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On normal modules

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ABSTRACT

Recall that a commutative ring R is said to be a *normal ring* if it is reduced and every two distinct minimal prime ideals are comaximal. A finitely generated reduced R -module M is said to be a *normal module* if every two distinct minimal prime submodules are comaximal. The concepts of normal modules and locally torsion free modules are different, whereas they are equal in theory of commutative rings. We give many properties and examples of normal modules, we use them to characterize locally torsion free modules and Baer modules. Also, we give the topological characterizations of normal modules.

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1. Introduction

Throughout this article, we focus only on commutative rings with a nonzero identity and nonzero unital modules. R will always represent such a ring and M will represent such an R -module. We denote the set of all maximal ideals and the set of all prime ideals of R by $Max(R)$ and $Spec(R)$, respectively. Recall that a commutative ring R is said to be a *locally integral domain* if R_P is an integral domain for each $P \in Spec(R)$. It is easy to see that every integral domain is also a locally integral domain but the converse is not true in general. For instance, take a von Neumann regular ring R that is not a field. Then, R is clearly a locally integral domain but not an integral domain. Let M be an R -module. The set of torsion elements of M is denoted by $T(M) = \{m \in M : rm = 0 \text{ for some nonzero } r \in R\}$ [3]. An R -module M is said to be a *torsion free module* if $T(M) = 0$. M is said to be a *torsion module* if $T(M) = M$. Otherwise, we say that M is a *non-torsion module*, that is, M is a module with a non-torsion element $m \in M$. In a recent paper, Jayaram et al. studied the concept of locally torsion free modules as a generalization of torsion free modules, and used them to characterize Baer modules and von Neumann regular rings. Recall from [16] that an R -module M is said to be a *locally torsion free module* if R_P -module M_P is a torsion free module for every $P \in Spec(R)$. Also, recall that a reduced ring (without nilpotent elements) R is said to be a *normal ring* if every two distinct minimal prime ideals P_1, P_2 of R are comaximal, that is, $P_1 + P_2 = R$ [11]. Our aim in this paper is to extend the concept of normal rings to modules and to use them in characterizing locally torsion free modules and Baer modules. For the sake of completeness, we give some notions and notations which will be used in the sequel. Let M be an R -module. Suppose that N is a submodule of M , K is a nonempty subset of M and L a nonempty subset of R . The *residuals* of N by K and L are denoted by

$$(N :_R K) = \{a \in R : aK \subseteq N\}$$

$$(N :_M L) = \{m \in M : Lm \subseteq N\}.$$

In particular, if $N = 0$, we will use $\text{ann}_R(K)$ and $\text{ann}_M(L)$ instead of $(0 :_R K)$ and $(0 :_R L)$, respectively. If $\text{ann}_R(M) = 0$, then M is said to be a *faithful module*. Also, recall from [5], an R -module M is said to be a *multiplication module* if every submodule N of M has the form IM for some ideal I of R . For more details on multiplication modules, the reader may consult [2] and [7]. M is said to be a *reduced module* if whenever $a^2m = 0$ for some $a \in R, m \in M$, then $am = 0$ [18]. A finitely generated reduced R -module M is said to be a *normal module* if every two distinct minimal prime submodules are comaximal. By [11, Theorem 8] and [16, Lemma 2.1], we know that the classes of normal rings and locally integral domains coincide. So, one can naturally ask whether the classes of normal modules and locally torsion free modules are equal or not. However, we show that these are different concepts (See, Examples 2.4 and 2.5). Also, we determine the conditions under which locally torsion free modules and normal modules are equal (See, Propositions 2.14, 2.15 and Theorem 2.16). Among the other things in this paper, we characterize normal modules in terms of normal rings (See, Lemma 2.11 and Proposition 2.12). Let S be a multiplicatively closed set of R and denote the set $\{m \in M : sm = 0 \text{ for some } s \in S\}$ by $O(S)$. Note that $O(S)$ is always a submodule of M . In particular, we prefer O_P to denote $O(R - P)$ for every $P \in \text{Spec}(R)$. Recall from [15] that a submodule N of M is said to be a **-submodule* if $N = O(S)$ for some multiplicatively closed set S of R . Also, recall from [14] that a submodule N of M is said to be a σ -submodule if for each $m \in N$, we have $\text{ann}_R(m) + (N :_R M) = R$. The set of all σ -submodules of M is denoted by $\sigma(M)$. N is called a *Baer submodule* if $m \in N$ implies that $\text{ann}_M(\text{ann}_R(m)) \subseteq N$ [14]. The authors in [14] and [15] studied some basic properties of *-submodules, σ -submodules and Baer submodules and they used them to characterize Baer modules, von Neumann regular modules, locally torsion free modules and weak quasi-regular modules. Here, we investigate further properties of *-submodules, σ -submodules and Baer submodules (See, Lemmas 3.1, 3.5, 3.7, 3.9, 3.13, Propositions 3.11, 3.12, 3.15, 3.16, 3.17 and Theorem 3.14). Also, we use *-submodules, σ -submodules and Baer submodules to give some new characterizations of Baer modules, locally torsion-free modules and normal modules (See, Theorems 3.2, 3.10 and 3.18). Finally, in Section 3, we study the Zariski topology of modules and give the topological characterizations of normal modules (See, Theorem 4.4).

2. Characterizations of normal modules

Let M be an R -module and N, K be two submodules of M . Recall that N and K are said to be comaximal if $N + K = M$. In particular, any two ideals I and J are *comaximal* if they are comaximal submodules of R -module R .

Recall that an element $e \in R$ is said to be a *weak idempotent* if $e - e^2 \in \text{ann}_R(M)$, or equivalently, $em = e^2m$ for each $m \in M$ [13]. Jayaram and Tekir, introduced von Neumann regular modules as a generalization of von Neumann regular rings and they gave many properties and characterizations of von Neumann regular modules in terms of weak idempotent elements. Recall from [13] that an R -module M is said to be a *von Neumann regular module* if for each $m \in M$, there exists $a \in R$ such that $Rm = aM = a^2M$. The authors in [13, Theorem 1] showed that a finitely generated multiplication module M is von Neumann regular module if $Rm = eM$ for some weak idempotent $e \in R$. Recently, Jayaram et al. introduced the class of Baer modules which is a generalization of finitely generated von Neumann regular modules. Recall from [14] that an R -module M is said to be a *Baer module* if for each $m \in M$, there exists a weak idempotent element $e \in R$ such that $\text{ann}_R(m)M = eM$. We warn the reader that there are other Baer module definitions in the literature (see for example [6], [23]). In this paper, we study the concept of Baer module in the sense of [14].

Now, we begin the study by giving a new property of minimal prime submodules in finitely generated Baer modules.

Proposition 2.1. *Let M be a finitely generated module. If M is a Baer module, then any two distinct minimal prime submodules are comaximal.*

Proof. From [19], we know that maximal submodules, prime submodules and minimal prime submodules exist in finitely generated modules. Let N and K be two distinct minimal prime submodules of a finitely generated Baer module M . Then, $(N :_R M)$ and $(K :_R M)$ are prime ideals containing $\text{ann}_R(M)$. Thus, $(N :_R M)$ contains P_1 and $(K :_R M)$ contains P_2 , where P_1 and P_2 are prime ideals which are minimal over $\text{ann}_R(M)$. This implies that $P_1/\text{ann}_R(M)$ and $P_2/\text{ann}_R(M)$ are minimal prime ideals of $R/\text{ann}_R(M)$. On the other hand, by [14, Proposition 2.12], $R/\text{ann}_R(M)$ is a Baer ring. Then we have $P_1/\text{ann}_R(M) + P_2/\text{ann}_R(M) = R/\text{ann}_R(M)$, which implies that $P_1 + P_2 = R$. Then we conclude that $(N :_R M) + (K :_R M) = R$ and so $N + K = M$ which completes the proof. \square

Definition 2.2. A finitely generated reduced module M is said to be a normal module if any two distinct minimal prime submodules are comaximal.

Remark 2.3. Note that, by [14, Proposition 2.7] and Proposition 2.1, every finitely generated Baer module is an example of a normal module. Also it can be easily shown that a finitely generated reduced module M is a normal module if and only if every prime submodule contains a unique minimal prime submodule.

Example 2.4 (A normal module that is not locally torsion free). Consider the \mathbb{Z} -module $M = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $P = \{(\bar{0}, \bar{0})\}$ is a prime submodule so is unique minimal prime. Since M is finitely generated reduced and has a unique minimal prime, M is a normal module. Let $a = 2$ and $m = (\bar{0}, \bar{1})$. Then $am = (\bar{0}, \bar{0})$ but $\text{ann}_{\mathbb{Z}}(a) + \text{ann}_{\mathbb{Z}}(m) = 2\mathbb{Z} \neq \mathbb{Z}$. Therefore, M is not a locally torsion free module by [16, Lemma 1].

Example 2.5 (A locally torsion free module that is not normal). Consider the \mathbb{Z} -module $M = \mathbb{Z}[X]$. Since M is not finitely generated, M is not a normal module. On the other hand, since M is a torsion free module, by [16, Example 2], M is a locally torsion free module.

Proposition 2.6. Let $n > 1$ be an integer. Then \mathbb{Z}_n is a normal \mathbb{Z} -module if and only if $n = p_1 p_2 \cdots p_r$ for some distinct prime numbers p_1, p_2, \dots, p_r .

Proof. Suppose that \mathbb{Z}_n is a normal \mathbb{Z} -module. Since it is reduced, one can easily see that n is square free integer, that is, $n = p_1 p_2 \cdots p_r$ for some distinct prime numbers p_1, p_2, \dots, p_r . For the converse, assume that $n = p_1 p_2 \cdots p_r$ for some distinct prime numbers p_1, p_2, \dots, p_r . If $r = 1$, then \mathbb{Z}_{p_1} has a unique minimal prime submodule which is also maximal. So, \mathbb{Z}_{p_1} is a trivially normal module. Assume that $r \geq 2$. It can be easily seen that every prime submodule of \mathbb{Z}_n is of the form (\bar{p}_i) which is also minimal prime. Let $i \neq j$. Since $\text{gcd}(p_i, p_j) = 1$, we have $(\bar{p}_i) + (\bar{p}_j) = \mathbb{Z}_n$. Therefore, \mathbb{Z}_n is a normal \mathbb{Z} -module. \square

It is easy to see that if M is a reduced R -module, then $R/\text{ann}_R(M)$ is a reduced ring. But the converse is not always true.

Example 2.7. Let p be a prime number. Consider $R = \mathbb{Z}$ -module $M = \prod_{n=1}^{\infty} \mathbb{Z}_{p^n}$. Then it is clear that $\text{ann}_R(M) = 0$ and so $R/\text{ann}_R(M) = \mathbb{Z}$ is a reduced ring. Now take $m = (\bar{1}, \bar{1}, \bar{0}, \bar{0}, \dots)$. Since $p^2 m = 0_M$ but $pm \neq 0_M$, it follows that M is not a reduced module.

Definition 2.8. Let M be a finitely generated R -module. Then according to [15],

- (i) M is said to satisfy the condition (#) if K is a minimal prime submodule, then $K = (K :_R M)M$,
- (ii) M is said to satisfy the condition (P) if $\bigcap (PM) = (\bigcap P)M$ for all prime ideals P minimal over $\text{ann}_R(M)$,
- (iii) M is said to satisfy the condition (# #) if it satisfies the condition (#) and (P).

The proof of the following two propositions can be found in [15, Proposition 2.16] and [15, Proposition 2.17]. Since they will be frequently used in the sequel, we will give them for the sake of completeness.

Proposition 2.9. *Let M be a finitely generated module satisfying the condition (#) and K a submodule of M . Then,*

- (i) *If P is a minimal prime ideal over $\text{ann}_R(M)$, then PM is a minimal prime submodule of M .*
- (ii) *If K is a minimal prime submodule of M , then $(K :_R M)$ is a minimal prime ideal over $\text{ann}_R(M)$.*

Proposition 2.10. *Let M be a finitely generated module and I be an ideal containing $\text{ann}_R(M)$. Assume that $N = (N :_R M)M$ for each prime submodule N of M minimal over IM . Then,*

- (i) *If P is a prime ideal minimal over I , then PM is a prime submodule minimal over IM .*
- (ii) *If K is a prime submodule minimal over IM , then $(K :_R M)$ is a prime ideal minimal over I .*

Lemma 2.11. *Suppose M is a finitely generated module satisfying the condition (# #). Then M is a reduced module if and only if $R/\text{ann}_R(M)$ is a reduced ring.*

Proof. The "if" part is clear. Now we will show that the "only if" part is also true. Suppose that $R/\text{ann}_R(M)$ is a reduced ring, that is, $\sqrt{\text{ann}_R(M)} = \text{ann}_R(M)$. Let $a^2m = 0$. If $a \in \sqrt{\text{ann}_R(M)} = \text{ann}_R(M)$, there is nothing to prove. Suppose that $a \notin \sqrt{\text{ann}_R(M)}$. Take a minimal prime P of $\text{ann}_R(M)$. Then $N = PM$ is a minimal prime submodule of M by Proposition 2.9. Assume that $am \notin PM = N$. Since $a(am) = 0 \in N$ and N is a prime submodule, we conclude that $a \in (PM :_R M) = P$ and thus $am \in aM \subseteq PM$ which is a contradiction. Since M satisfies (# #) condition, we have

$$0 = \text{ann}_R(M)M = \left(\bigcap_{P \in \text{Min}(\text{ann}_R(M))} P \right)M = \bigcap_{P \in \text{Min}(\text{ann}_R(M))} (PM).$$

Then we get $am \in \bigcap_{P \in \text{Min}(\text{ann}_R(M))} (PM) = 0$ and so $am = 0$ which completes the proof. \square

Proposition 2.12. *Let M be a finitely generated R -module satisfying (# #). Then M is a normal module if and only if $R/\text{ann}_R(M)$ is a normal ring.*

Proof. (\Rightarrow) : Suppose that M is a normal module. Since M is a reduced module, so is $R/\text{ann}_R(M)$ by Lemma 2.11. Take two minimal prime ideals $P_1/\text{ann}_R(M)$ and $P_2/\text{ann}_R(M)$ of $R/\text{ann}_R(M)$. Then P_1 and P_2 are minimal prime ideals of $\text{ann}_R(M)$. By Proposition 2.9, P_1M, P_2M are minimal prime submodules of M and thus $P_1M + P_2M = (P_1 + P_2)M = M$. Since M is finitely generated and $P_1 + P_2 \supseteq \text{ann}_R(M)$, by [4, Corollary 2.5], $P_1 + P_2 = R$. This implies that $P_1/\text{ann}_R(M) + P_2/\text{ann}_R(M) = R/\text{ann}_R(M)$. Therefore, $R/\text{ann}_R(M)$ is a normal ring.

(\Leftarrow) : It is similar to the proof of Proposition 2.1. \square

Lemma 2.13. ([16, Lemma 2.1]) *Let M be an R -module. The following statements are equivalent.*

- (i) *M is a locally torsion free module.*
- (ii) *$am = 0$ implies $\text{ann}_R(a) + \text{ann}_R(m) = R$ for each $a \in R$ and $m \in M$.*
- (iii) *$\text{ann}_R(am) = \text{ann}_R(a) + \text{ann}_R(m)$ for each $a \in R$ and $m \in M$.*
- (iv) *R_P -module M_P is a torsion free module for each $P \in \text{Max}(R)$.*

Proposition 2.14. *Let M be a finitely generated faithful reduced module satisfying the condition (#). Further assume that for any m in M , $\text{ann}_R(m) = \text{ann}_R(a)$ for some a in R . If M is a normal module, then M is a locally torsion-free module.*

Proof. Suppose $rm = 0$ for some r in R and m in M . Assume that $\text{ann}_R(r) + \text{ann}_R(m)$ contained in P for some prime ideal P of R . By assumption, we have $\text{ann}_R(m) = \text{ann}_R(a)$ for some a in R . Then $\text{ann}_R(r), \text{ann}_R(a)$ are contained in P , so there exist prime ideals P_0 and P_1 minimal over $\text{ann}_R(r), \text{ann}_R(a)$ and contained in P . Since R is a reduced ring, by [9, Lemma 2.8], P_0 and P_1 are minimal prime ideals. If $P_0 = P_1$, then r and a are not in P_0 , a contradiction. So by Proposition 18 and Proposition 8 of [20], P_0M and P_1M are distinct minimal prime submodules. Since M is normal $P_0M + P_1M = M$ and so $PM = M$, a contradiction. Therefore, $\text{ann}_R(r) + \text{ann}_R(m) = R$, that is, M is a locally torsion-free module. \square

Proposition 2.15. *Let M be a finitely generated module satisfying (#) property. If M is a locally torsion-free module, then M is a normal module.*

Proof. Let N_1, N_2 be two minimal prime submodules of M . Then by Proposition 2.9, $(N_1 :_R M), (N_2 :_R M)$ are distinct two minimal prime ideals over $\text{ann}_R(M)$. Take $r \in (N_1 :_R M) - (N_2 :_R M)$. Then it is easy to see that $\text{ann}_R(r)$ is a subset of $(N_2 :_R M)$. Since $r \in (N_1 :_R M)$, we get $yr^n \in \text{ann}_R(M)$ for some $y \in R - (N_1 :_R M)$ and $n \in \mathbb{N}$. Then we get $yr^nM = 0$. Since M is a locally torsion-free module, it is a reduced module and so $r(yM) = 0$. Let $m' \in M - N_1$. Since N_1 is a prime submodule and y is not in $(N_1 :_R M)$, we have ym' is not in N_1 . Thus we have $\text{ann}_R(ym')$ is a subset of $(N_1 :_R M)$. Note that $r(ym') = 0$. Then by Lemma 2.13, we have $\text{ann}_R(r) + \text{ann}_R(ym') = R$. Since $\text{ann}_R(r) + \text{ann}_R(ym') \subseteq (N_2 :_R M) + (N_1 :_R M)$, we have $N_1 + N_2 = M$ which completes the proof. \square

Theorem 2.16. *Let M be a finitely generated faithful reduced module satisfying (#) property. Further assume that for any m in M , $\text{ann}_R(m) = \text{ann}_R(a)$ for some a in R . Then M is a normal module if and only if M is a locally torsion-free module.*

Proof. Follows from Propositions 2.14 and 2.15. \square

Theorem 2.17. *Let M_i be an R_i -module for each $i = 1, 2, \dots, n$. Suppose that $R = \prod_{i=1}^n R_i$ and $M = \prod_{i=1}^n M_i$. The following statements are equivalent.*

- (i) M is a normal R -module.
- (ii) M_i is a normal R_i -module for each $i = 1, 2, \dots, n$.

Proof. First note that M is a finitely generated reduced module if and only if M_i is a finitely generated reduced R_i -module for each $i = 1, 2, \dots, n$.

(\Rightarrow) : Suppose that M is a normal module and choose $i \in \{1, 2, \dots, n\}$. Let $P_{i,1}, P_{i,2}$ be two distinct minimal prime submodules of M_i . Then $N_1 = M_1 \times M_2 \times \dots \times M_{i-1} \times P_{i,1} \times M_{i+1} \times \dots \times M_n$ and $N_2 = M_1 \times M_2 \times \dots \times M_{i-1} \times P_{i,2} \times M_{i+1} \times \dots \times M_n$ are two minimal prime submodules of M . Since M is normal, we have $N_1 + N_2 = M$ which implies that $P_{i,1} + P_{i,2} = M_i$. Therefore, M_i is a normal R_i -module.

(\Leftarrow) : Suppose that M_i is a normal R_i -module for each $i = 1, 2, \dots, n$. Let P, K be two minimal prime submodules of M . Then $P = M_1 \times M_2 \times \dots \times P_t \times \dots \times M_n$ and $K = M_1 \times M_2 \times \dots \times K_j \times \dots \times M_n$ for some minimal prime submodules P_t of M_t and K_j of M_j . If $j \neq t$, we have $P + K = M$. So assume that $j = t$. Since $K \neq P$ and M_t is normal, we have $P_t \neq K_t$ so that $P_t + K_t = M_t$ which implies that $P + K = M$. Therefore, M is a normal module. \square

3. Baer modules, locally torsion-free modules and normal modules

We begin this section by giving a characterization of locally torsion-free modules in terms of $(*)$ -submodules and σ -submodules.

Lemma 3.1. (i) Let M be a reduced module and $N = O(S)$ be a proper submodule for some multiplicatively closed set S of R . Then $\sqrt{(N :_R M)} = (N :_R M)$. In particular, $(N :_R M) \cap S = \emptyset$.

(ii) Let M be a finitely generated faithful reduced module satisfying the condition (#). Then minimal prime submodules are $*$ -submodules.

Proof. (i) : It is clear.

(ii) : Let N be minimal prime submodule and let $(N :_R M) = P$. By [Proposition 2.9](#), P is a minimal prime ideal of R . Denote $O_P = \{m \in M : rm = 0 \text{ for some } r \notin P\}$. First note that, by condition (#), $N = PM$. Let $m \in N$. Then $m = r_1 m_1 + \cdots + r_n m_n$ for some $r_i \in P$ and $m_i \in M$. Since R is a reduced ring and P is a minimal prime ideal, we get $y_i r_i = 0$ for some $y_i \notin P$. Now take $y = y_1 \cdots y_n$. Then $y \notin P$ and $ym = 0$. This yields that $m \in O_P$. Also it can be easily seen that $O_P \subseteq N$. i.e., $N = O_P$ is a $*$ -submodule. \square

Theorem 3.2. Let M be a finitely generated faithful reduced module satisfying the condition (#). Then M is a locally torsion-free module if and only if $*$ -submodules are σ -submodules if and only if minimal prime submodules are σ -submodules.

Proof. Suppose M is a locally torsion-free module and let $N = O(S)$ be a $*$ -submodule. Let $m \in N$. Then $am = 0$ for some $a \in S$. By hypothesis, $\text{ann}_R(a) + \text{ann}_R(m) = R$. Clearly, $\text{ann}_R(a)$ is contained in $(N :_R M)$, so N is a σ -submodule. By [Lemma 3.1](#) (ii), minimal prime submodules are σ -submodules.

Now, assume that minimal prime submodules are σ -submodules. Suppose $rm = 0$ and $\text{ann}_R(r) + \text{ann}_R(m)$ is contained in P for some prime ideal P of R . Then $\text{ann}_R(r)$ is contained in $P_0 \subseteq P$ for some minimal prime ideal P_0 of R . Note that r does not belong to P_0 . By [Proposition 2.9](#), $P_0 M$ is a minimal prime submodule, so m is in $P_0 M$. Therefore $\text{ann}_R(m) + (P_0 M :_R M) = R$. Then by [Proposition 8](#) of [20], $(P_0 M :_R M) = P_0$. So $P = R$, a contradiction. Therefore, M is a locally torsion-free module. \square

Definition 3.3. Let N be a submodule of M . Then

- (i) N is called an m -submodule if $N = (N :_R M)M$.
- (ii) An m -submodule N is said to be a strong m -submodule if all prime submodules minimal over N are m -submodules.

Lemma 3.4. Suppose that M is a finitely generated reduced R -module. Let N be a strong m -submodule and $N = O(S)$ for some multiplicatively closed subset S of R . If N' is a prime submodule minimal over N , then N' is a minimal prime submodule of M .

Proof. By [Proposition 2.10](#), $(N' :_R M)$ is a minimal prime ideal over $(N :_R M)$. We claim that $(N' :_R M) \cap S = \emptyset$. Suppose that $d \in (N' :_R M) \cap S$. Then there exists $d' \notin (N' :_R M)$ such that $dd' \in (N :_R M)$ as $\sqrt{(N :_R M)} = (N :_R M)$ and $(N' :_R M)$ is minimal over $(N :_R M)$. Thus $dd'M \subseteq O(S)$. It can be easily shown that $d'M \subseteq O(S) = N$, so $d' \in (N :_R M) \subseteq (N' :_R M)$ which is a contradiction. Therefore $(N' :_R M) \cap S = \emptyset$. We claim that $(N' :_R M)$ is minimal over $\text{ann}_R(M)$. As M is a finitely generated module, $M = \sum_{i=1}^n Rm_i$. So $\text{ann}_R(M) = \bigcap_{i=1}^n \text{ann}_R(m_i)$. It is enough, if we show that for any $a \in (N' :_R M)$ there exists $d \notin (N' :_R M)$ such that $ad \in \text{ann}_R(M)$. Let $a \in (N' :_R M)$. Then there exists $b \notin (N' :_R M)$ such that $ab \in (N :_R M)$ and so $abM \subseteq N$. Since $M = \sum_{i=1}^n Rm_i$, $abm_i \in O(S)$ for each i . Then for each i , there exists $c_i \in S$ such that $abc_i m_i = 0$. Put $c = c_1 c_2 \cdots c_n$. Then $c \in S$ and $abc m_i = 0$ for $i = 1, 2, \dots, n$. Therefore $abc \in \text{ann}_R(M)$. Since $(N' :_R M) \cap S = \emptyset$, it follows that $c \notin (N' :_R M)$ and so $bc \notin (N' :_R M)$. Put $d = bc$. Thus $d \notin (N' :_R M)$ such that $ad \in \text{ann}_R(M)$. So $(N' :_R M)$ is minimal over $\text{ann}_R(M)$. Consequently, by [Proposition 2.10](#), N' is a minimal prime submodule of M . \square

Lemma 3.5. *Let M be a finitely generated reduced module satisfying the condition (P) and $N = O_P = \{m \in M : am = 0 \text{ for some } a \notin P\}$, where $P \in \text{Spec}(R)$, i.e, N is a $*$ -submodule. Assume that N is a strong m -submodule of M . Then $N = O_P = \bigcap_{\alpha \in \Delta} N_\alpha$, where each N_α is a minimal prime submodule of M .*

Proof. By Lemma 3.1 and [15, Lemma 2.20], $\text{rad}N = N$. Then by Lemma 3.4, $N = \bigcap_{\alpha \in \Delta} N_\alpha$, where each N_α is a minimal prime submodule containing N . \square

Theorem 3.6. *Suppose that M is a finitely generated faithful reduced module satisfying the condition (P). Suppose for each $P \in \text{Spec}(R)$, O_P is a strong m -submodule. Consider the following conditions:*

- (i) M is a locally torsion-free module.
- (ii) O_P is a prime submodule for each $P \in \text{Spec}(R)$.
- (iii) Any two distinct minimal prime submodules are comaximal.

Then (i) \Rightarrow (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) if for any $m \in M$, $\text{ann}_R(m) = \text{ann}_R(r)$ for some $r \in R$.

Proof. (i) \Rightarrow (ii) : Suppose (i) holds. Suppose $am \in O_P$ with $m \notin O_P$, where P is prime ideal. Then $a'am = 0$ for some $a' \notin P$. By (i), $\text{ann}_R(a) + \text{ann}_R(a'm) = R$. Note that $\text{ann}_R(a'm) \subseteq P$, so $\text{ann}_R(a) \not\subseteq P$, and thus $ra = 0$ for some $r \notin P$. Clearly $a \in (O_P :_R M)$. Thus, O_P is a prime submodule.

(ii) \Rightarrow (iii) : Suppose (ii) holds. By Lemma 3.4, for each prime ideal P of R , O_P is a minimal prime submodule. Again for each prime submodule K of M , $O_{(K:_R M)}$ is the only minimal prime submodule contained in K , so any two distinct minimal prime submodules are comaximal.

(iii) \Rightarrow (i) : Suppose that P is a minimal prime submodule of M . Then P is minimal over $O_{(P:M)}$. Since $O_{(P:M)}$ is a strong m -submodule, P is an m -submodule, that is, $P = (P :_R M)M$. Thus, M satisfies the condition (#). So by Proposition 2.14, M is a locally torsion-free module. \square

Lemma 3.7. *Let M be a reduced module and N be a Baer submodule. Then $(N :_R M) = \sqrt{(N :_R M)}$.*

Proof. Let $a \in \sqrt{(N :_R M)}$. Then $a^k \in (N :_R M)$ for some $k \in \mathbb{N}$. This implies that $a^k m \in N$ for each $m \in M$. Since N is a Baer submodule and $\text{ann}_R(a^k m) \subseteq \text{ann}_R(am)$, we get $am \in \text{ann}_M(\text{ann}_R(am)) \subseteq \text{ann}_M(\text{ann}_R(a^k m)) \subseteq N$, and so $a \in (N :_R M)$. \square

Definition 3.8. Let M be an R -module and N a proper σ -submodule of M .

- (i) N is called a maximal σ -submodule if N is a maximal element in $\sigma(M)$.
- (ii) N is called a prime σ -submodule if $N_1 \cap N_2 \subseteq N$ implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$ for every $N_1, N_2 \in \sigma(M)$.

Note that a maximal σ -submodule is a prime σ -submodule and if M is finitely generated, then every σ -submodule is contained in a maximal σ -submodule.

Lemma 3.9. *Let M be a finitely generated reduced R -module. Consider the following cases:*

- (i) Every minimal prime submodule is a σ -submodule.
- (ii) Every minimal prime submodule is a maximal σ -submodule.
- (iii) Every maximal submodule contains a unique minimal prime submodule.
- (iv) M is a normal module, i.e, every prime submodule contains a unique minimal prime submodule.

Then (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).

Proof. (i) \Rightarrow (ii) : Let N be a minimal prime submodule. Then by (i), N is a σ -submodule. Since M is finitely generated and N is a proper submodule, there exists a proper maximal σ -submodule

K containing N . Let $m \in K$. Assume that $m \notin N$. This implies that $\text{ann}_R(m) \subseteq (N :_R M)$ since N is a prime submodule. As K is a σ -submodule and $m \in K$, it follows that $\text{ann}_R(m) + (K :_R M) = R$ and this gives $(N :_R M) + (K :_R M) = (K :_R M) = R$ and so $K = M$ which is a contradiction. So that $N = K$, i.e., N is a maximal σ -submodule.

(ii) \Rightarrow (iii) : Let Q be a maximal submodule of M . Assume that Q contains two distinct minimal prime submodules K, N of M . By assumption K, N are maximal σ -submodules. Since $K + N$ is also a σ -submodule, we have $K + N = M \subseteq Q$, a contradiction.

(iii) \Rightarrow (iv) : It is clear. \square

Theorem 3.10. *Let M be a finitely generated faithful reduced module satisfying (#) property. Further assume that for any m in M , $\text{ann}_R(m) = \text{ann}_R(a)$ for some a in R . Then the following statements are equivalent:*

- (i) Every maximal submodule contains a unique minimal prime submodule.
- (ii) M is a normal module.
- (iii) M is a locally torsion-free module.
- (iv) Every $*$ -submodule is a σ -submodule.
- (v) Every minimal prime submodule is a σ -submodule.
- (vi) Every minimal prime submodule is a maximal σ -submodule.

Proof. Follows from Lemmas 3.9, 2.13, Theorems 3.2 and 2.16. \square

Now, we characterize σ -submodules in terms of $*$ -submodules.

Proposition 3.11. *Let M be an R -module. Then a submodule N of M is a σ -submodule if and only if $N = \bigcap_Q O_Q$, where Q denotes any maximal ideal containing $(N :_R M)$.*

Proof. \Rightarrow : Let $m \in N$. Since N is a σ -submodule, it follows that $\text{ann}_R(m) + (N :_R M) = R$. Let P be a maximal ideal containing $(N :_R M)$. Then $\text{ann}_R(m) \not\subseteq P$ and so $m \in O_P$, i.e., $m \in \bigcap_Q O_Q$, where Q denotes any maximal ideal containing $(N :_R M)$. Conversely, assume that $m \in \bigcap_Q O_Q$, where Q is any maximal ideal containing $(N :_R M)$. Let P be a maximal ideal of R . If P contains $(N :_R M)$, then $m \in O_P$ and so $rm = 0$ for some $r \notin P$. This implies that $\text{ann}_R(m)_P = R_P$. Otherwise, we would have $(N :_R M) \not\subseteq P$ and so $(N :_R M)_P = R_P$. In both cases we have $(\text{ann}_R(m) + (N :_R M))_P = R_P$ for all maximal ideal P of R and so $\text{ann}_R(m) + (N :_R M) = R$. This gives $x + y = 1$ for some $x \in \text{ann}_R(m)$ and $y \in (N :_R M)$ and so $m = xm + ym = ym \in N$. Thus $N = \bigcap_Q O_Q$.

\Leftarrow : Assume that $N = \bigcap_Q O_Q$, where Q denotes any maximal ideal containing $(N :_R M)$. Now we show that N is a σ -submodule of M . Let $m \in N$. Assume that $\text{ann}_R(m) + (N :_R M) \neq R$. Then there exists a maximal ideal Q' of R such that $\text{ann}_R(m) + (N :_R M) \subseteq Q'$. Since $m \in N = \bigcap_Q O_Q$, we have $m \in O_{Q'}$ and so $r'm = 0$ for some $r' \notin Q'$, i.e., $\text{ann}_R(m) \not\subseteq Q'$ which is a contradiction. \square

Recall from [13], that a submodule N of M is a *complemented submodule* if there exists a submodule K of M such that $N + K = M$ and $N \cap K = 0$. We say that a submodule N of M is a *locally complemented* if N_Q is a complemented submodule of R_Q -module M_Q for each $Q \in \text{Max}(R)$.

Proposition 3.12. *Let N be a σ -submodule of M . Then N is a locally complemented submodule.*

Proof. Assume that N is a σ -submodule of M . Then by previous proposition, $N = \bigcap_Q O_Q$, where Q denotes any maximal ideal containing $(N :_R M)$. Let P be a maximal ideal of R . If $(N :_R M) \not\subseteq P$, then

$N_P = M_P$. Assume that $(N :_R M) \subseteq P$. Let $m \in N$, then $m \in O_P$ and so $rm = 0$ for some $r \notin P$, i.e., $\frac{m}{1} = 0$. Thus $N_P = 0_P$. \square

Recall that an R -module M is called a *colon distributive module* if $(N + K :_R M) = (N :_R M) + (K :_R M)$ for each $N, K \in L(M)$. In fact, colon distributive module was first studied by Smith in [24] as μ -module. Note that a finitely generated module M is a colon distributive module if and only if M is a multiplication module [24, Theorem 3.8]. We call an R -module M a *locally faithful* (locally colon distributive) module if R_Q -module M_Q is a faithful (colon distributive) for all maximal ideal Q of R . Note that if M is a finitely generated module, then M is locally faithful (locally colon distributive) module if and only if it is a faithful module (colon distributive).

Lemma 3.13. *Let M be a locally faithful and locally colon distributive R -module. If N is a locally complemented submodule, then N is a σ -submodule.*

Proof. Let N be a locally complemented submodule of a locally faithful and locally colon distributive R -module M . Assume that N is not a σ -submodule. Then there exists $m \in N$ such that $m \notin O_Q$ for some maximal ideal $Q \supseteq (N :_R M)$. This implies that for all $r \notin Q$, $rm \neq 0$ and so $\frac{m}{1} \neq 0$ and so $N_Q \neq 0$. Since $(N :_R M) \subseteq Q$, we have $N_Q \neq M_Q$. As N is a locally complemented and M_Q is a colon distributive, by [13, Lemma 3], $N_Q = \frac{e}{s}M_Q$ for some weak idempotent $\frac{e}{s} \in R_Q$. Since M_Q is faithful, then $\frac{e}{s}$ is idempotent in R_Q . Note that only idempotents are $0_Q, 1_Q$ in R_Q and so $N_Q = M_Q$ or $N_Q = 0_Q$, a contradiction. \square

Theorem 3.14. *Let M be a locally faithful and locally colon distributive module. Then the following statements are equivalent for any submodule N of M :*

- (i) N is a locally complemented submodule.
- (ii) N is a σ -submodule.
- (iii) $N = \bigcap_Q O_Q$, where Q denotes any maximal ideal containing $(N :_R M)$.

Proof. Follows from Lemma 3.13 and Proposition 3.12. \square

Proposition 3.15. *Let M be a faithful colon distributive normal module. Further assume that for any m in M , $\text{ann}_R(m) = \text{ann}_R(a)$ for some a in R . Then a submodule N of M is a maximal σ -submodule if and only if N is a minimal prime submodule.*

Proof. Let N be a minimal prime submodule. Then by Theorem 3.10, N is a maximal σ -submodule. Let N be a maximal σ -submodule of M . First note that M is a multiplication module since it is colon distributive. Take a maximal submodule K containing N . Then $Q = (K :_R M)$ is a maximal ideal containing $(N :_R M)$. Since N is a σ -submodule, N_Q is a complemented in R_Q -module M_Q by Proposition 3.12 and so by the proof of Lemma 3.13, $N_Q = 0$. This implies that $N \subseteq O_Q$. Since O_Q is a $*$ -submodule and M is a normal module, so O_Q is a σ -submodule of M by Theorem 3.10. By the maximality of N , $N = O_Q$. Since M_Q is torsion-free, it follows that $N = O_Q$ is a prime submodule. Let N' be a minimal prime submodule of M which is contained in N . Then N' is a σ -submodule by Theorem 3.10. Let $m \in N - N'$. Then $\text{ann}_R(m) \subseteq (N' :_R M)$ and also $\text{ann}_R(m) + (N :_R M) = R = (N' :_R M) + (N :_R M) = (N :_R M)$ a contradiction so that $N = N'$. \square

Proposition 3.16. *Let M be a finitely generated R -module. Then the following statements are satisfied:*

- (i) *If N is a prime submodule and also a σ -submodule, then N is a prime σ -submodule.*
- (ii) *If $M \neq T(M)$ and N is a prime submodule which is also a σ -submodule, then $(N :_R M)$ is a prime σ -ideal.*

Proof. (i) : Let N be a prime submodule which is also a σ -submodule. Since M is finitely generated, there exists a maximal σ -submodule K containing N . Then $(N :_R M) \subseteq (K :_R M)$. Let $m \in K - N$. Since N is a prime submodule, it follows that $ann_R(m) \subseteq (N :_R M)$. Also note that $ann_R(m) + (K :_R M) = R$ since K is a σ -submodule. Thus we have $(N :_R M) + (K :_R M) = (K :_R M) = R$ which is a contradiction. Hence $N = K$ is a maximal σ -submodule so that it is a prime σ -submodule.

(ii) : By [14, Proposition 3.18], $(N :_R M)$ is a σ -ideal. Since N is a prime submodule, $(N :_R M)$ is a prime ideal. By similar argument in (i), $(N :_R M)$ is a maximal σ -ideal so that it is a prime σ -ideal. \square

Proposition 3.17. *Let M be a finitely generated non-torsion module satisfying (#) property and R is a reduced ring. Then every minimal prime submodule is a Baer submodule.*

Proof. Suppose that N is a minimal prime submodule. By Proposition 2.9, $(N :_R M)$ is a minimal prime ideal. So $(N :_R M)$ is a $*$ -ideal. Since M is a non-torsion module and N is a prime m -submodule, by [15, Lemma 2.5], N is a $*$ -submodule. Then by [15, Proposition 2.7], N is a Baer submodule. \square

We now characterize Baer modules in terms of $*$ -submodules and σ -submodules.

Theorem 3.18. *Let M be a finitely generated non-torsion reduced multiplication module. Then M is a Baer module if and only if every Baer submodule is a $*$ -submodule and every $*$ -submodule is a σ -submodule.*

Proof. \Rightarrow : Suppose that M is a Baer module. Let N be a Baer submodule. Then by [15, Lemma 2.23], $N = \bigcap_{\alpha \in \Delta} N_\alpha$, where N_α 's are prime Baer submodules. This implies that $(N :_R M) = \bigcap_{\alpha \in \Delta} (N_\alpha :_R M)$. By [15, Lemma 2.24], $(N_\alpha :_R M)$'s are prime Baer ideals. Since M is finitely generated faithful Baer module, by [14, Proposition 2.12], R is a Baer ring so is a quasi-regular ring. Then by [12, Theorem 3], $(N_\alpha :_R M)$'s are minimal prime ideals. Again by [12, Corollary 3], $(N :_R M)$ is a $*$ -ideal. Suppose that $(N :_R M) = O(S)$ for some multiplicatively closed subset S of R . Then $(N :_R M) \cap S = \emptyset$. Let $\overrightarrow{O(S)} = \{m \in M : rm = 0 \text{ for some } r \in S\}$. Now we will show that $N = \overrightarrow{O(S)}$. Let $m \in N$. Then $m = \sum_{i=1}^n a_i m_i$, where $a_i \in (N :_R M)$. So $a_i s_i = 0$ for some $s_i \in S$. Put $s = s_1 s_2 \cdots s_n$. Then $sm = 0$. So that $N \subseteq \overrightarrow{O(S)}$. Note that by [16, Theorem 3], M is a locally torsion-free module. Suppose that $m \in \overrightarrow{O(S)}$. Then $rm = 0$ for some $r \in S$. This implies that $ann_R(r) + ann_R(m) = R$ and so $1 = s + r'$ for some $s \in ann_R(r)$ and $r' \in ann_R(m)$. Thus $m = sm$. Since $sr = 0$, $s \in O(S) = (N :_R M)$. So $m \in (N :_R M)M = N$. Then we have $N = \overrightarrow{O(S)}$. Thus N is a $*$ -submodule. By Theorem 3.10, every $*$ -submodule is a σ -submodule. The converse part follows from [14, Theorem 3.6]. \square

4. Topological characterizations of normal modules

Let R be a ring and I be an ideal of R . The variety of I , denoted by $V_R(I)$, is defined by $V_R(I) = \{p \in Spec(R) : I \subseteq p\}$. Then the collection $\zeta(R) = \{V_R(I) : I \leq R\}$ satisfies the axioms for closed sets in a topological space. The resulting topology is known as the Zariski topology on $Spec(R)$. For any element $r \in R$, the set $D_r = Spec(R) - V_R(rR)$ is open in $Spec(R)$ and it is well-known that the family $F = \{D_r : r \in R\}$ forms a base for the Zariski topology on $Spec(R)$.

The set of minimal prime ideals of a ring R will be denoted by $Min(R)$. We consider $Min(R)$ as a subspace of $Spec(R)$ with the Zariski topology. For an ideal I of R , we denote the set $V_R(I) \cap Min(R)$ by $V_R^o(I)$. Clearly, $\{V_R^o(I) : I \leq R\}$ is the family of closed sets of $Min(R)$ and the family $\{D_r^o = D_r \cap Min(R) : r \in R\}$ forms a base for the Zariski topology on $Min(R)$. More information about the space of minimal prime ideals can be found in [1] and [9].

The set of prime submodules of an R -module M is called the prime spectrum of M and denoted by $Spec(M)$. The Zariski topology on $Spec(M)$ is described by taking the set $\{V(N) : N \leq M\}$ as the family of closed sets of $X = Spec(M)$ where $V(N) = \{P \in Spec(M) : (N :_R M) \subseteq (P :_R M)\}$.

Let M be an R -module. The ring $R/\text{ann}_R(M)$ will be denoted by \bar{R} and \bar{I} will denote the ideal $I/\text{ann}_R(M)$ where I is an ideal of R containing $\text{ann}_R(M)$. The map $\psi : \text{Spec}(M) \longrightarrow \text{Spec}(\bar{R})$, defined by $\psi(P) = \overline{(P :_R M)}$, is called the natural map of $\text{Spec}(M)$ [21]. If M is a finitely generated R -module, then the natural map ψ is surjective by [21, Proposition 3.5].

Lemma 4.1. *Let M be an R -module and $\psi : \text{Spec}(M) \longrightarrow \text{Spec}(\bar{R})$ be the natural map.*

- (1) [21, Proposition 3.1] ψ is continuous for the Zariski topologies; more precisely, $\psi^{-1}(V_{\bar{R}}(\bar{I})) = V(IM)$ for every ideal I of R containing $\text{ann}_R(M)$.
- (2) [21, Theorem 3.6] If ψ is surjective, then ψ is both open and closed map; more precisely, for every $N \leq M$, $\psi(V(N)) = V_{\bar{R}}(\overline{(N :_R M)})$ and $\psi(\text{Spec}(M) - V(N)) = \text{Spec}(\bar{R}) - V_{\bar{R}}(\overline{(N :_R M)})$.
- (3) [21, Proposition 4.3] Let M be an R -module and $X_r = \text{Spec}(M) - V(rM)$ for any $r \in R$. Then the set $\{X_r : r \in R\}$ forms a base for the Zariski topology on $\text{Spec}(M)$.
- (4) [21, Proposition 4.1] Let M be an R -module, $r \in R$ and $\psi : \text{Spec}(M) \longrightarrow \text{Spec}(\bar{R})$ be the natural map. Then

- (a) $\psi^{-1}(D_{\bar{r}}) = X_r$
- (b) $\psi(X_r) \subseteq D_{\bar{r}}$. If ψ is surjective, then $\psi(X_r) = D_{\bar{r}}$.

The set of all minimal prime submodules of an R -module M will be denoted by $\text{Min}(M)$. Consider $\text{Min}(M)$ as a subspace of $\text{Spec}(M)$ with the Zariski topology. For a submodule N of M , we denote the set $V(N) \cap \text{Min}(M)$ by $V^o(N)$. Clearly, $\{V^o(N) : N \leq M\}$ is the family of closed sets of $\text{Min}(M)$. Since the family $\{X_r : r \in R\}$ is a base for the Zariski topology on $\text{Spec}(M)$, the subspace $\text{Min}(M)$ has $\{X_r^o = X_r \cap \text{Min}(M) : r \in R\}$ as a base of open sets.

Let M be a finitely generated R -module satisfying condition (#). Then, by Proposition 3, we can define the map $\psi^o : \text{Min}(M) \longrightarrow \text{Min}(\bar{R})$ by $\psi^o(P) = \overline{(P :_R M)}$. Clearly, ψ^o is the restriction map of the natural map ψ on $\text{Min}(M)$. Also, ψ^o is surjective by Proposition 3.

Lemma 4.2. *Let M be a finitely generated R -module satisfying condition (#). Consider the map $\psi^o : \text{Min}(M) \longrightarrow \text{Min}(\bar{R})$ defined as above. Then the following statements are true.*

- (1) The map ψ^o is continuous.
- (2) The map ψ^o is both closed and open.
- (3) $(\psi^o)^{-1}(D_{\bar{r}}^o) = X_r^o$ for any $r \in R$.
- (4) $\psi^o(X_r^o) = D_{\bar{r}}^o$ for any $r \in R$.
- (5) Assume that \bar{R} is a reduced ring. Then, $X_r^o = V^o(IM)$ where $I/\text{ann}_R(M) = \text{ann}_{\bar{R}}(\bar{r})$. In particular, $V^o(rM)$ and X_r^o are disjoint open and closed sets.

Proof. (1) This is clear since ψ^o is the restriction of the continuous map $\psi : \text{Spec}(M) \longrightarrow \text{Spec}(\bar{R})$.

(2) Let $N \leq M$. Then we have $\psi^o(V(N) \cap \text{Min}(M)) = \psi(V(N) \cap \text{Min}(M)) = \psi(V(N)) \cap \psi(\text{Min}(M)) = V_{\bar{R}}(\overline{(N :_R M)}) \cap \text{Min}(\bar{R})$ by Lemma 4.1-(2). Thus ψ^o is a closed map. Similarly,

$\psi^o(\text{Min}(M) - (\text{Min}(M) \cap V(N))) = \psi(\text{Min}(M) - (\text{Min}(M) \cap V(N))) = \psi(\psi^{-1}(\text{Min}(\bar{R})) - \psi^{-1}(\text{Min}(\bar{R}) \cap V_{\bar{R}}(\overline{(N :_R M)}))) = \psi\psi^{-1}(\text{Min}(\bar{R}) - (\text{Min}(\bar{R}) \cap V_{\bar{R}}(\overline{(N :_R M)}))) = \text{Min}(\bar{R}) - (\text{Min}(\bar{R}) \cap V_{\bar{R}}(\overline{(N :_R M)}))$ since ψ is surjective. Thus ψ^o is an open set.

(3) $(\psi^o)^{-1}(D_{\bar{r}}^o) = \psi^{-1}(D_{\bar{r}} \cap \text{Min}(\bar{R})) = \psi^{-1}(D_{\bar{r}}) \cap \psi^{-1}(\text{Min}(\bar{R})) = X_r \cap \text{Min}(M) = X_r^o$ by Lemma 4.1-(4).

(4) Since ψ^o is surjective, this follows from part (3).

(5) By part (3), we have $X_r^o = (\psi^o)^{-1}(D_{\bar{r}}^o)$. By [9, Theorem 2.3], $D_{\bar{r}}^o = V_{\bar{R}}^o(\text{ann}_{\bar{R}}(\bar{r}))$. There exists an ideal I of R containing $\text{ann}_R(M)$ such that $\text{ann}_{\bar{R}}(\bar{r}) = I/\text{ann}_R(M)$. Thus $D_{\bar{r}}^o = V_{\bar{R}}^o(\text{ann}_{\bar{R}}(\bar{r})) = V_{\bar{R}}(\text{ann}_{\bar{R}}(\bar{r})) \cap \text{Min}(\bar{R}) = V_{\bar{R}}(\bar{I}) \cap \text{Min}(\bar{R})$. It follows that $X_r^o = (\psi^o)^{-1}(D_{\bar{r}}^o) = (\psi^o)^{-1}(V_{\bar{R}}(\bar{I}) \cap \text{Min}(\bar{R}))$

$Min(\bar{R}) = \psi^{-1}(V_{\bar{R}}(\bar{I}) \cap Min(\bar{R})) = \psi^{-1}(V_{\bar{R}}(\bar{I})) \cap \psi^{-1}(Min(\bar{R})) = V(IM) \cap Min(M) = V^o(IM)$ by Lemma 4.1-(1). \square

Corollary 4.3. *Let M be a finitely generated R -module satisfying condition (#) and \bar{R} be a reduced ring. Then $Min(M)$ is a Hausdorff space with a base of clopen sets.*

Proof. Let P and P' be distinct minimal prime submodules of M . Since M satisfies the condition (#), $(P :_R M) \neq (P' :_R M)$. Let $a \in (P :_R M) - (P' :_R M)$. Then, by Lemma 4.2-(5), $V^o(aM)$ and X_a^o are disjoint open sets containing P and P' , respectively. Hence $Min(M)$ is a Hausdorff space. \square

Theorem 4.4. *Let M be a finitely generated multiplication R -module. Then the following are equivalent.*

- (1) M is a Baer module.
- (2) M is a normal module and $Min(M)$ is a compact space.
- (3) \bar{R} is a reduced ring and there exists a continuous function δ from $Spec(M)$ onto $Min(\bar{R})$ which is the identity on $Min(M)$.

Proof. (1) \implies (2) M is a normal module by Remark 1. \bar{R} is a Baer ring by [14, Proposition 2.12]. Since M is a reduced R -module, \bar{R} is a reduced ring. According to [1, Theorem on p. 83], $Min(\bar{R})$ is a compact space. Since $\{X_r^o : r \in R\}$ is a base for the Zariski topology on $Min(M)$, for any open cover of $Min(M)$, there is a family $\{r_\lambda : \lambda \in \Lambda\}$ of elements of R such that $Min(M) \subseteq \cup_{\lambda \in \Lambda} X_{r_\lambda}^o$. By Lemma 4.2-(4), $\psi^o(Min(M)) = Min(\bar{R}) \subseteq \cup_{\lambda \in \Lambda} \psi^o(X_{r_\lambda}^o) = \cup_{\lambda \in \Lambda} D_{r_\lambda}^o$. Since $Min(\bar{R})$ is a compact space, there exists a finite subset Λ' of Λ such that $Min(\bar{R}) \subseteq \cup_{\lambda \in \Lambda'} D_{r_\lambda}^o$ whence $(\psi^o)^{-1}(Min(\bar{R})) = Min(M) \subseteq \cup_{\lambda \in \Lambda'} (\psi^o)^{-1}(D_{r_\lambda}^o) = \cup_{\lambda \in \Lambda'} X_{r_\lambda}^o$ by Lemma 4.2. Thus $Min(M)$ is a compact space.

(2) \implies (3) By Proposition 2.12, \bar{R} is a normal ring, hence it is reduced and every prime ideal of \bar{R} contains a unique minimal prime ideal. Also, $Min(\bar{R})$ is the image of the compact space $Min(M)$ under the continuous function ψ^o . So $Min(\bar{R})$ is also a compact space. By [1, Theorem on p. 83], there exists a continuous function φ from $Spec(\bar{R})$ onto $Min(\bar{R})$ which is the identity on $Min(\bar{R})$. Consider the natural map $\psi : Spec(M) \longrightarrow Spec(\bar{R})$. Since ψ is continuous, the function $\varphi \circ \psi$ is a continuous map from $Spec(M)$ onto $Min(\bar{R})$ which is the identity on $Min(M)$.

(3) \implies (1) Define the map $\phi : Spec(\bar{R}) \longrightarrow Spec(M)$ by $\phi(\bar{P}) = PM$. Then it can be easily seen that $\phi^{-1}(V(N)) = V_{\bar{R}}((N :_R M))$ for any submodule N of M . Thus ϕ is a continuous map. By (3), there is a continuous function $\delta : Spec(M) \longrightarrow Min(\bar{R})$ which is the identity on $Min(M)$. It follows that $\delta \circ \phi : Spec(\bar{R}) \longrightarrow Min(\bar{R})$ is a continuous map which is the identity on $Min(\bar{R})$. By [1, Theorem on p. 83], \bar{R} is a Baer ring. Hence, M is a Baer module by [14, Theorem 2.14]. \square

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