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Inverse scattering on the half-line for generalized ZS-AKNS system with general boundary conditions

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The inverse scattering problem for a first order system of three equations on the half-line with nonsingular diagonal matrix multiplying the derivative and general boundary conditions is considered. It is focused the case of two repeated diagonal elements of diagonal matrix. The scattering matrix on the half line is defined and a unique restoration of the potential from the scattering matrix is proved. The possible application to integration of integro-differential four-wave interaction problem is also focused.

Keywords: Inverse scattering problem; generalized ZS-AKNS systems; scattering matrix on the half line.

2000 Mathematics Subject Classification: Primary 35R30; Secondary 35L50, 35P25, 37K15

1. Introduction and Problem Formulation

Let us consider the system of first-order ordinary differential equations of the form

$$L(\psi) \equiv i\sigma \frac{d}{dx} \psi + Q(x)\psi = \lambda \psi, \quad x \geq 0, \quad (1.1)$$

where

$$\sigma = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad Q(x) = \begin{bmatrix} 0 & Q_1(x) \\ Q_2(x) & 0 \end{bmatrix},$$

I_{m_k} is $m_k \times m_k$ identity matrix, Q_1 is an $m_1 \times m_2$ and Q_2 is an $m_2 \times m_1$ matrix functions. In scattering theory of differential operators, the matrix coefficient Q is always called the potential and it is assumed that it is measurable matrix with measurable complex-valued rapidly decreasing entries.

The present paper deals with the inverse scattering problem (ISP) for the ZS-AKNS system (1.1). There are many publications on the ISP for the system (1.1), where $m_1 = m_2$ [1, 2, 7, 8, 10, 16, 22, 23] and also the Weyl and spectral theory of this system with $m_1 = m_2$ dealt with, for instance, in [4, 9, 24] (see also various references therein). In contrast to these cases, the ISP in half line for the system (1.1) with $m_1 \neq m_2$ is not intensively investigated. The principal difficulty is to determine the sufficiently many scattering problems to ensure the unique solution of the ISP on the half line under consideration. We note that Weyl theory for the system (1.1) with $m_1 \neq m_2$ was also

much less studied. The direct and inverse problems for the system (1.1) with self-adjoint potential on the semi-axis is studied in [5, 6], that is, the Weyl function is constructed and the $m_1 \times m_2$ potential Q_1 is recovered from the Weyl function for the case $Q_2 = Q_1^*$. A uniqueness of solutions of the inverse problem based on the Weyl matrix is shown in [26] for the first order system with the diagonal matrix σ which has distinct complex eigenvalues. The system (1.1) on a finite interval for more general nonsingular diagonal matrix σ which has repeated eigenvalues and for potential matrix Q where the diagonal is zero in its block representation, is considered in [18, 19]. It is proved that the matrix Q is uniquely determined by the monodromy matrix. In the case $\sigma = \sigma^*$, the minimum number of matrix entries of monodromy matrix sufficient to uniquely determine Q is also found.

Namely, we assume in this paper that $m_1 = 2, m_2 = 1$. Two different problems will be considered for the system (1.1) with the boundary condition at $x = 0$. First is the problem with the boundary condition

$$\psi_3(0) = h_{11}\psi_1(0) + h_{12}\psi_2(0) \tag{1.2}$$

and second is the problem with the boundary condition

$$\psi_3(0) = h_{21}\psi_1(0) + h_{22}\psi_2(0) \tag{1.3}$$

where $\det \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \neq 0$ and $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$. For this case, the ISP of recovering non-stationary potential on the semi-plane for the nonstationary analogue of the system (1.1) has been studied in [11]. This work determines the nonstationary approach to the ISP for the system (1.1) on the half line. By applying the nonstationary approach, the ISP for the system (1.1) on the half line is reduced to the ISP for the same system on the whole line [12, 13] with the potential extending to whole line by zero. The some approach is used in [14] where the nonstationary results on the ISP for the first order strictly hyperbolic system on the half-plane [15] which is converted to the analogue stationary system on the half line.

The system (1.1) is called Dirac-type, ZS (Zakharov-Shabat), AKNS (Ablowitz-Kaup-Newell-Segur) or canonical system in the various literature. The name ZS-AKNS is used because of the fact that this system is an auxiliary linear system for many important nonlinear integrable wave equations and as such it was studied. For example, in the case $m_1 = 2, m_2 = 1$, the equation (1.1) with self-adjoint potential arises in [20] as auxiliary linear equation of some nonlinear evolution system of equations with $1 + 1$ dimensions.

In present paper we consider the system (1.1) with the potential $Q(x)$ where the entries q_{ij} which are are complex valued measurable functions satisfying the Naimark-type of condition [21]:

$$\|Q(x)\| \leq c e^{-\varepsilon|x|}, \quad c \text{ is constant, } \varepsilon > 0. \tag{1.4}$$

The suitability of this type condition in theory of ISP on the half-line for the equations with the non-selfadjoint potential is presented in [17, 22]. This condition assumes the analyticity of the scattering matrix in the strip $|\text{Im } \lambda| < \varepsilon_0$ for some ε_0 , and it guarantees also that the point spectrum and spectral singularities remain discrete and do not accumulate on the real axis. The similar results

in the case of polynomial-type decreasing potential

$$\|Q(x)\| \leq c(1 + |x|)^{-1-\varepsilon}, \quad c \text{ is constant, } \varepsilon > 0$$

which is considered for two component Dirac equation in [25], can be also easily obtained for the system (1.1), but the spectral singularities can accumulate on the real axis.

The paper is organized as follows: In Chapter 2, we recall some auxiliary results on the ISP for the system (1.1) on the whole line, since ISP on the half-line is reduced to the ISP on the whole line for the system (1.1) with the coefficients which are zero in $x < 0$. In Chapter 3, the scattering matrix on the half-line is introduced, the factorizational results for the components of the scattering matrix and corresponding Riemann-Hilbert problems to the ISP on the half line are obtained and the uniqueness are proved by the techniques in [11] under additional conditions that the corresponding Riemann-Hilbert problems are regular. This Chapter includes an example that a single problem for the system (1.1) on the half-line is not sufficient for the unique restoration of the potential. In last Chapter, two concluding remarks on the non-regular Riemann-Hilbert case of ISP and the possible application of the system (1.1) to the integration of the nonlinear integro-differential evolution equation are presented, which suggest the lines for further investigation.

Notations: As usual \mathbb{R} stands for the real axis and \mathbb{C} stands for complex plane. Throughout the paper, we shall write $\mathbf{A}_+(\lambda)$ and $\mathbf{A}_-(\lambda)$ for the functions which are analytic in the complex upper-half plane and lower half-plane, respectively.

2. Inverse Scattering Problem on the Whole Line

In this section we recall some auxiliary results on the ISP for the system (1.1) with $m_1 = 2, m_2 = 1$ on the whole line [12, 13].

Consider the system (1.1) on the whole axis $(-\infty, +\infty)$. Introduce the following boundary conditions at infinity:

$$\lim_{x \rightarrow -\infty} \psi_k e^{i\lambda x} = b_k, \quad k = 1, 2, \tag{2.1}$$

$$\lim_{x \rightarrow -\infty} \psi_3 e^{-i\lambda x} = a_3. \tag{2.2}$$

$$\lim_{x \rightarrow +\infty} \psi_k e^{i\lambda x} = a_k, \quad k = 1, 2, \tag{2.3}$$

$$\lim_{x \rightarrow +\infty} \psi_3 e^{-i\lambda x} = b_3, \tag{2.4}$$

Lemma 2.1 (Theorem 1, [12]). *Let λ be real number and the coefficients of system (1.1) satisfy condition (1.2). Then the following statements hold:*

- 1) *There exists a unique solution in the class of bounded function of problems (1.1), (2.1), (2.2) and (1.1), (2.3), (2.4).*
- 2) *For any bounded solution $\psi(x, \lambda)$ of the system (1.1) there exist limits (2.1)–(2.4).*

The following theorem allows us to determine the scattering matrix for the system (1.1) on the whole line.

Lemma 2.2 (Theorem 2, [12]). *Let λ be real number and the coefficients of system (1.1) satisfy condition (1.2). Then the solution of problem (1.1), (2.1), (2.2) can be represented in the form*

$$\begin{aligned} \psi_k(x, \lambda) &= b_k e^{-i\lambda x} + \sum_{j=1}^2 b_j \int_{-\infty}^x B_{kj}(x, t) e^{-i\lambda t} dt + a_3 \int_{-\infty}^x B_{k3}(x, t) e^{i\lambda t} dt, \quad k = 1, 2, \\ \psi_3(x, \lambda) &= a_3 e^{i\lambda x} + \sum_{j=1}^2 b_j \int_{-\infty}^x B_{3j}(x, t) e^{-i\lambda t} dt + a_3 \int_{-\infty}^x B_{33}(x, t) e^{i\lambda t} dt, \end{aligned} \tag{2.5}$$

and solution of problem (1.1), (2.3), (2.4) can be represented in the form

$$\begin{aligned} \psi_k(x, \lambda) &= a_k e^{-i\lambda x} + \sum_{j=1}^2 a_j \int_x^{+\infty} A_{kj}(x, t) e^{-i\lambda t} dt + b_3 \int_x^{+\infty} A_{k3}(x, t) e^{i\lambda t} dt, \quad k = 1, 2, \\ \psi_3(x, \lambda) &= b_3 e^{i\lambda x} + \sum_{j=1}^2 a_j \int_x^{+\infty} A_{3j}(x, t) e^{-i\lambda t} dt + b_3 \int_x^{+\infty} A_{33}(x, t) e^{i\lambda t} dt, \end{aligned} \tag{2.6}$$

where the kernels satisfy the estimation

$$|A_{kj}(x, t)|, |B_{kj}(x, t)| \leq C_1 e^{-\varepsilon \frac{x+t}{2}}, \quad 0 \leq x \leq t, \quad k, j = 1, 2, 3. \tag{2.7}$$

Now from (2.5) and (2.6) we have

$$(I + \mathbf{B}_-(x, \lambda)) e^{i\lambda \sigma x} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = (I + \mathbf{A}_+(x, \lambda)) e^{i\lambda \sigma x} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \tag{2.8}$$

where

$$\begin{aligned} \mathbf{B}_-(x, \lambda) &= \int_{-\infty}^0 \begin{bmatrix} B_{11}(x, x+t) & B_{12}(x, x+t) & A_{13}(x, x-t) \\ B_{21}(x, x+t) & B_{22}(x, x+t) & A_{23}(x, x-t) \\ B_{31}(x, x+t) & B_{32}(x, x+t) & A_{33}(x, x-t) \end{bmatrix} e^{i\lambda t} dt, \\ \mathbf{A}_+(x, \lambda) &= \int_0^{+\infty} \begin{bmatrix} A_{11}(x, x+t) & A_{12}(x, x+t) & B_{13}(x, x-t) \\ A_{21}(x, x+t) & A_{22}(x, x+t) & B_{23}(x, x-t) \\ A_{31}(x, x+t) & A_{32}(x, x+t) & B_{33}(x, x-t) \end{bmatrix} e^{i\lambda t} dt, \\ e^{i\lambda \sigma x} &= \begin{bmatrix} e^{-i\lambda t} & 0 & 0 \\ 0 & e^{-i\lambda t} & 0 \\ 0 & 0 & -e^{i\lambda t} \end{bmatrix}. \end{aligned}$$

It is easy to see that matrix functions $\mathbf{B}_-(x, \lambda)$ and $\mathbf{A}_+(x, \lambda)$ admit analytical extension to lower ($\text{Im } \lambda \leq 0$) and upper ($\text{Im } \lambda \geq 0$) half-plane, respectively. If we suppose that matrix functions $\mathbf{B}_-(x, \lambda)$ and $\mathbf{A}_+(x, \lambda)$ nowhere degenerate in their domains of analyticity, i.e.

$$\begin{aligned} \det(I + \mathbf{B}_-(x, \lambda)) &\neq 0, \quad \text{Im } \lambda \leq 0, \\ \det(I + \mathbf{A}_+(x, \lambda)) &\neq 0, \quad \text{Im } \lambda \geq 0, \end{aligned} \tag{2.9}$$

then from (2.8) we obtain

$$e^{i\lambda \sigma x} \mathbf{S}(\lambda) e^{-i\lambda \sigma x} = (I + \mathbf{A}_+(x, \lambda))^{-1} (I + \mathbf{B}_-(x, \lambda)), \tag{2.10}$$

It is easy to see that

$$\mathbf{S}(\lambda) : \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

The matrix $\mathbf{S}(\lambda)$, $\lambda \in \mathbb{R}$ is called the *scattering matrix on the whole line* for the system (1.1). If the scattering matrix $\mathbf{S}(\lambda)$, $\lambda \in \mathbb{R}$ is known, then we can consider the formula (2.10) and as matrix Riemann-Hilbert problem of finding the matrix functions $\mathbf{B}_-(x, \lambda)$ and $\mathbf{A}_+(x, \lambda)$ for every $x \in (-\infty, +\infty)$. Under the conditions (2.9), we obtain the regular Riemann-Hilbert problem with the normalization to unit matrix at infinity.

Theorem 2.1 (Theorem 3, [12]). *Let $S(\lambda)$ be scattering matrix for the system of equation (1.1) with the coefficients satisfying condition (1.2). Then coefficients of the system (1.1) on the whole axis are uniquely determined from $S(\lambda)$ under the condition (2.9).*

Consider the system (1.1) on the whole line under the self-adjoint potential ($Q_1^* = Q_2$). In this case the operator defined in the space $L_2(\mathbb{R}, \mathbb{C}^3)$ by the differential expression $l(\psi) \equiv i\sigma \frac{d}{dx}\psi + Q(x)\psi$ is self-adjoint. It is shown in [16] that the conditions (2.9), which are sufficient for uniqueness for ISP in general case is satisfied automatically for the self-adjoint case of problem.

Corollary 2.1 (Theorem 2, [13]). *Let $S(\lambda)$ be scattering matrix for the system of equations (1.1) with the self-adjoint potential satisfying condition (1.2). Then the coefficients of the system (1.1) on the whole line are uniquely determined from $S(\lambda)$.*

3. Inverse Scattering Problem on the Half Line

We will study the inverse scattering problem (ISP) for the system (1.1) in the half-axis $x \geq 0$. Consider system (1.1) under boundary condition (1.2) with $|h_{11}| + |h_{12}| \neq 0$.

3.1. Scattering Matrix on the Half-line

It will be used the transformation operator method for the solution of the inverse scattering problem for the system (1.1) in the semi-axis. One of these will be used in the determination of the scattering matrix in the half axis. Other transformation operators will be used for additional properties of the scattering matrix.

We begin the definition of scattering matrix for the system (1.1) in the half-axis. For this reason, we will use the integral representation (2.6) of the bounded solutions of the system (1.1).

Suppose that, for some real λ , $\psi(x, \lambda)$ is a bounded solution of the system (1.1) with the boundary condition (1.2). Then by Lemma 2 the solution $\psi(x, \lambda)$ can be represented in the form (2.6). Hence from (2.6) and (1.2) have

$$(1 + \mathbf{M}_{1-}(\lambda))b = (h_{11} + \mathbf{M}_{1+}(\lambda))a_1 + (h_{12} + \mathbf{M}_{3+}(\lambda))a_2,$$

where

$$\begin{aligned} \mathbf{A}_{33-}(\lambda) - h_{11}\mathbf{A}_{13-}(\lambda) - h_{12}\mathbf{A}_{23-}(\lambda) &= \mathbf{M}_{1-}(\lambda), \\ h_{11}\mathbf{A}_{11+}(\lambda) + h_{12}\mathbf{A}_{21+}(\lambda) - \mathbf{A}_{31+}(\lambda) &= \mathbf{M}_{1+}(\lambda), \\ h_{11}\mathbf{A}_{12+}(\lambda) + h_{12}\mathbf{A}_{22+}(\lambda) - \mathbf{A}_{32+}(\lambda) &= \mathbf{M}_{3+}(\lambda). \end{aligned} \tag{3.1}$$

and

$$\mathbf{A}_+(\lambda) = \int_0^{+\infty} A(0,t)e^{i\lambda t} dt, \quad \mathbf{A}_-(\lambda) = \int_0^{+\infty} A(0,t)e^{-i\lambda t} dt.$$

The simple examples (such an example is given in last subsection of this section) show that one scattering problem on the semi-axis is not enough for unique determination of potential. Let us consider the new scattering problem for the system (1.1) under boundary condition (1.3) with

$$\det \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \neq 0.$$

From (2.6) and (1.3) we have

$$(1 + \mathbf{M}_{2-}(\lambda))b = (h_{21} + \mathbf{M}_{4+}(\lambda))a_1 + (h_{22} + \mathbf{M}_{2+}(\lambda))a_2$$

where

$$\begin{aligned} \mathbf{A}_{33-}(\lambda) - h_{21}\mathbf{A}_{13-}(\lambda) - h_{22}\mathbf{A}_{23-}(\lambda) &= \mathbf{M}_{2-}(\lambda), \\ h_{21}\mathbf{A}_{11+}(\lambda) + h_{22}\mathbf{A}_{21+}(\lambda) - \mathbf{A}_{31+}(\lambda) &= \mathbf{M}_{4+}(\lambda), \\ h_{21}\mathbf{A}_{12+}(\lambda) + h_{22}\mathbf{A}_{22+}(\lambda) - \mathbf{A}_{32+}(\lambda) &= \mathbf{M}_{2+}(\lambda). \end{aligned} \tag{3.2}$$

We introduce the 2×2 matrix function

$$\mathbf{S}_H(\lambda) = \begin{bmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{R}$$

with

$$\begin{aligned} s_{11}(\lambda) &= \frac{h_{11} + \mathbf{M}_{1+}(\lambda)}{1 + \mathbf{M}_{1-}(\lambda)}, & s_{12}(\lambda) &= \frac{h_{12} + \mathbf{M}_{3+}(\lambda)}{1 + \mathbf{M}_{1-}(\lambda)}, \\ s_{21}(\lambda) &= \frac{h_{21} + \mathbf{M}_{4+}(\lambda)}{1 + \mathbf{M}_{2-}(\lambda)}, & s_{22}(\lambda) &= \frac{h_{22} + \mathbf{M}_{2+}(\lambda)}{1 + \mathbf{M}_{2-}(\lambda)}. \end{aligned} \tag{3.3}$$

under the condition that $1 + \mathbf{M}_{1-}(\lambda) \neq 0$ and $1 + \mathbf{M}_{2-}(\lambda) \neq 0$. We call $\mathbf{S}_H(\lambda)$ the *scattering matrix* for the system (1.1) on the half line.

3.2. Riemann-Hilbert Problems for the Inverse Scattering Problem on the Half Line

From the estimation (2.7) of the kernels $A_{kj}(x,t)$ ($k, j = \overline{1, 3}$) follows that $|A_{kj}(0,t)| \leq Ce^{-\varepsilon \frac{t}{2}}$. It means that the functions $\mathbf{A}_{kj+}(\lambda) = \int_0^{+\infty} A_{kj}(0,t)e^{i\lambda t} dt$ and $\mathbf{A}_{kj-}(\lambda) = \int_0^{+\infty} A_{kj}(0,t)e^{-i\lambda t} dt$ are analytic for $\text{Im } \lambda > -\frac{\varepsilon}{2}$ and $\text{Im } \lambda < \frac{\varepsilon}{2}$, respectively, and tend to zero as $|\lambda| \rightarrow \infty$ in their domains of analyticity. It is easy to imply that the functions $\mathbf{M}_{k+}(\lambda)$, $k = \overline{1, 4}$ are analytic for $\text{Im } \lambda > -\frac{\varepsilon}{2}$ and the functions $\mathbf{M}_{k-}(\lambda)$, $k = 1, 2$ are analytic for $\text{Im } \lambda < \frac{\varepsilon}{2}$. The asymptotic relations $\mathbf{M}_{k+}(\lambda) = o(1)$, ($\text{Im } \lambda > -\frac{\varepsilon}{2}$), $\mathbf{M}_{k-}(\lambda) = o(1)$, ($\text{Im } \lambda < \frac{\varepsilon}{2}$) also hold at $|\lambda| \rightarrow \infty$. Then the functions $1 + \mathbf{M}_{k+}(\lambda)$ and $1 + \mathbf{M}_{k-}(\lambda)$ have finite number of zeros.

If the scattering matrix $\mathbf{S}_H(\lambda)$, $\lambda \in \mathbb{R}$ is known. Then we can consider the formulas (3.3) as scalar Riemann problems for half-plane (with respect to real axis) of finding the functions $\mathbf{M}_{k+}(\lambda)$, $k = \overline{1, 4}$ analytical for $\text{Im } \lambda > 0$ and the functions $\mathbf{M}_{k-}(\lambda)$, $k = 1, 2$ analytical for $\text{Im } \lambda < 0$. It means that the matrix elements of scattering matrix $\mathbf{S}_H(\lambda)$ admit factorizations.

Since $\det H \neq 0$ we can suppose $h_{11} \neq 0, h_{22} \neq 0$ or $h_{12} \neq 0, h_{21} \neq 0$. Without loss of generality let $h_{11} \neq 0, h_{22} \neq 0$. Then

$$h_{11}^{-1} s_{11}(\lambda) = \frac{1 + h_{11}^{-1} \mathbf{M}_{1+}(\lambda)}{1 + \mathbf{M}_{1-}(\lambda)}, \quad h_{22}^{-1} s_{22}(\lambda) = \frac{1 + h_{22}^{-1} \mathbf{M}_{2+}(\lambda)}{1 + \mathbf{M}_{2-}(\lambda)} \quad (3.4)$$

are Riemann problems with the canonical normalization. Under the conditions $1 + \mathbf{M}_{k-}(\lambda) \neq 0, k = 1, 2$ (in the case of regular Riemann problems) these Riemann problems are uniquely solvable by the functions $\mathbf{M}_{k-}(\lambda)$ and $\mathbf{M}_{k+}(\lambda), k = 1, 2$. Thus the functions $\mathbf{M}_{k-}(\lambda), k = 1, 2$ and $\mathbf{M}_{k+}(\lambda), k = \overline{1, 4}$ are uniquely determined by scattering matrix $\mathbf{S}_H(\lambda), \lambda \in \mathbb{R}$ in (3.1) and (3.2).

Some of analytical properties of scattering matrix directly follows from the representation (2.7). It needs other transformation operators of the system (1.1) with the complicated boundary conditions in $x = 0$ and in infinity for the other analytical properties of the scattering matrix. Consider system (1.1) with the boundary data $\{a_1, a_2, \psi_3(0, \lambda)\}$ and $\{\psi_1(0, \lambda), \psi_2(0, \lambda), b_3\}$. The equivalent integral equations are

$$\begin{aligned} \psi_k(x, \lambda) &= a_k e^{-i\lambda x} - i \int_x^{+\infty} e^{-i\lambda(x-s)} q_{k3}(s) \psi_3(s, \lambda) ds, \quad k = 1, 2, \\ \psi_3(x, \lambda) &= \psi_3(0, \lambda) e^{i\lambda x} - i \int_0^x e^{i\lambda(x-s)} (q_{31}(s) \psi_1(s) + q_{32}(s) \psi_2(s)) ds. \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \psi_k(x, \lambda) &= \psi_k(0, \lambda) e^{-i\lambda x} + i \int_0^x e^{-i\lambda(x-s)} q_{k3}(s) \psi_3(s, \lambda) ds, \quad k = 1, 2, \\ \psi_3(x, \lambda) &= b_3 e^{i\lambda x} + i \int_x^{+\infty} e^{i\lambda(x-s)} (q_{31}(s) \psi_1(s) + q_{32}(s) \psi_2(s)) ds. \end{aligned} \quad (3.6)$$

Let λ real number and $\psi(x, \lambda)$ be a bounded solution of the system (3.6), then the following representation holds

$$\begin{aligned} \psi_k(x, \lambda) &= a_k e^{i\lambda x} + \sum_{j=1}^2 a_j \int_x^{+\infty} D_{kj}(x, t) e^{i\lambda t} dt + \psi_3(0, \lambda) \int_0^x D_{k3}(x, t) e^{-i\lambda t} dt, \quad k = 1, 2, \\ \psi_3(x, \lambda) &= \psi_3(0, \lambda) e^{-i\lambda x} + \sum_{j=1}^2 a_j \int_x^{+\infty} D_{3j}(x, t) e^{i\lambda t} dt + \psi_3(0, \lambda) \int_0^x D_{33}(x, t) e^{-i\lambda t} dt, \end{aligned} \quad (3.7)$$

where the kernels satisfy the estimation

$$|D_{kj}(x, t)| \leq C_2 e^{-\varepsilon \frac{x+t}{2}}, \quad 0 \leq x \leq t, \quad k, j = 1, \dots, 3.$$

Let λ real number and $\psi(x, \lambda)$ be a bounded solution of the system (3.7), then the following representation holds

$$\begin{aligned} \psi_k(x, \lambda) &= \psi_k(0, \lambda) e^{i\lambda x} + \sum_{j=1}^2 \psi_j(0, \lambda) \int_0^x C_{kj}(x, t) e^{i\lambda t} dt + b \int_x^{+\infty} C_{k3}(x, t) e^{-i\lambda t} dt, \quad k = 1, 2, \\ \psi_3(x, \lambda) &= b e^{-i\lambda x} + \sum_{j=1}^2 \psi_j(0, \lambda) \int_0^x C_{3j}(x, t) e^{i\lambda t} dt + b \int_x^{+\infty} C_{33}(x, t) e^{-i\lambda t} dt, \end{aligned} \quad (3.8)$$

where the kernels satisfy the estimation

$$|C_{kj}(x, t)| \leq C_3 e^{-\varepsilon \frac{x+t}{2}}, \quad 0 \leq x \leq t, \quad k, j = 1, \dots, 3.$$

Following the results in [11], from the representation (3.7) and (3.8) in $x = 0$ we obtain the following factorizations related to scattering matrix $\mathbf{S}_H(\lambda)$, $\lambda \in \mathbb{R}$:

Theorem 3.1. Let $\mathbf{S}_H(\lambda) = \begin{bmatrix} \mathbf{s}_{11}(\lambda) & \mathbf{s}_{12}(\lambda) \\ \mathbf{s}_{21}(\lambda) & \mathbf{s}_{22}(\lambda) \end{bmatrix}$, $\lambda \in \mathbb{R}$ be the scattering matrix for the system (1.1) on the half-axis and the matrix $H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$ satisfies the condition $\det H \neq 0$, with $h_{11} \neq 0$, $h_{22} \neq 0$ and $h_{22} - h_{12} \neq 0$. Then the functions $h_{11}^{-1}\mathbf{s}_{11}(\lambda)$, $h_{22}^{-1}\mathbf{s}_{22}(\lambda)$, $h_2 h_{11}^{-1} h_{22} \hat{\gamma}_{22}(\lambda)$ and $-h_{22} \hat{\gamma}_{21}(\lambda)$ are factorizable, where $\hat{\gamma}_{22}(\lambda)$ and $\hat{\gamma}_{21}(\lambda)$ are defined as in formulas $\begin{bmatrix} \hat{\gamma}_{11}(\lambda) & \hat{\gamma}_{12}(\lambda) \\ \hat{\gamma}_{21}(\lambda) & \hat{\gamma}_{22}(\lambda) \end{bmatrix} = \begin{bmatrix} \mathbf{s}_{11}(\lambda) - \mathbf{s}_{21}(\lambda) & \mathbf{s}_{12}(\lambda) - \mathbf{s}_{22}(\lambda) \\ \mathbf{s}_{11}(\lambda) & \mathbf{s}_{12}(\lambda) \end{bmatrix}^{-1}$ and $h_2 = (h_{22} - h_{12})^{-1} \det H$.

The functions $\gamma_{22}(\lambda)$ and $\gamma_{21}(\lambda)$ admit factorizations in the form of

$$\gamma_{22}(\lambda) = \frac{1 + \mathbf{B}_+(\lambda)}{1 + \mathbf{C}_-(\lambda)}, \quad \gamma_{21}(\lambda) = \frac{1 + \mathbf{B}_-(\lambda)}{1 + \mathbf{C}_+(\lambda)}, \quad (3.9)$$

where the functions $\mathbf{B}_\pm(\lambda)$ and $\mathbf{C}_\pm(\lambda)$ are the functions which are organized by the kernels of the representations (3.7) and (3.8). There exists a number $\varepsilon_0 > 0$ such that the functions $\mathbf{B}_+(\lambda)$ and $\mathbf{C}_+(\lambda)$ are analytic for $\text{Im } \lambda > -\varepsilon_0$ and the functions $\mathbf{C}_-(\lambda)$ and $\mathbf{B}_-(\lambda)$ are analytic for $\text{Im } \lambda < \varepsilon_0$. The following asymptotic relations also hold at $|\lambda| \rightarrow \infty$:

$$\begin{aligned} \mathbf{B}_+(\lambda), \mathbf{C}_+(\lambda) &= o(1), \quad (\text{Im } \lambda > -\varepsilon_0), \\ \mathbf{B}_-(\lambda), \mathbf{C}_-(\lambda) &= o(1), \quad (\text{Im } \lambda < \varepsilon_0). \end{aligned}$$

If the scattering matrix $\mathbf{S}_H(\lambda)$, $\lambda \in \mathbb{R}$ is known, then we can consider the formulas (2.6) and (3.9) as Riemann problems for half-plane (with respect to real axis) with the canonical normalizations of finding the functions $\mathbf{B}_+(\lambda)$ and $\mathbf{C}_+(\lambda)$ analytical for $\text{Im } \lambda > 0$ and the functions $\mathbf{C}_-(\lambda)$ and $\mathbf{B}_-(\lambda)$ analytical for $\text{Im } \lambda < 0$.

3.3. Inverse Scattering Problem on the Half Line

The inverse scattering problem for the system (1.1) on the half-axis is the problem of finding the potential from the known scattering matrix $\mathbf{S}_H(\lambda)$.

The ISP for the system (1.1) in the half-axis is reduced to the ISP for the system on the whole axis obtained by the system (1.1) in the half-axis by the extension of the coefficients are equal to zero for $x < 0$. Let us introduce the transmission matrix $\mathbf{T}(\lambda)$, $\lambda \in \mathbb{R}$ for the bounded solutions

$\psi(x, \lambda)$ of the system (1.1) as follows $\mathbf{T}(\lambda) \begin{bmatrix} a_1 \\ a_2 \\ \psi_3(0, \lambda) \end{bmatrix} = \begin{bmatrix} \psi_1(0, \lambda) \\ \psi_2(0, \lambda) \\ b_3 \end{bmatrix}$. Thus, transforming the

final results in [11] we obtain the result about the ISP for the system (1.1) on the half-line.

Theorem 3.2. Let $S_H(\lambda)$ be scattering matrix for the system (1.1) on the half-line with the coefficients satisfying condition (1.4) and the matrix $H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$ satisfies the condition $\det H \neq 0$, with $h_{11} \neq 0$, $h_{22} \neq 0$ and $h_{22} - h_{12} \neq 0$. Then the transmission matrix $\mathbf{T}(\lambda)$ is uniquely determined

by scattering matrix $S_H(\lambda)$ and the inverse scattering problem on the semi-axis is reduced to the inverse problem on the whole axis, when the Riemann problems for the system (1.1) on the semi-axis are uniquely solvable.

Proof. From the representation (2.7) we determine that

$$\mathbf{T}(\lambda) = \begin{bmatrix} 1 + \mathbf{A}_{11+}(\lambda) & \mathbf{A}_{12+}(\lambda) & \mathbf{A}_{13-}(\lambda) \\ \mathbf{A}_{21+}(\lambda) & 1 + \mathbf{A}_{22+}(\lambda) & \mathbf{A}_{23-}(\lambda) \\ -[1 + \mathbf{A}_{33-}(\lambda)]^{-1} \mathbf{A}_{31+}(\lambda) & -[1 + \mathbf{A}_{33-}(\lambda)]^{-1} \mathbf{A}_{32+}(\lambda) & [1 + \mathbf{A}_{33-}(\lambda)]^{-1} \end{bmatrix}.$$

From (3.2) and (3.4) we find following important formulas for the ISP on the half-line:

$$\begin{aligned} \mathbf{A}_{13+}(\lambda) + \frac{h_{12} - h_{22}}{\det H} \mathbf{A}_{33+}(\lambda) &= h_{12} \mathbf{M}_{2+}(\lambda) - h_{22} \mathbf{M}_{1+}(\lambda), \\ \mathbf{A}_{23+}(\lambda) + \frac{h_{21} - h_{11}}{\det H} \mathbf{A}_{33+}(\lambda) &= h_{21} \mathbf{M}_{1+}(\lambda) - h_{11} \mathbf{M}_{2+}(\lambda), \\ \mathbf{A}_{11-}(\lambda) + \frac{h_{22} - h_{12}}{\det H} \mathbf{A}_{31-}(\lambda) &= h_{22} \mathbf{M}_{1-}(\lambda) - h_{12} \mathbf{M}_{2-}(\lambda), \\ \mathbf{A}_{21-}(\lambda) + \frac{h_{21} - h_{11}}{\det H} \mathbf{A}_{31-}(\lambda) &= h_{11} \mathbf{M}_{2-}(\lambda) - h_{21} \mathbf{M}_{1-}(\lambda), \\ \mathbf{A}_{12-}(\lambda) + \frac{h_{22} - h_{12}}{\det H} \mathbf{A}_{32-}(\lambda) &= h_{22} \mathbf{M}_{3-}(\lambda) - h_{12} \mathbf{M}_{4-}(\lambda), \\ \mathbf{A}_{22-}(\lambda) + \frac{h_{21} - h_{11}}{\det H} \mathbf{A}_{32-}(\lambda) &= h_{11} \mathbf{M}_{3-}(\lambda) - h_{21} \mathbf{M}_{4-}(\lambda). \end{aligned} \tag{3.10}$$

If the Riemann problems (3.4) are uniquely solvable, then the functions $\mathbf{M}_{k-}(\lambda)$, $k = 1, 2$ and $\mathbf{M}_{k+}(\lambda)$, $k = \overline{1, 4}$ are uniquely determined by scattering matrix $\mathbf{S}_H(\lambda)$, $\lambda \in \mathbb{R}$ in (3.5). It means that, the linear combinations of the functions $\mathbf{A}_{ij+}(\lambda)$, $i = \overline{1, 3}$, $j = 1, 2$ and $\mathbf{A}_{i3-}(\lambda)$, $i = \overline{1, 3}$ such as in (3.1) and (3.2) are uniquely determined by $\mathbf{S}_H(\lambda)$. By using the formulas (3.10) it can be clear that those function also are uniquely determined by $\mathbf{S}_H(\lambda)$ if the Riemann problems (3.9) are uniquely solvable. Immediately, the matrix $\mathbf{T}(\lambda)$ is uniquely determined by the scattering matrix $\mathbf{S}_H(\lambda)$. \square

Because the transmission matrix $\mathbf{T}(\lambda)$ is a scattering matrix for the system (1.1) on the whole line with the coefficients which are extended by zero in $x < 0$ the following corollary of the theorem about the ISP on the half-line is true.

Corollary 3.1. *Let $\mathbf{S}_H(\lambda)$ be scattering matrix for the system (1.1) on the half-line with the potential (1.4) and the boundary conditions (1.2), (1.3) in the case $\det H \neq 0$. Then the coefficients of the system (1.1) are uniquely determined by scattering matrix $\mathbf{S}_H(\lambda)$, when the Riemann-Hilbert problems (3.4) and (3.9) for the system (1.1) on the half-line and on the whole line, respectively, are uniquely solvable.*

Because the ISP on the whole line for the system (1.1) with self-adjoint potential is solvable without any additional condition, the following corollary of the theorem about the ISP on the semi-axis is true.

Corollary 3.2. *Let $\mathbf{S}_H(\lambda)$ be scattering matrix for the system (1.1) with self-adjoint potential on the half-axis with the boundary conditions (1.2), (1.3) in the case $\det H \neq 0$. Then the coefficients of the system (1.1) are uniquely determined by scattering matrix $\mathbf{S}_H(\lambda)$, when the Riemann problems for the system (1.1) on the half-axis are uniquely solvable.*

3.4. Example

The following example shows that the single problem for the system (1.1) on the half-line is not sufficient for the unique restoration of the potential.

Let us consider the following equation in $x \geq 0$:

$$\begin{aligned} -i \frac{d\psi_k}{dx} + q_{k3}(x)\psi_3 &= \lambda \psi_k, \quad k = 1, 2, \\ i \frac{d\psi_3}{dx} &= \lambda \psi_3, \end{aligned}$$

with the boundary condition

$$\psi_3(0, \lambda) = \psi_1(0, \lambda).$$

This problem has following explicit solution:

$$\begin{aligned} \psi_k(x, \lambda) &= a_k e^{i\lambda x} - ib \int_x^{+\infty} q_{k3}(s) e^{i\lambda(x-2s)} ds, \quad k = 1, 2, \\ \psi_3(x, \lambda) &= b e^{-i\lambda x}, \end{aligned}$$

where

$$b = \frac{1}{1 + R_-(\lambda)} a_1$$

with

$$R_-(\lambda) = \int_0^{+\infty} \frac{1}{2} q_{13}\left(\frac{\tau}{2}\right) e^{-i\lambda\tau} d\tau.$$

By comparing with the definition of the scattering matrix we take

$$s_{11}(\lambda) = \frac{1}{1 + R_-(\lambda)}, \quad s_{12}(\lambda) = 0.$$

Let us consider the inverse scattering problem for the system which we consider in this example. Let $s_{11}(\lambda)$ are known integral operators. Then we take

$$R_-(\lambda) = \frac{1 - s_{11}(\lambda)}{s_{11}(\lambda)}.$$

It means that the function $\frac{1 - s_{11}(\lambda)}{s_{11}(\lambda)}$ is Laplace transform of the function $\frac{1}{2} q_{13}\left(\frac{\tau}{2}\right)$, $\tau \geq 0$. Then

$$q_{13}\left(\frac{\tau}{2}\right) = \frac{1}{\pi i} \int_{-\infty + ix}^{+\infty + ix} \frac{1 - s_{11}(\lambda)}{s_{11}(\lambda)} e^{i\lambda\tau} d\lambda, \quad x < \frac{\varepsilon}{2}$$

holds by Mellin's formula. It means that the same scattering matrix corresponds to the potential with arbitrary entry q_{23} .

4. Concluding Remarks

1. Some features of the theory for real potential with smoothness and decay are the standard scattering matrix contains all the necessary information (apart from finite discrete data) and the inverse problem can be solved via an integral equation of standard type, known as the Gelfand-Levitan-Marchenko (GLM) equation. A scattering theory for non-selfadjoint Sturm–Liouville equation and for the first order 2×2 system (ZS-AKNS system) on the half-line was developed by Lyance [17] and Nizhnik [22]. The key to the general problem is the observation that the inverse problem is a matrix Riemann-Hilbert problem. For full development of this idea one must identify appropriate normalized eigenfunctions for the spectral problem and determine their existence and properties; determine the form and properties of the associated scattering data; recover the potential Q from its scattering data (the inverse problem). The inverse problem can be solved by solving a matrix Riemann-Hilbert factorization problem with data on \mathbb{R} . The operators in Hilbert space $L_2(0, +\infty)$ associated by $L(y)$ and one of the boundary conditions (1.2) and (1.3) we mean L_1 and L_2 , respectively. Since the potential Q is not generally symmetric, the operators $L_k, k = 1, 2$ can not be self-adjoint and it makes difficult the spectral analysis of these operators. As noted in Sect. 3, for a Newton type potential Q , the operators $L_k, k = 1, 2$ have finitely many singularities. If Z is the set of singularities of L_1 and L_2 , there is a corresponding matrix functions $\mathbf{S}_H(\lambda) : \mathbb{R} \cup Z \rightarrow M_3(\mathbb{C})$ which plays the role of scattering data and which necessarily satisfies certain algebraic and analytic constraints. The transformation to data on the line is made by solving the matrix Riemann-Hilbert problems on the half-line. The obstruction to solving these problems is skipped by the assumption that they are regular (the singular numbers is absent). The problem on the line consists of winding-number relationships between principal minors of scattering matrix on \mathbb{R} and the singularities. For details in the case of singularities of special form, see [3, 28]. It is shown in these literature, for example, that in the non-regular case the Riemann–Hilbert problem can be augmented by certain discrete data so as to characterize the potential uniquely on the whole line. The same sort of results should be true in this case, which suggests a line for further investigation.
2. In the case $m_1 = 2, m_2 = 1$, rewrite the the problem (1.1) in the form

$$-i \frac{d}{dx} \psi - \Lambda \psi = \lambda \sigma^{-1} \psi, \tag{4.1}$$

where $\sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $\Lambda = \sigma^{-1} Q = \begin{bmatrix} 0 & 0 & q_{13} \\ 0 & 0 & q_{23} \\ -q_{31} & -q_{32} & 0 \end{bmatrix}$. This system is used in solving an important system of differential equations. In addition to (1.1), consider the equation

$$-i \frac{d}{dt} \psi - \Omega \psi = \lambda \tau \psi, \tag{4.2}$$

where

$$\tau = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0 & v_1 q_{12} & v_2 q_{13} \\ -v_1 q_{21} & 0 & v_3 q_{23} \\ -v_2 q_{31} & -v_3 q_{32} & 0 \end{bmatrix}$$

with distinct and nonzero constants $a_k, k = 1, 2, 3, v_1 = \frac{a_1 - a_2}{2}, v_2 = \frac{a_1 - a_3}{2}, v_3 = \frac{a_2 - a_3}{2}$ and

$$q_{12} = -\iota \int_x^{+\infty} q_{13} q_{32} ds, \quad q_{21} = -\iota \int_x^{+\infty} q_{23} q_{31} ds.$$

Differentiating (4.1) with respect to t , and (4.2) with respect to x , and then equating the second derivatives with respect to λ , we obtain a system of equations

$$\Lambda_t - \Omega_x + i[\Lambda, \Omega] = 0 \tag{4.3}$$

or

$$\begin{cases} (q_{13})_t - v_2(q_{13})_x = i v_1 q_{12} q_{23}, \\ (q_{23})_t - v_3(q_{23})_x = -i v_1 q_{21} q_{13}, \\ (q_{31})_t - v_2(q_{31})_x = i v_1 q_{32} q_{21}, \\ (q_{32})_t - v_3(q_{32})_x = -i v_1 q_{31} q_{12}. \end{cases} \tag{4.4}$$

This system admits, as it can be easily verified, Lax representation

$$\frac{d}{dt} L = [L, A],$$

where the corresponding Lax pair consists of the operators $L = i\sigma \frac{d}{dx} + Q, A = \tau^{-1}(i \frac{d}{dt} + \Omega)$.

As is mentioned in [27] that the system (4.3) admits physical interpretation, if the potential Q in (1.1) is symmetric. In this case the the number of independent functions in (4.4) reduces to half:

$$\begin{cases} (q_{13})_t - v_2 (q_{13})_x = v_1 q_{23} \int_x^{+\infty} q_{13} q_{23} ds, \\ (q_{23})_t - v_3 (q_{23})_x = v_1 q_{13} \int_x^{+\infty} q_{23} q_{13} ds. \end{cases} \tag{4.5}$$

Thus, under an assumption that the potential Q depends on an additional evolution parameter t , the ISP for the generalized ZS-AKNS system (1.1) and its self-adjoint case can be applied to the integration of the the nonlinear integro-differential evolution equation in the form (4.4) and (4.5). The soliton sort of solutions should be exist in this case, which also suggests a line for further investigation.

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