(ii) *Every radical submodule of M is radically S-finite.*
(iii) *Every prime submodule of M is radically S-finite.*

Proof $(i) \Rightarrow (ii)$: Obvious.

$(ii) \Rightarrow (iii)$: Follows from the fact that every prime submodule is radical.

$(iii) \Rightarrow (i)$: Assume that there exists a submodule of M which is not radically S-finite. Let $\mathcal{A}$ be the family of submodules of M which are not radically S-finite. Thus, $\mathcal{A} \neq \emptyset$. Then, by Proposition 2.5, there exists a prime submodule of M which is not radically S-finite, which contradicts the hypothesis. □

Suppose that I is a proper ideal of R. Then, the canonical homomorphism $\pi : R \to R/I$ is defined by $\pi(r) = r + I$ for each $r \in R$. Also it is easy to see that if S is a multiplicatively closed subset of R, then $\pi(S)$ is a multiplicatively closed subset of the factor ring R/I. In the following, we assume that $\pi : R \to R/\mathrm{ann}(M)$.

Theorem 2.7 *Let M be a finitely generated multiplication R-module and $S \subseteq R$ be a multiplicatively closed subset of R. The following statements are equivalent.*

(i) *M satisfies the S-Noetherian spectrum condition.*

(ii) *$R/ann(M)$ satisfies the $\pi(S)$-Noetherian spectrum condition.*

Proof $(i) \Rightarrow (ii)$: Let $\mathfrak{J}$ be an ideal of $R/\text{ann}(M)$. Then, there exists an ideal I of R containing $\text{ann}(M)$ such that $\mathfrak{J} = I/\text{ann}(M)$. Put $N = IM$. Since M satisfies the S-Noetherian spectrum condition, there exist a finitely generated submodule F of N and an $s \in S$ such that $sN \subseteq \text{rad}_M(F)$. Since M is multiplication module and F is finitely generated submodule, $F = KM$ for some finitely generated ideal K of R. As M is finitely generated multiplication, by [15, Theorem 4], we have $sN = sIM \subseteq \text{rad}_M(F) = \text{rad}_M(KM) = \sqrt{(KM : M)}M$. Since M is finitely generated multiplication, $(KM : M) = K + ann(M)$, we get $sIM \subseteq \sqrt{K + ann(M)}M$. Now by [22, Theorem 9 Corollary], we get $sI \subseteq \sqrt{K + \text{ann}(M)}$. Since K is finitely generated, we can write $K = Ra_1 + Ra_2 + \cdots + Ra_n$ for some $a_i \in K$. Also note that $K \subseteq I$ and

$$\sqrt{K + \text{ann}(M)}/\text{ann}(M) = \sqrt{K + \text{ann}(M)/\text{ann}(M)} = \sqrt{(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n})},$$

where $\overline{a_i} = a_i + \text{ann}(M)$. This implies that $(s + \text{ann}(M))\mathfrak{J} \subseteq \sqrt{(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n})}$ for some $a_i + \text{ann}(M) \in \mathfrak{J}$. Thus, $\mathfrak{J}$ is radically $\pi(S)$-finite. Hence, $R/\text{ann}(M)$ satisfies the $\pi(S)$-Noetherian spectrum condition.

$(ii) \Rightarrow (i)$: Let N be a submodule of M. Since $R/\text{ann}(M)$ satisfies the $\pi(S)$-Noetherian spectrum condition, there exist an $s \in S$ and $a_1, a_2, \ldots, a_n \in (N : M)$ such that $(s + \text{ann}(M)) [(N : M)/\text{ann}(M)] \subseteq \sqrt{(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n})}$, where $\overline{a_i} = a_i + \text{ann}(M)$. First note that

$$\sqrt{(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n})} = \sqrt{Ra_1 + Ra_2 + \cdots + Ra_n + \text{ann}(M)}/\text{ann}(M).$$

Then, we conclude that $s(N : M) \subseteq \sqrt{Ra_1 + Ra_2 + \cdots + Ra_n + \text{ann}(M)}$. As M is multiplication, we get $sN \subseteq \sqrt{Ra_1 + Ra_2 + \cdots + Ra_n + \text{ann}(M)}M$. Since M is finitely generated multiplication, by [15, Theorem 4] and [22, Theorem 9 Corollary],

$$\begin{aligned} sN &\subseteq \sqrt{Ra_1 + Ra_2 + \cdots + Ra_n + \text{ann}(M)}M \\ &= \text{rad}\left((Ra_1 + Ra_2 + \cdots + Ra_n + \text{ann}(M))\, M\right) \\ &= \text{rad}\left((Ra_1 + Ra_2 + \cdots + Ra_n)\, M\right). \end{aligned}$$

As M is finitely generated, note that $(Ra_1 + Ra_2 + \cdots + Ra_n)\, M$ is finitely generated and also $(Ra_1 + Ra_2 + \cdots + Ra_n)\, M \subseteq N$. Thus N is radically S-finite. Hence, M satisfies the S-Noetherian spectrum condition. □

Corollary 2.8 *Let M be a finitely generated faithful multiplication R-module and $S \subseteq R$ be a multiplicatively closed subset of R. Then, M satisfies the S-Noetherian spectrum condition if and only if R satisfies the S-Noetherian spectrum condition.*

Proof Directly from Theorem 2.7. □

Let M be an R-module and $\mathfrak{p}$ a prime ideal of R. Then, $S_{\mathfrak{p}} = R - \mathfrak{p}$ is a multiplicatively closed subset of R. If M satisfies the $S_{\mathfrak{p}}$-Noetherian spectrum condition, then we say that M satisfies the $\mathfrak{p}$-Noetherian spectrum condition. Now, we characterize modules having Noetherian spectrum in terms of modules satisfying the S-Noetherian spectrum condition.

Theorem 2.9 *The following statements are equivalent for an R-module M.*
(i) *M has Noetherian spectrum.*
(ii) *M satisfies the $\mathfrak{p}$-Noetherian spectrum condition for each $\mathfrak{p} \in \mathrm{Spec}(R)$.*
(iii) *M satisfies the $\mathfrak{m}$-Noetherian spectrum condition for each $\mathfrak{m} \in \mathrm{Max}(R)$.*

Proof $(i) \Rightarrow (ii)$: Follows from Example 2.2 (ii).

$(ii) \Rightarrow (iii)$: Follows from the fact that every maximal ideal is prime.

$(iii) \Rightarrow (i)$: Assume that M satisfies the $\mathfrak{m}$-Noetherian spectrum condition for each $\mathfrak{m} \in \mathrm{Max}(R)$. Let N be a submodule of M, we want to show that $V(F) = V(N)$ for some finitely generated submodule F of N. Since M satisfies the $\mathfrak{m}$-Noetherian spectrum condition, for each $\mathfrak{m} \in \mathrm{Max}(R)$, there exist $r_{\mathfrak{m}} \notin \mathfrak{m}$ and a finitely generated submodule $F_{\mathfrak{m}}$ of N such that $r_{\mathfrak{m}}N \subseteq \mathrm{rad}_M(F_{\mathfrak{m}})$. Now set $J = \{r_{\mathfrak{m}} : \exists \mathfrak{m} \in \mathrm{Max}(R),$ $r_{\mathfrak{m}} \notin \mathfrak{m}$ and a finitely generated submodule $F_{\mathfrak{m}}$ of N such that $r_{\mathfrak{m}}N \subseteq \mathrm{rad}_M(F_{\mathfrak{m}})\}$. It is easy to see that $(J) = R$ and so there exists $r_{\mathfrak{m}_i} \in J - \mathfrak{m}_i$ such that $(r_{\mathfrak{m}_1}) + \cdots + (r_{\mathfrak{m}_n}) = R$. Then, there exists a finitely generated submodule $F_{\mathfrak{m}_i}$ of N such that $r_{\mathfrak{m}_i}N \subseteq \mathrm{rad}_M(F_{\mathfrak{m}_i})$ for each $i = 1, 2, \ldots, n$. Then, we have

$$N = RN = [(r_{\mathfrak{m}_1}) + \cdots + (r_{\mathfrak{m}_n})]N \subseteq \sum_{i=1}^{n} \mathrm{rad}_M(F_{\mathfrak{m}_i}).$$

Let $F = \sum_{i=1}^{n} F_{\mathfrak{m}_i}$. Then, F is a finitely generated submodule of N and so $V(N) \subseteq V(F)$. Let $P \in V(F)$, then $\sum_{i=1}^{n} F_{\mathfrak{m}_i} \subseteq P$ and so for each i, $F_{\mathfrak{m}_i} \subseteq P$. This implies that $\mathrm{rad}_M(F_{\mathfrak{m}_i}) \subseteq P$ and so $\sum_{i=1}^{n} \mathrm{rad}_M(F_{\mathfrak{m}_i}) \subseteq P$. Hence, $N \subseteq \sum_{i=1}^{n} \mathrm{rad}_M(F_{\mathfrak{m}_i}) \subseteq P$, namely $P \in V(N)$. Thus, $V(F) = V(N)$ and M has Noetherian spectrum. □

Let $f : M \to M'$ be an R-homomorphism. It is clear that $f(\mathrm{rad}_M(N)) \subseteq \mathrm{rad}_{M'}(f(N))$ for any submodule N of M. To see this, take $m' \in f(\mathrm{rad}_M(N))$. Then, there exists $m \in \mathrm{rad}_M(N)$ such that $m' = f(m)$. Now, take a prime submodule P^* of M' containing $f(N)$. If $f^{-1}(P^*) \neq M$ then $f^{-1}(P^*)$ is a prime submodule of M containing N. This implies that $m \in f^{-1}(P^*)$ and so $m' = f(m) \in P^*$. Thus, we have $f(\mathrm{rad}_M(N)) \subseteq \mathrm{rad}_{M'}(f(N))$. Also, in [16, Corollary 1.3], McCasland and Moore proved that for any submodule $Ker(f) \subseteq N \subseteq M$, $N' \subseteq M'$, $f(\mathrm{rad}_M(N)) = \mathrm{rad}_{M'}(f(N))$ and $f^{-1}(\mathrm{rad}_{M'}(N')) = \mathrm{rad}_M(f^{-1}(N'))$ provided that f is surjective.

Proposition 2.10 (i) *Suppose that $f : M \to M'$ is an R-epimorphism and $S \subseteq R$ is a multiplicatively closed subset. If M satisfies the S-Noetherian spectrum condition, so does M'.*

(ii) *Assume that M is an R-module satisfying the S-Noetherian spectrum condition, where $S \subseteq R$ is a multiplicatively closed subset. For any proper submodule N of M, the factor R-module M/N satisfies the S-Noetherian spectrum condition.*

Proof (i): Let N' be a submodule of M'. Then, there exists a submodule N of M such that $f(N) = N'$. As M satisfies the S-Noetherian spectrum condition, $sN \subseteq \text{rad}_M(F)$ for some $s \in S$ and a finitely generated submodule F of N. This implies that $f(sN) = sf(N) = sN' \subseteq f(\text{rad}_M(F)) \subseteq \text{rad}_{M'}(f(F))$. As $f(F)$ is a finitely generated submodule of N', N' is radically S-finite. Thus, M' satisfies the S-Noetherian spectrum condition.

(ii): Follows from (i). □

Let M_i be an R_i-module for each $i = 1, 2, \ldots, n$. Assume that $R = R_1 \times R_2 \times \cdots \times R_n$ and $M = M_1 \times M_2 \times \cdots \times M_n$. Then, it is clear that M is an R-module and all submodules of M have the form $N = N_1 \times N_2 \times \cdots \times N_n$, where N_i is a submodule of M_i for each $i = 1, 2, \ldots, n$. Also, if S_i is a multiplicatively closed subset of R_i, then $S = S_1 \times S_2 \times \cdots \times S_n$ is a multiplicatively closed subset of R.

Proposition 2.11 *Let M_i be an R_i-module for $i = 1, 2$ and $R = R_1 \times R_2$. The following statements are equivalent.*

(i) *N is a prime submodule of the R-module $M_1 \times M_2$.*

(ii) *N has the form $N_1 \times M_2$ or $M_1 \times N_2$, where N_1 and N_2 are prime submodules of M_1 and M_2, respectively.*

Proof $(i) \Rightarrow (ii)$: Let $N = N_1 \times N_2$ be a prime submodule of R-module $M_1 \times M_2$ and $I = 0 \times R_2$. Then, $I(M_1 \times 0) \subseteq N$. Since N is prime, either $M_1 \times 0 \subseteq N$ or $I \subseteq (N : M_1 \times M_2)$. Then, we have $M_1 \times 0 \subseteq N$ or $0 \times M_2 \subseteq N$ and so $N = M_1 \times N_2$ or $N = N_1 \times M_2$. It is easy to check that N_1 and N_2 are prime submodules of M_1 and M_2, respectively.

$(ii) \Rightarrow (i)$: It is clear. □

Theorem 2.12 *Let M_i be an R_i-module for each $i = 1, 2, \ldots, n$. Suppose that $R = R_1 \times R_2 \times \cdots \times R_n$ and $M = M_1 \times M_2 \times \cdots \times M_n$. The following statements are equivalent.*

(i) *N is a prime submodule of R-module M.*

(ii) *N has the form $M_1 \times M_2 \times \cdots \times M_{j-1} \times N_j \times M_{j+1} \times \cdots \times M_n$, where N_j is a prime submodule of M_j.*

Proof It follows from Proposition 2.11, using induction on n. □

Corollary 2.13 *Let M_i be an R_i-module for each $i = 1, 2, \ldots, n$. Suppose that $R = R_1 \times R_2 \times \cdots \times R_n$ and $M = M_1 \times M_2 \times \cdots \times M_n$. Assume that $N = N_1 \times N_2 \times \cdots \times N_n$, where N_i is a submodule of M_i. Then $rad_M(N) = rad_{M_1}(N_1) \times rad_{M_2}(N_2) \times \cdots \times rad_{M_n}(N_n)$.*

Proof Let $(m_1, m_2, \ldots, m_n) \in \mathrm{rad}_M(N)$. Now, we will show that $m_i \in \mathrm{rad}_{M_i}(N_i)$. Take a prime submodule P_i of M_i containing N_i. Put $P = M_1 \times M_2 \times \cdots \times M_{i-1} \times P_i \times M_{i+1} \times \cdots \times M_n$. Then by previous theorem, P is a prime submodule containing N. This implies that $(m_1, m_2, \ldots, m_n) \in P$ and so $m_i \in P_i$ and this yields that $m_i \in \mathrm{rad}_{M_i}(N_i)$. Then, we have $\mathrm{rad}_M(N) \subseteq \mathrm{rad}_{M_1}(N_1) \times \mathrm{rad}_{M_2}(N_2) \times \cdots \times \mathrm{rad}_{M_n}(N_n)$. Now, assume that $(m_1, m_2, \ldots, m_n) \in \mathrm{rad}_{M_1}(N_1) \times \mathrm{rad}_{M_2}(N_2) \times \cdots \times \mathrm{rad}_{M_n}(N_n)$. Then, we have $m_k \in \mathrm{rad}_{M_k}(N_k)$ for each $k = 1, 2, \ldots, n$. Take a prime submodule P of M containing N. Then, there exists a prime submodule P_t of M_t such that $P = M_1 \times M_2 \times \cdots \times M_{t-1} \times P_t \times M_{t+1} \times \cdots \times M_n$. Since $N \subseteq P$, $N_t \subseteq P_t$ and so $m_t \in P_t$. This implies that $(m_1, m_2, \ldots, m_n) \in M_1 \times M_2 \times \cdots \times M_{t-1} \times P_t \times M_{t+1} \times \cdots \times M_n = P$. Hence, $(m_1, m_2, \ldots, m_n) \in \mathrm{rad}_M(N)$ and so we conclude that $\mathrm{rad}_M(N) = \mathrm{rad}_{M_1}(N_1) \times \mathrm{rad}_{M_2}(N_2) \times \cdots \times \mathrm{rad}_{M_n}(N_n)$. □

Theorem 2.14 *Let M_i be an R_i-module and S_i be a multiplicatively closed subset of R_i for each $i = 1, 2, \ldots, n$. Assume that $R = R_1 \times R_2 \times \cdots \times R_n$, $S = S_1 \times S_2 \times \cdots \times S_n$ and $M = M_1 \times M_2 \times \cdots \times M_n$ is an R-module. The following statements are equivalent.*
(i) *M satisfies the S-Noetherian spectrum condition.*
(ii) *M_i satisfies the S_i-Noetherian spectrum condition for each $i = 1, 2, \ldots, n$.*

Proof $(i) \Rightarrow (ii)$: Let M satisfy the S-Noetherian spectrum condition and N_i be a submodule of M_i. Then $N = 0 \times 0 \times \cdots \times 0 \times N_i \times 0 \times \cdots \times 0$ is a submodule of M, so there exist $s = (s_1, \ldots, s_n) \in S$ and a finitely generated submodule $F = F_1 \times F_2 \times \cdots \times F_n$ of N such that $sN \subseteq \mathrm{rad}_M(F) = \mathrm{rad}_{M_1}(F_1) \times \mathrm{rad}_{M_2}(F_2) \times \cdots \times \mathrm{rad}_{M_n}(F_n)$. Hence, $s_i N_i \subseteq \mathrm{rad}_{M_i}(F_i)$ and F_i is a finitely generated submodule of N_i. Thus, M_i satisfies the S_i-Noetherian spectrum condition for each $i = 1, 2, \ldots, n$.
$(ii) \Rightarrow (i)$: Let M_i satisfy the S_i-Noetherian spectrum condition for each $i = 1, 2, \ldots, n$. Suppose that $L = N_1 \times N_2 \times \cdots \times N_n$ is a submodule of M, where N_i is a submodule of M_i. Hence, for each $i = 1, 2, \ldots, n$, there exist $s_i \in S_i$ and a finitely generated submodule F_i of N_i such that $s_i N_i \subseteq \mathrm{rad}_{M_i}(F_i)$. Let $s = (s_1, \ldots, s_n) \in S$, we have

$$\begin{aligned} sL &= s[N_1 \times N_2 \times \cdots \times N_n] = s_1 N_1 \times s_2 N_2 \times \cdots \times s_n N_n \\ &\subseteq \mathrm{rad}_{M_1}(F_1) \times \mathrm{rad}_{M_2}(F_2) \times \cdots \times \mathrm{rad}_{M_n}(F_n) = \mathrm{rad}_M(F), \end{aligned}$$

where $F = F_1 \times F_2 \times \cdots \times F_n$ is a finitely generated submodule of L. Therefore, M satisfies the S-Noetherian spectrum condition. □

Definition 2.15 Let $\mathfrak{F}$ be a nonempty family of submodules of an R-module M. $\mathfrak{F}$ is said to be radically S-saturated if whenever $sN \subseteq K$ for some $s \in S$, $K \in \mathfrak{F}$ and a radical submodule N of M, then $N \in \mathfrak{F}$.

Definition 2.16 Let $\mathfrak{F}$ be a nonempty family of submodules of an R-module M. $K \in \mathfrak{F}$ is said to be an S-maximal element if there exists an $s \in S$ and whenever $K \subseteq N$ for some $N \in \mathfrak{F}$, then $sN \subseteq K$.

Let M be a finitely generated multiplication R-module. Then for any submodule N of M, $(\mathrm{rad}_M(N) : M) = \sqrt{(N : M)}$ by proof of [15, Theorem 4]. Assume that

N is a radical submodule of M, i.e., $\mathrm{rad}_M(N) = N$. Then, $\sqrt{(N : M)} = (\mathrm{rad}_M(N) : M) = (N : M)$ and so $(N : M)$ is a radical ideal of R.

Theorem 2.17 *Let M be a finitely generated multiplication R-module and $S \subseteq R$ be a multiplicatively closed subset. The following statements are equivalent*:

(i) *M satisfies the S-Noetherian spectrum condition.*
(ii) *Every radical submodule is radically S-finite.*
(iii) *Every prime submodule is radically S-finite.*
(iv) *Every nonempty radically S-saturated family $\mathfrak{F}$ of radical submodules has a maximal element.*
(v) *Every nonempty family $\mathfrak{F}$ of radical submodules has an S-maximal element.*

Proof $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$: Follows from Theorem 2.6.

$(ii) \Rightarrow (iv)$: Suppose that $\mathfrak{F}$ is a nonempty radically S-saturated family of radical submodules of M. Then, it is clear that $(\mathfrak{F}, \subseteq)$ is a partially ordered set. We will apply Zorn's lemma. Take a chain $\{N_i\}_{i\in\Delta}$ of $\mathfrak{F}$. Now, we will show that $N = \bigcup\limits_{i\in\Delta} N_i$ is a radical submodule of M. Let $m \in rad_M(N)$. Since M is a finitely generated multiplication module, $Rm = IM$ for some finitely generated ideal I of R. Since M is multiplication, by [2, Theorem 3.1], $I^k M \subseteq N$ for some $k \in \mathbb{N}$. Since I and M are finitely generated, $I^k M$ is a finitely generated submodule. So, we can write $I^k M = Rm_1 + Rm_2 + \cdots + Rm_n$ for some $m_j \in M$. Then, for all $j = 1, 2, \ldots, n$, we have $m_j \in N = \bigcup\limits_{i\in\Delta} N_i$. Then, there exists $t \in \Delta$ such that $m_j \in N_t$ for all $j = 1, 2, \ldots, n$. This implies that $I^k M \subseteq N_t$ and so $I \subseteq \sqrt{(N_t : M)} = (N_t : M)$ since N_t is a radical submodule. Thus, we have $Rm = IM \subseteq N_t \subseteq N$. Hence, N is a radical submodule of M. Then, by (ii), N is radically S-finite. Then, there exist $s \in S$, and $x_1, x_2, \ldots, x_l \in N$ such that $sN \subseteq rad_M(Rx_1 + Rx_2 + \cdots + Rx_l)$. Since $x_1, x_2, \ldots, x_l \in N$, there exists $v \in \Delta$ such that $x_1, x_2, \ldots, x_l \in N_v$ so that $sN \subseteq rad_M(Rx_1 + Rx_2 + \cdots + Rx_l) \subseteq rad_M(N_v) = N_v$. As $N_v \in \mathfrak{F}$ and $\mathfrak{F}$ is radically S-saturated, $N \in \mathfrak{F}$. Then, the supremum of the chain $\bigvee\limits_{i\in\Delta} N_i = N \in \mathfrak{F}$. By Zorn's lemma, $\mathfrak{F}$ has a maximal element.

$(iv) \Rightarrow (v)$: Suppose that $\mathfrak{F}$ is a nonempty family of radical submodules of M. Consider $\mathfrak{F}^\star = \{N : N$ is a radical submodule, $\exists s \in S$ and $N' \in \mathfrak{F}$ such that $sN \subseteq N'\}$. Now, we will show that $\mathfrak{F}^\star$ is a radically S-saturated family of radical submodules of M. Let $N^\star \in \mathfrak{F}^\star$ and $s \in S$ with $sK \subseteq N^\star$ for some radical submodule K of M. As $N^\star \in \mathfrak{F}^\star$, there exist $s_1 \in S$ and $K' \in \mathfrak{F}$ such that $s_1 N^\star \subseteq K'$ and so $ss_1 K \subseteq s_1 N^\star \subseteq K'$. Then, $K \in \mathfrak{F}^\star$ and hence $\mathfrak{F}^\star$ is radically S-saturated. Thus by (iv), $\mathfrak{F}^\star$ has a maximal element $L \in \mathfrak{F}^\star$. Then, there exist $s^\star \in S$ and a submodule $L' \in \mathfrak{F}$ such that $s^\star L \subseteq L'$. Now, we will show that L' is an S-maximal element of $\mathfrak{F}$. Assume that $L' \subseteq K'$ for some $K' \in \mathfrak{F}$. Let $m \in rad_M(L + K')$. Then $I^{n_0} M \subseteq L + K'$ for some ideal I of R and $n_0 \in \mathbb{N}$, where $Rm = IM$. This implies that $s^\star I^{n_0} M \subseteq s^\star L + s^\star K' \subseteq K'$. Then, we get $s^\star I \subseteq \sqrt{(K' : M)} = (K' : M)$, and so $s^\star m \in s^\star Rm = s^\star IM \subseteq K'$. Thus, we have $s^\star \mathrm{rad}_M(L + K') \subseteq K'$. As $K' \in \mathfrak{F}$ and $rad_M(L + K')$ is a radical submodule, we get $rad_M(L + K') \in \mathfrak{F}^\star$. Since $L \subseteq rad_M(L + K')$, by the maximality of L, we have $K' \subseteq rad_M(L + K') = L$ and so $s^\star K' \subseteq s^\star L \subseteq L'$. Hence, L' is an S-maximal element of $\mathfrak{F}$.

$(v) \Rightarrow (i)$: Suppose that N is a submodule of M. Consider the family $\mathfrak{F} = \{rad_M(Rx_1 + Rx_2 + \cdots + Rx_n) : x_1, x_2, \ldots, x_n \in N \text{ and } n \in \mathbb{N}\}$ of radical submodules of M. Then by (v), $\mathfrak{F}$ has an S-maximal element $K \in \mathfrak{F}$. So there exists $s \in S$ such that whenever $K \subseteq L$ for some $L \in \mathfrak{F}$, then $sL \subseteq K$. As $K \in \mathfrak{F}$, we can write $K = rad_M(Rx_1 + Rx_2 + \cdots + Rx_k)$ for some $x_1, x_2, \ldots, x_k \in N$. Now, take $m \in N$. As $K \subseteq rad_M(Rx_1 + Rx_2 + \cdots + Rx_k + Rm)$ and $rad_M(Rx_1 + Rx_2 + \cdots + Rx_k + Rm) \in \mathfrak{F}$, we have $sm \in srad_M(Rx_1 + Rx_2 + \cdots + Rx_k + Rm) \subseteq K$. This implies that $sN \subseteq K = rad_M(Rx_1 + Rx_2 + \cdots + Rx_k)$. Hence, N is radically S-finite and M satisfies S-Noetherian spectrum condition. □

Following [1], an increasing sequence of submodules $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ of M is said to be S-stationary, if there exist $s \in S$ and $k \in \mathbb{N}$ such that $sN_n \subseteq N_k$ for all $n \geq k$.

Corollary 2.18 *Let M be a finitely generated multiplication R-module such that the set of all radical submodules of M is countable and $S \subseteq R$ be a multiplicatively closed subset. The following statements are equivalent*:

(i) *M satisfies the S-Noetherian spectrum condition.*
(ii) *Every radical submodule is radically S-finite.*
(iii) *Every prime submodule is radically S-finite.*
(iv) *Every increasing sequence of radical submodules of M is S-stationary.*
(v) *Every nonempty radically S-saturated family $\mathfrak{F}$ of radical submodules has a maximal element.*
(vi) *Every nonempty family $\mathfrak{F}$ of radical submodules has an S-maximal element.*

Proof $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (v) \Leftrightarrow (vi)$: Follows from Theorem 2.17.

$(vi) \Rightarrow (iv)$: Consider the increasing sequence of radical submodules $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ of M. Put $\mathfrak{F} = \{N_i\}_{i \in \mathbb{N}}$. Then by (vi), $\mathfrak{F}$ has an S-maximal element $N_k \in \mathfrak{F}$. Then, there exists $s \in S$ such that $sN_n \subseteq N_k$ for each $n \geq k$ since $N_k \subseteq N_n$. Hence, $\{N_i\}_{i \in \mathbb{N}}$ is S-stationary.

$(iv) \Rightarrow (v)$: Suppose that $\mathfrak{F}$ is a nonempty radically S-saturated family of radical submodules of M. Then it is clear that $(\mathfrak{F}, \subseteq)$ is a partially ordered set. Take a chain $\mathcal{C}$ in $\mathfrak{F}$. By assumption, $\mathcal{C}$ is countable and so we can write $\mathcal{C} = \{L_i\}_{i \in \mathbb{N}}$. We show that $\mathcal{C}$ has an upper bound. We define an increasing sequence $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots \subseteq N_k \subseteq \cdots$ as follows: Let $N_1 = L_1$. Having defined N_{k-1}, let $N_k = L_k$ if $L_k \supseteq N_{k-1}$ and let $N_k = N_{k-1}$ otherwise. Then, it is easily seen that $\{N_i\}_{i \in \mathbb{N}} \subseteq \mathcal{C}$ and $\bigcup_{i \in \mathbb{N}} L_i = \bigcup_{i \in \mathbb{N}} N_i$. Put $N = \bigcup_{i \in \mathbb{N}} N_i$. By using a similar argument as in the proof of previous theorem, we can show that N is a radical submodule of M. Since $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots \subseteq N_k \subseteq \cdots$ is an increasing sequence of radical submodules, by (iv), there exist $s \in S$ and $m \in \mathbb{N}$ such that $sN_n \subseteq N_m$ for all $n \in \mathbb{N}$. This implies that $sN \subseteq N_m$. As $N_m \in \mathfrak{F}$ and $\mathfrak{F}$ is radically S-saturated, $N \in \mathfrak{F}$. Thus, $\mathcal{C}$ has an upper bound N in $\mathfrak{F}$. By Zorn's lemma, $\mathfrak{F}$ has a maximal element. □

Corollary 2.19 *Let R be a ring and $S \subseteq R$ be a multiplicatively closed subset. The following statements are equivalent*:

(i) *R satisfies the S-Noetherian spectrum condition.*
(ii) *Every radical ideal is radically S-finite.*
(iii) *Every prime ideal is radically S-finite.*
(iv) *Every nonempty family $\mathfrak{F}$of radical ideals has an S-maximal element.*

Proof In Theorem 2.17, take $M = R$. □

Let M be an R-module. The idealization or trivial extension $R \propto M = R \oplus M$ is a commutative ring with componentwise addition and multiplication $(a, m)(b, m') = (ab, am' + bm)$ for each $a, b \in R$ and $m, m' \in M$ [3]. The prime ideals of $R \propto M$ have the form $\mathfrak{p} \propto M$ for some prime ideal $\mathfrak{p}$ of R [3, Theorem 3.2]. Suppose that I is an ideal of R and N is a submodule of M. Then, $I \propto N$ is an ideal of $R \propto M$ if and only if $IM \subseteq N$ [3, Theorem 3.1]. In that case, $I \propto N$ is called a homogeneous ideal of $R \propto M$. Note that by [3, Theorem 3.2], $\sqrt{I \propto N} = \sqrt{I} \propto M$. Assume that $S \subseteq R$ is a multiplicatively closed subset and N is a submodule of M. Then $S \propto N$ is a multiplicatively closed subset of $R \propto M$ [3, Theorem 3.8]. Now, we determine when the trivial extension $R \propto M$ satisfies $S \propto M$-Noetherian spectrum condition.

Theorem 2.20 *Let M be an R-module and $S \subseteq R$ be a multiplicatively closed subset. The following statements are equivalent.*

(i) *R satisfies the S-Noetherian spectrum condition.*
(ii) *$R \propto M$ satisfies the $S \propto 0$-Noetherian spectrum condition.*
(iii) *$R \propto M$ satisfies the $S \propto M$-Noetherian spectrum condition.*

Proof $(i) \Rightarrow (ii)$: Suppose that J is a prime ideal of $R \propto M$ and R satisfies S-Noetherian spectrum condition. Then there exists a prime ideal $\mathfrak{p}$ of R such that $J = \mathfrak{p} \propto M$. As R satisfies the S-Noetherian spectrum condition, $\mathfrak{p}$ is radically S-finite, i.e., there exist $s \in S$ and $x_1, x_2, \ldots, x_n \in \mathfrak{p}$ such that $s\mathfrak{p} \subseteq \sqrt{Rx_1 + Rx_2 + \cdots + Rx_n}$. Also, it is clear that $(s, 0)(\mathfrak{p} \propto M) = s\mathfrak{p} \propto sM$ and $(s, 0) \in S \propto 0$. Moreover, one can easily see that $[R \propto M](x_i, 0) = Rx_i \propto x_i M$ and so $\sqrt{\sum_{i=1}^{n}[R \propto M](x_i, 0)} = \sqrt{\sum_{i=1}^{n} Rx_i \propto \sum_{i=1}^{n} x_i M} = \sqrt{\sum_{i=1}^{n} Rx_i} \propto M$ by [3, Theorem 3.2]. This implies that

$$\begin{aligned}(s, 0)(\mathfrak{p} \propto M) &= s\mathfrak{p} \propto sM \subseteq s\mathfrak{p} \propto M \\ &\subseteq \sqrt{\sum_{i=1}^{n} Rx_i} \propto M \\ &= \sqrt{\sum_{i=1}^{n}[R \propto M](x_i, 0)}.\end{aligned}$$

As $(x_i, 0) \in \mathfrak{p} \propto M$ for all $i = 1, 2, \ldots, n$, it follows that $\mathfrak{p} \propto M$ is radically $S \propto 0$-finite. By Corollary 2.19, $R \propto M$ satisfies the $S \propto 0$-Noetherian spectrum condition.

$(ii) \Rightarrow (iii)$: Since $S \propto 0 \subseteq S \propto M$, the rest follows from Proposition 2.4.

$(iii) \Rightarrow (i)$: Assume that $R \propto M$ satisfies the $S \propto M$-Noetherian spectrum condition. Take an ideal I of R. Then put $J = I \propto M$. Since J is radically $S \propto M$-finite, there exist $(s, m) \in S \propto M$ and $(x_1, m_1), (x_2, m_2), \ldots, (x_n, m_n) \in I \propto M$ such that $(s, m)(I \propto M) \subseteq \sqrt{\sum_{i=1}^{n} [R \propto M](x_i, m_i)}$. As $\sqrt{[R \propto M](x_i, m_i)} = \sqrt{Rx_i} \propto M$, we have

$$\begin{aligned}(s, m)(I \propto M) &\subseteq \sqrt{\sum_{i=1}^{n} [R \propto M](x_i, m_i)} \\ &= \sqrt{\sum_{i=1}^{n} \sqrt{[R \propto M](x_i, m_i)}} \\ &= \sqrt{\sum_{i=1}^{n} \left(\sqrt{Rx_i} \propto M\right)} \\ &= \sqrt{\left(\sum_{i=1}^{n} \sqrt{Rx_i}\right) \propto M} \\ &= \sqrt{\sum_{i=1}^{n} Rx_i} \propto M.\end{aligned}$$

This implies that $sI \subseteq \sqrt{\sum_{i=1}^{n} Rx_i}$ for some $x_1, x_2, \ldots, x_n \in I$. Hence, R satisfies the S-Noetherian spectrum condition. □

Corollary 2.21 *Let M be an R-module. Then, the following statements are equivalent.*

(i) *R has Noetherian spectrum.*
(ii) *$R \propto M$ has Noetherian spectrum.*

Proof Take $S = \{1\}$ and apply Theorem 2.20. □

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