

Generalization of Kim's estimates in terms of the trace of divergence-free symmetric tensor

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Communicated by: P. Agarwal

Funding information

TUBITAK The Scientific and Technological Research Council of Turkey, Grant/Award Number: 120F109

In this paper, we generalized E.C. Kim's estimates by taking into account the trace of the divergence-free symmetric nonzero tensor. We have also shown that E.C. Kim's estimates still valid in the case of the trace of the divergence-free symmetric tensor vanished identically. In the equality case, we characterized eta-Killing spinor with Killing pair over Sasakian spin manifolds.

KEYWORDS

Dirac operator, estimation of eigenvalues, optimization, spin and spinc geometry

MSC CLASSIFICATION

53C25; 53C27; 34L40

1 | INTRODUCTION

The Dirac operator is one of the fundamental tools of the geometry and topology for mathematicians and previous works.^{1–3} The Dirac operator owes its fame to the fact that Witten made basic proof of the positive mass theorem on the basis of this operator.³ The self-adjointing of the Dirac operator makes it possible to study its eigenvalues.^{2,4–8} One of these studies is the estimating lower bound for the eigenvalues of the Dirac operator.^{9–14} The lower bound estimates corresponding to the square of the first eigenvalue of the Dirac operator defined on a closed Riemannian spin manifolds have been studied intensively in terms of the scalar curvature, energy–momentum tensor, and divergence-free symmetric.^{5,15–17} In this direction, the first estimates is given in 1963 by Lichnerowicz.¹⁷ The author used an integration of the Schrödinger–Lichnerowicz formula defined on a closed Riemannian spin manifold to get the following lower bound:

$$\lambda^2 \geq \frac{Scal}{4}, \quad (1.1)$$

where $Scal$ is the scalar curvature of the manifold. Of course, this result is interesting when the scalar curvature is positive, but the estimate can be improved. Accordingly, the first sharp estimate is obtained in 1980 by Friedrich as follows¹⁸:

$$\lambda^2 \geq \frac{m}{4(m-1)} Scal. \quad (1.2)$$

This proof is based on a modification of the spinorial Levi–Civita connection. The equality case is characterized by a nontrivial real–Killing spinor with positive scalar curvature.

In 1986, Hijazi obtained the following optimal estimates for $m \geq 3$ -dimension in terms of the Yamabe operator⁷:

$$\lambda^2 \geq \frac{m}{4(m-1)} \mu_1, \quad (1.3)$$

where μ_1 is the eigenvalue of the Yamabe operator. Later on, Bar obtained an estimate in term of the Euler characteristic of M denoted by $\chi(M)$ as follows⁴:

$$\lambda^2 \geq \frac{2\Pi\chi(M)}{Area(M)}. \tag{1.4}$$

After this point, mathematicians choose to use different geometric invariants such as energy–momentum tensor and divergence-free symmetric tensor to bring an optimal lower bound for the eigenvalues of the Dirac operator. In this paper, we optimized Kim’s estimates in terms of the trace of divergence-free symmetric tensor.

Before giving an optimal lower bound, let’s give some basic information about the B -twist Dirac operator defined as follows.

Consider an m -dimensional closed Riemannian spin manifold with a spinor bundle \mathbb{S} over (M, g) and a spinorial Levi–Civita connection denoted by ∇ lifted to the spinor bundle by using the Levi–Civita connection given on M . Also, the Levi–Civita connection given on M is denoted by ∇ . With respect to the spinorial Levi–Civita connection ∇ , Dirac operator is locally expressed as

$$D\Phi = \sum_{i=1}^m e_i \cdot \nabla_{e_i} \Phi, \tag{1.5}$$

where $\{e_1, \dots, e_m\}$ is an orthonormal frame field on M , “ \cdot ” denotes the Clifford multiplication and $\Phi \in \Gamma(\mathbb{S})$. On the spinor bundle, \mathbb{S} Hermitian inner product is defined by¹⁹

$$(V \cdot \Phi, V \cdot \Psi) = |V|^2 (\Phi, \Psi), \tag{1.6}$$

where $V \in \Gamma(TM)$ and $\Phi, \Psi \in \Gamma(\mathbb{S})$. The spinorial Levi–Civita connection is satisfied the following properties for all vector fields $V, W \in \Gamma(TM)$ and spinor fields $\Phi, \Psi \in \Gamma(\mathbb{S})$,

$$\begin{aligned} V(\Phi, \Psi) &= (\nabla_V \Phi, \Psi) + (\Phi, \nabla_V \Psi), \\ \nabla_V(W \cdot \Phi) &= \nabla_V W \cdot \Phi + W \cdot \nabla_V \Phi. \end{aligned}$$

In 2009, Kim is obtained two estimates on an m -dimensional closed Riemannian spin manifold for the eigenvalues of the Dirac operator in terms of the eigenvalue of the B -twist Dirac operator define as⁵

$$D_B \Phi = \sum_{i=1}^m B^{-1}(e_i) \cdot \nabla_{e_i} \Phi = \sum_{i=1}^m e_i \cdot \nabla_{B^{-1}(e_i)} \Phi, \tag{1.7}$$

where B is a nondegenerate symmetric $(0, 2)$ tensor field on (M, g) identified with the induced $(1, 1)$ -tensor field B via $B(V, W) = g(V, B(W))$. Also, divergence-free symmetric tensor is defined by

$$div(B^{-1}) = \sum_{i=1}^m (\nabla_{e_i} B)(e_i) = 0.$$

Kim’s⁵ estimate obtained in Theorem 2.1 is describing with $tr(B^{-1}) = 0$ and $div(B^{-1}) = 0$. Accordingly, it is impossible to comment on the geometry of the manifold in case $tr(B^{-1})$ is nonzero. In this case, by expanding Kim’s twistor-like operator

$$T_V(\Phi) = \nabla_V \Psi + pV \cdot \Phi + qB^{-1}(V) \cdot D_B \Phi \tag{1.8}$$

to

$$T_V(\Phi) = \nabla_V \Phi + pV \cdot \Phi + qB^{-1}(V) \cdot D_B \Phi + rtr(B^{-1})V \cdot D_B \Phi, \tag{1.9}$$

we generalized Kim’s estimates as follows:

$$\lambda_1^2 \geq \supinf_p \left(\frac{1}{(mp^2 - 2p + 1)} \left(\frac{Scal}{4} + \frac{\tilde{\lambda}_1^2}{\alpha} + \frac{\Delta\alpha}{2\alpha} \right) \right), \tag{1.10}$$

where $\tilde{\lambda}_1 \in \mathbb{R}$ is the smallest nonzero eigenvalue of the D_B , $p < \frac{1}{2m}$ and $\alpha : M \rightarrow \mathbb{R}$ is a positive real-valued function defined as

$$\alpha = \frac{((1 - pm)|B^{-1}|^2 + p|tr(B^{-1})|^2)^2}{|B^{-1}|^2(1 - pm)^2 - mp^2|tr(B^{-1})|^2}. \tag{1.11}$$

Recall that, if $tr(B^{-1}) = 0$, twistor-like operator given in (1.9) induces to (1.8) which is given by Kim.⁵ The estimates are still valid. Finally, in the case $tr(B^{-1}) \neq 0$, we apply our estimates to the Sasakian spin manifolds which lead us to describe eta-Killing spinor.

Theorem 1.1. *Assume that β is a nondegenerate symmetric tensor defined on an m -dimensional closed Riemannian spin manifold (M, g) such that $div(B^{-1}) = 0$. Then, the first nonzero eigenvalue λ_1 of D associated with the eigenspinor Φ_1 , satisfied*

$$\lambda_1^2 \geq \sup_{p, M} \inf \left(\frac{1}{(mp^2 - 2p + 1)} \left(\frac{Scal}{4} + \frac{\tilde{\lambda}_1^2}{\alpha} + \frac{\Delta\alpha}{2\alpha} \right) \right), \tag{1.12}$$

where $\tilde{\lambda}_1$ is the smallest nonzero eigenvalue of the D_B , $p < \frac{1}{2m}$ and $\alpha : M \rightarrow \mathbb{R}$ is a positive real-valued function defined as

$$\alpha = \frac{((1 - pm)|B^{-1}|^2 + p|tr(B^{-1})|^2)^2}{|B^{-1}|^2(1 - pm)^2 - mp^2|tr(B^{-1})|^2}. \tag{1.13}$$

The equality case of (1.12) is satisfied if and only if $\Delta\alpha$ vanishes identically and if the spinorial Levi-Civita connection satisfies

$$\nabla_V \Phi_1 = -p\lambda V \cdot \Phi_1 - q\tilde{\lambda}B^{-1}(V) \cdot \Phi_1 - rtr(B^{-1})V \cdot \Phi_1, \tag{1.14}$$

for some constants $\lambda, \tilde{\lambda}$ and for all vector fields V . Here p, q, r are real-valued functions and Φ_1 is the first eigenspinor field of both D and D_B .

Proof. For any real-valued functions p, q, r and spinor field $\Phi \in \Gamma(\mathbb{S})$, define the following modified spinorial Levi-Civita connection T on $\Gamma(\mathbb{S})$ by

$$T_i \Phi = \nabla_i \Phi + pe_i \cdot D\Phi + qB^{-1}(e_i) \cdot D_B \Phi + rtr(B^{-1})e_i \cdot D_B \Phi.$$

One can easily compute,

$$\begin{aligned} \sum_{i=1}^m (T_i \Phi, T_i \Phi) &= div \left[\sum_{i=1}^m (\Phi, e_i \cdot D\Phi + \nabla_{e_i} \Phi) e_i \right] + (mp^2 - 2p + 1) |D\Phi|^2 \\ &\quad + (q^2|B^{-1}|^2 - 2q + (2qr + mr^2)|tr(B^{-1})|^2) |D_B \Phi|^2 \\ &\quad - \frac{Scal}{4} |\Phi|^2 + (2pq - 2r + 2prm) tr(B^{-1}) (D\Phi, D_B \Phi). \end{aligned} \tag{1.15}$$

Remember that the real-valued function α with the eigenspinor Φ_1 corresponding to the eigenvalue λ_1 of D satisfies the following equation⁵:

$$\int_M \alpha div \left[\sum_{i=1}^m (\Phi_1, e_i \cdot D\Phi_1 + \nabla_{e_i} \Phi_1) e_i \right] \mu = -\frac{1}{2} \int_M \Delta(\alpha) |\Phi_1|^2 \mu. \tag{1.16}$$

To vanish the last term of Equation (1.15), we set r as

$$r = \frac{pq}{1 - pm}. \tag{1.17}$$

We still use the parameter r for ease of the calculation obtained below. Then, taking integral of (1.15) over M by considering Equation (1.16), we get

$$\int_M \alpha |T\Phi_1|^2 \mu = -\frac{1}{2} \int_M (\Delta(\alpha) |\Phi_1|^2 + (mp^2 - 2p + 1) \alpha \lambda_1^2 |\Phi_1|^2 + (q^2 |B^{-1}|^2 - 2q + (2qr + mr^2) |tr(B^{-1})|^2) \alpha |D_B \Phi_1|^2 - \alpha \frac{Scal}{4} |\Phi_1|^2) \mu. \quad (1.18)$$

Let's define the positive function P_1 as below to get an optimal result:

$$\begin{aligned} P_1 &:= \int_M [(D_B \Phi_1, D_B \Phi_1) - \tilde{\lambda}_1^2 (\Phi_1, \Phi_1)] \mu + \int_M \left[\alpha \sum_{i=1}^m (T_i \Phi_1, T_i \Phi_1) \right] \mu \\ &= \int_M ((mp^2 - 2p + 1) \alpha \lambda_1^2 |\Phi_1|^2) \mu - \frac{1}{2} \int_M \Delta(\alpha) |\Phi_1|^2 \mu - \int_M \tilde{\lambda}_1^2 |\Phi_1|^2 \mu \\ &\quad + \int_M \left[((q^2 |B^{-1}|^2 - 2q + (2qr + mr^2) |tr(B^{-1})|^2) \alpha + 1) |D_B \Phi_1|^2 - \alpha \frac{Scal}{4} |\Phi_1|^2 \right] \mu \geq 0. \end{aligned} \quad (1.19)$$

Set the free parameters q and α as

$$q = \frac{1-r|tr(B^{-1})|^2}{|B^{-1}|^2} \quad \text{and} \quad \alpha = \frac{((1-pm)|B^{-1}|^2+p|tr(B^{-1})|^2)^2}{|B^{-1}|^2(1-pm)^2-mp^2|tr(B^{-1})|^2}.$$

Taking into account (1.17), one can obtain the following equality:

$$q = \frac{1-r|tr(B^{-1})|^2}{|B^{-1}|^2} = \frac{r(1-pm)}{p}. \quad (1.20)$$

Solving (1.20), we get $r = \frac{p}{(1-pm)|B^{-1}|^2+p|tr(B^{-1})|^2}$. Then, in terms of the parameter p , P_1 can be rewritten as

$$\begin{aligned} P_1 &= \int_M ((mp^2 - 2p + 1) \alpha \lambda_1^2 |\Phi_1|^2) \mu - \frac{1}{2} \int_M \Delta(\alpha) |\Phi_1|^2 \mu - \int_M \tilde{\lambda}_1^2 |\Phi_1|^2 \mu \\ &\quad + \int_M \left[\left(\left(\frac{mp^2 |tr(B^{-1})|^2 - (1-pm)^2 |B^{-1}|^2}{|B^{-1}|^2 (|B^{-1}|^2 (1-pm)^2 + p |tr(B^{-1})|^2)} \right) \alpha + 1 \right) |D_B \Phi_1|^2 - \alpha \frac{Scal}{4} |\Phi_1|^2 \right] \mu \geq 0. \end{aligned} \quad (1.21)$$

Finally, if $|B^{-1}|^2 \geq \frac{|tr(B^{-1})|^2}{m}$ and $\frac{1}{2m} > p$ are chosen to make α positive, then one can get the desired inequality given in (1.12). \square

Note that, if $tr(B^{-1})$ vanishes identically, the spinorial Levi-Civita connection (1.9) reduces to (1.8). In this case, all results are same with Kim's estimations.⁵

Consider the ratio of $\lambda_1 \neq 0$ and $\tilde{\lambda}_1$ which are first eigenvalues of D and D_B , respectively, and denote this ration by $\kappa_1 := \frac{\tilde{\lambda}_1}{\lambda_1}$. Then under the same conditions as in Theorem 1.1, Equation (1.21) can be rewritten as follows:

$$P_2 := \int_M \left(\alpha \lambda_1^2 \left(mp^2 - 2p + 1 - \frac{\kappa_1^2}{\alpha} \right) |\Phi_1|^2 - \frac{1}{2} \Delta(\alpha) |\Phi_1|^2 - \alpha \frac{Scal}{4} |\Phi_1|^2 \right) \mu \geq 0. \quad (1.22)$$

This give us the following corollary:

Corollary 1.2. Under the same conditions as in Theorem 1.1, one has

$$\lambda_1^2 \geq \sup_{\zeta} \inf_{\tilde{M}} \left(\frac{Scal}{4\zeta} + \frac{\Delta\alpha}{2\alpha\zeta} \right), \tag{1.23}$$

where $\zeta = mp^2 - 2p + 1 - \frac{\kappa_1^2}{\alpha}$ and $\tilde{M} \subset M$ is given by $\tilde{M} := \{x \in M : \zeta(x) > 0\}$.

Corollary 1.3. Under the same conditions as in Theorem 1.1, if real-valued functions $p = \frac{2}{m}$ and $r = -\frac{2q}{m}$ are taken in Equation (1.19), one gets

$$\lambda_1^2 \geq \inf_M \left(\frac{Scal}{4} + \frac{\tilde{\lambda}_1^2}{|B^{-1}|^2} + \frac{\Delta\alpha}{2\alpha} \right), \tag{1.24}$$

where $\alpha : M \rightarrow \mathbb{R}$ is a real-valued function defined by $\alpha = |B^{-1}|^2$.

Proof. Inserting $p = \frac{2}{m}$ into Equation (1.20), one gets $r = -\frac{2q}{m}$. This means that the term $(2qr + mr^2) |tr(B^{-1})|^2$ given in Equation (1.19) is vanished. Then P_1 induces to

$$\begin{aligned} P_1 := & \int_M \left(\alpha \lambda_1^2 |\Phi_1|^2 - \frac{1}{2} \Delta(\alpha) |\Phi_1|^2 - \tilde{\lambda}_1^2 |\Phi_1|^2 \right. \\ & \left. + ((q^2 |B^{-1}|^2 - 2q)\alpha + 1) |D_B \Phi_1|^2 - \alpha \frac{Scal}{4} |\Phi_1|^2 \right) \mu \geq 0. \end{aligned} \tag{1.25}$$

If the free parameters q and α are taken as follows:

$$q = \frac{1}{|B^{-1}|^2} \quad \text{and} \quad \alpha = |B^{-1}|^2. \tag{1.26}$$

one gets the desired result given in (1.24). □

As in Corollary 1.2, consider the ratio of $\lambda_1 \neq 0$ and $\tilde{\lambda}_1$ which are the first eigenvalues of D and D_B , respectively, and denotes this ration by $\kappa_1 := \frac{\tilde{\lambda}_1}{\lambda_1}$. Then under the same conditions as in Corollary 1.3, Equation (1.25) can be rewritten as follows:

$$\int_M \left(\alpha \lambda_1^2 \left(1 - \frac{\kappa_1^2}{|B^{-1}|^2} \right) |\Phi_1|^2 - \frac{1}{2} \Delta(\alpha) |\Phi_1|^2 - \alpha \frac{Scal}{4} |\Phi_1|^2 \right) \mu \geq 0. \tag{1.27}$$

This proves the following:

Corollary 1.4. In the notations of Corollary 1.3, we have

$$\lambda_1^2 \geq \sup_{\zeta} \inf_{\tilde{M}} \left(\frac{Scal}{4\zeta} + \frac{\Delta\alpha}{2\alpha\zeta} \right), \tag{1.28}$$

where $\zeta = 1 - \frac{\kappa_1^2}{|B^{-1}|^2}$ and $\tilde{M} \subset M$ is given by $\tilde{M} := \{x \in M : \zeta(x) > 0\}$.

In the next theorem, we construct a spinorial Levi-Civita connection with respect to the nondegenerate symmetric tensor and its trace to give a lower bound estimate.

Theorem 1.5. Assume that β is a nondegenerate symmetric tensor defined on an m -dimensional closed Riemannian manifold (M, g) , such that $\text{div}(B^{-1}) = 0$. Let κ_1 be the ratio of $\tilde{\lambda}_1 \in \mathbb{R}$, $\lambda_1 \neq 0 \in \mathbb{R}$ which are the first eigenvalues of D and D_B , respectively. Then, for any real-valued functions $q, r : M \rightarrow \mathbb{R}$ satisfying both

$$\frac{1}{2|B^{-1}|^2} > q > 0 \quad \text{and} \quad q(1 + q|B^{-1}|^2) > (2qr + mr^2) |tr(B^{-1})|^2, \tag{1.29}$$

we have

$$\lambda_1^2 \geq \sup_{\kappa(q,r,\kappa_1)} \inf_{\tilde{M}} \left(\frac{Scal}{4\kappa(q,r,\kappa_1)} + \frac{\Delta(\alpha)}{2\kappa(q,r,\kappa_1)} \right), \tag{1.30}$$

where $\kappa(q, r, \kappa_1)$ and α are real-valued functions defined on M by

$$\begin{aligned} \kappa(q, r, \kappa_1) = & \left(\frac{(m-1)(q|B^{-1}|^2(2m + mq|B^{-1}|^2 + 2q|tr(B^{-1})|^2))}{(q - 2q|B^{-1}|^2)(m + mq|B^{-1}|^2 + q|tr(B^{-1})|^2)^2} \right. \\ & + \frac{(m-1)(q|tr(B^{-1})|^2(1 - m|tr(B^{-1})|^2) - m)}{(q - 2q|B^{-1}|^2)(m + mq|B^{-1}|^2 + q|tr(B^{-1})|^2)^2} \\ & + \frac{(mr^2 + 2q^2)|tr(B^{-1})|^4 + (q^3 - m^2qr^2 - 2mq^2r)|tr(B^{-1})|^4|B^{-1}|^2}{(q - 2q|B^{-1}|^2)(m + mq|B^{-1}|^2 + q|tr(B^{-1})|^2)^2} \\ & \left. + \frac{mq^3|tr(B^{-1})|^2|B^{-1}|^4 - (mqr^2 + 2q^2r)|tr(B^{-1})|^6}{(q - 2q|B^{-1}|^2)(m + mq|B^{-1}|^2 + q|tr(B^{-1})|^2)^2} - \kappa_1^2 \right) \\ \alpha = & \frac{1}{q - 2q^2|B^{-1}|^2}, \end{aligned} \tag{1.31}$$

respectively, and $\tilde{M} = \{x \in M | \kappa(q, r, \kappa_1)(x) > 0\}$.

Equality case of (1.30) is satisfied if and only if the spinorial Levi-Civita connection satisfies

$$\nabla_V \Phi_1 = -p\lambda V \cdot \Phi_1 - q\tilde{\lambda}B^{-1}(V) \cdot \Phi_1 - rtr(B^{-1})V \cdot \Phi_1 \tag{1.32}$$

for some constant $\lambda, \tilde{\lambda} \in \mathbb{R}, \tilde{\lambda} \neq 0$ and for all vector fields V . Here, p, q, r are real-valued functions given in (1.45) and (1.47) and Φ_1 is the first eigenspinor of both D and D_B .

Proof. Consider the following modified spinorial Levi-Civita connection T defined on $\Gamma(\mathbb{S})$ by

$$T_V \Phi = \nabla_V \Phi + pV \cdot D\Phi + qB^{-1}(V) \cdot D_B \Phi + rtr(B^{-1})V \cdot D_B \Phi, \tag{1.33}$$

where p, q, r are real-valued functions defined on M . Assume that λ_1 is the first eigenvalue of D associated with the eigenspinor Φ_1 and $\tilde{\lambda}_1$ is the first eigenvalue of D_B such that $\tilde{\lambda}_1 = \kappa_1 \lambda_1$. Using (1.15)–(1.16) with positive real-valued function α and free functions $\gamma, \kappa : M \rightarrow \mathbb{R}$ to define the following positive real-valued function P_5 as follows:

$$\begin{aligned} P_5 := & \int_M [(D_B \Phi_1, D_B \Phi_1) - \kappa_1^2 \lambda_1^2 (\Phi_1, \Phi_1)] \mu + \int_M \alpha \sum_{i=1}^m |T_i \Phi_1|^2 \\ & + \gamma^2 (D_B \Phi_1 - \kappa D \Phi_1, D_B \Phi_1 - \kappa D \Phi_1) \mu \\ = & \int_M [(mp^2 - 2p + 1)\alpha + \gamma^2 \kappa^2 - \kappa_1^2 \lambda_1^2] |\Phi_1|^2 - \frac{Scal}{4} \alpha |\Phi_1|^2 - \frac{\Delta \alpha}{2} |\Phi_1|^2 \\ & + 2((pq - r + prm)tr(B^{-1})\alpha - \gamma^2 \kappa) \lambda_1 (D_B \Phi_1, \Phi_1) \\ & + ((q^2|B^{-1}|^2 - 2q + (2qr + mr^2)|tr(B^{-1})|^2)\alpha + \gamma^2 + 1) |D_B \Phi_1|^2 \geq 0. \end{aligned} \tag{1.34}$$

Using the relation $D_B \Phi_1 = \kappa D \Phi_1$, then multiplying modified spinorial Levi-Civita connection defined in (1.33) with e_i and $B^{-1}(e_i)$ and then summing over $i = 1, \dots, m$ gives Equations (1.35) and (1.36), respectively. To vanish the last two lines of (1.34), the real-valued functions $p, q, r, \alpha, \gamma, \kappa$ have to satisfy the following four equations:

$$(1 - mp) - (q + mr)tr(B^{-1})\kappa = 0, \tag{1.35}$$

$$(1 + q|B^{-1}|^2 - r(tr(B^{-1}))^2)\kappa - ptr(B^{-1}) = 0, \tag{1.36}$$

$$(pq - r + prm) \operatorname{tr}(B^{-1})\alpha - \gamma^2 \kappa = 0, \quad (1.37)$$

$$(q^2 |B^{-1}|^2 - 2q + (2qr + mr^2) |\operatorname{tr}(B^{-1})|^2) \alpha + \gamma^2 + 1 = 0. \quad (1.38)$$

Solving Equation (1.36), one has

$$p = \frac{(1 + q|B^{-1}|^2 - r|\operatorname{tr}(B^{-1})|^2) \kappa}{\operatorname{tr}(B^{-1})}. \quad (1.39)$$

Inserting (1.35) into Equation (1.37), we have

$$(pq - r(1 - pm)) \operatorname{tr}(B^{-1})\alpha - \gamma^2 \kappa = (pq - r\kappa(q + mr)\operatorname{tr}(B^{-1})) \operatorname{tr}(B^{-1})\alpha - \gamma^2 \kappa. \quad (1.40)$$

Inserting p into Equation (1.40), we get

$$\gamma^2 = \alpha (q(1 + q|B^{-1}|^2) - (2qr + mr^2)|\operatorname{tr}(B^{-1})|^2). \quad (1.41)$$

Putting γ^2 into Equation (1.38), we obtain

$$\alpha = \frac{1}{q(1 - 2q|B^{-1}|^2)}. \quad (1.42)$$

Note that Equations (1.35) and (1.36) together give

$$\kappa = \frac{1 - mp}{(q + mr)\operatorname{tr}(B^{-1})} = \frac{p\operatorname{tr}(B^{-1})}{(1 + q|B^{-1}|^2 - r|\operatorname{tr}(B^{-1})|^2)}. \quad (1.43)$$

Also solving Equations (1.35) and (1.36), we have

$$\kappa = \frac{\operatorname{tr}(B^{-1})}{(m + mq|B^{-1}|^2 + q|\operatorname{tr}(B^{-1})|^2)}. \quad (1.44)$$

This means

$$p = \frac{(1 + q|B^{-1}|^2 - r|\operatorname{tr}(B^{-1})|^2)}{(m + mq|B^{-1}|^2 + q|\operatorname{tr}(B^{-1})|^2)}. \quad (1.45)$$

So the relations

$$\begin{aligned} \alpha &= \frac{1}{q(1 - 2q|B^{-1}|^2)} > 0, \gamma^2 = \frac{(q(1 + q|B^{-1}|^2) - (2qr + mr^2)|\operatorname{tr}(B^{-1})|^2)}{q(1 - 2q|B^{-1}|^2)} > 0, \\ \kappa^2 &= \frac{p(1 - mp)}{(q + mr)(1 + q|B^{-1}|^2 - r|\operatorname{tr}(B^{-1})|^2)} \\ &= \frac{(1 - mp)}{(q + mr)(m + mq|B^{-1}|^2 + q|\operatorname{tr}(B^{-1})|^2)} > 0 \end{aligned} \quad (1.46)$$

imply the restriction

$$\frac{1}{2|B^{-1}|^2} > q > 0 \text{ and } q(1 + q|B^{-1}|^2) > (2qr + mr^2)|\operatorname{tr}(B^{-1})|^2. \quad (1.47)$$

Inserting (1.45) and (1.46) into (1.34) gives the desired estimates given in (1.30). The limiting case of (1.30) is easy to check. \square

2 | ESTIMATING OVER SASAKIAN SPIN MANIFOLDS

In this section, we get an eigenvalue estimate with the aid of the divergence-free symmetric nondegenerate tensor $B^{-1} = \frac{I}{m} - \frac{1}{m}\xi \otimes \eta$ by applying Theorem 1.1 and Theorem 1.5 to the Sasakian spin manifolds. Then, by using the relations between the eta-Killing pair (u, v) corresponding to the eta-Killing spinor and the decomposition property of the spinor bundle built over Sasakian spin manifold, we investigate the geometric properties of the equality case.

On a $2n + 1$ dimensional manifold M , Sasakian structure is defined by an almost contact metric structure of M . An almost contact metric structure is expressed by (ϕ, ξ, η, g) . Here, ϕ is a $(1, 1)$ -tensor field, ξ is a vector field, η is a 1-form, and g is the metric. An almost contact metric structure satisfies

$$\eta(\xi) = 1, \phi^2 = -V + \eta(V)\xi, g(\phi V, \phi W) = g(V, W) - \eta(V)\eta(W). \tag{2.1}$$

On an almost contact metric manifold a fundamental 2-form Θ is defined as

$$\phi(V, W) = g(V, \phi(W)),$$

where V, W is a vector fields. In addition, if the following condition is satisfied

$$(\nabla_V \phi)W = g(V, W) - \eta(W)V \tag{2.2}$$

for all vector fields V, W , then an almost contact metric structure is called Sasakian structure and with this structure manifold is called Sasakian. Moreover, if the Ricci curvature tensor Ric defined on the Sasakian manifold (M, ϕ, ξ, η, g) satisfies

$$Ric = u g + v \eta \otimes \eta \tag{2.3}$$

for some constants $u, v \in \mathbb{R}$ with $u + v = 2n$, then the Sasakian manifold (M, ϕ, ξ, η, g) is called eta-Einstein. Friedrich and Kim showed that any Spinor bundle constructed on an almost contact metric manifold is split under the action of the fundamental 2-form. For more information, see Friedrich and Kim.¹⁵

Definition. On an $(m = 2n + 1)$ -dimensional Sasakian spin manifold (M, ϕ, ξ, η, g) , eta-Killing spinor with Killing pair (u, v) is characterized by a nontrivial spinor field Φ satisfying

$$\nabla_V \Phi = uV \cdot \Phi + v\eta(V)\xi \cdot \Phi \tag{2.4}$$

for some real numbers $u, v \in \mathbb{R}, u \neq 0$ and $\forall V \in \Gamma(TM)$.

Since eta-Killing spinor with Killing pair (u, v) is an eigenspinor of the Dirac operator with eigenvalue $\lambda = -(2n + 1)u - v$, Killing pair (u, v) can be expressed in terms of the scalar curvature by considering some special decomposition of the spinor bundle.¹⁵

Theorem 2.1. *Let (M, ϕ, ξ, η) be an $2n + 1$ -dimensional a closed Sasakian spin manifold $n \geq 1$. Let $B^{-1} = \frac{I}{m} - \frac{1}{m}\xi \otimes \eta, \operatorname{div}(B^{-1}) = 0$. Let λ_1 and $\tilde{\lambda}_1$ be the first eigenvalue of D and D_B , respectively. Then we have*

$$\lambda_1^2 \geq \sup_{p, \kappa(p)} \inf_M \frac{1}{(mp^2 - 2p + 1)} \left(\frac{\operatorname{Scal}}{4} + \frac{m^2(mp^2 - 2pm + 1)\tilde{\lambda}_1^2}{(m - 1)(p - 1)^2} \right), \tag{2.5}$$

where $p < \frac{1}{2m}$ and $\tilde{\lambda}_1 \in \mathbb{R}$ is the first eigenvalue of D_B . Also, if $\lambda_1 \neq 0$, inequality (2.5) can be rewritten

$$\lambda_1^2 \geq \sup_p \inf_M \frac{1}{(mp^2 - 2p + 1)} \left(\frac{\operatorname{Scal}}{4\kappa(p)} \right), p < \frac{1}{2m}, \tag{2.6}$$

where $\kappa(p)$ and μ_1 are real-valued functions on M defined by

$$\kappa(p) = \frac{(m-1)(p-1)^2(mp^2 - 2p + 1) - m^2(mp^2 - 2pm + 1)\mu_1^2}{(m-1)(p-1)^2(mp^2 - 2p + 1)}, \quad (2.7)$$

$$\mu_1 = \frac{\tilde{\lambda}_1}{\lambda_1}.$$

The equality case of (2.5) is satisfied in two cases if and only if there exist an eta-Killing spinor Φ_1 with the following Killing pairs

1.

$$\left(\frac{1}{2}, -\frac{m}{4} + \frac{Scal}{4(m-1)}\right), \left(-\frac{1}{2}, \frac{m}{4} - \frac{Scal}{4(m-1)}\right), \quad (2.8)$$

where $m \geq 5$ and Φ_1 is the first eigenspinor of D_B .

2.

$$\left(\frac{-2 + \sqrt{4 + 2Scal}}{4}, \frac{4 - \sqrt{4 + 2Scal}}{4}\right), \quad (2.9)$$

where $m = 3$ and such Φ_1 is the first eigenspinor of D_B .

Proof. Let's define the nondegenerate symmetric tensor field B^{-1} on M by $B^{-1} = \frac{I}{m} - \frac{1}{m}\xi \otimes \eta$. So, the positive function given in (1.13) can be rewritten as follows:

$$\alpha = \frac{(m-1)(p-1)^2}{m^2(mp^2 - 2pm + 1)}. \quad (2.10)$$

Inserting the function α into (1.12), one gets inequality (2.5) and (2.6), respectively. Considering $B^{-1} = \frac{I}{m} - \frac{1}{m}\xi \otimes \eta$, we get

$$tr(B^{-1}) = \frac{m-1}{m}, |B^{-1}|^2 = \frac{m-1}{m^2}. \quad (2.11)$$

In limiting case, by taking into account (1.14), one can rewrite spinorial Levi-Civita connection as follows:

$$\begin{aligned} \nabla_{e_i}\Phi_1 &= -\left(p\lambda_1 + \frac{mp}{1-p}\tilde{\lambda}_1\right)e_i \cdot \Phi_1 - \frac{(1-pm)m^2}{(m-1)(1-p)}\tilde{\lambda}_1 B^{-1}(e_i) \cdot \Phi_1 \\ &= -\left(p\lambda_1 + \frac{mp}{1-p}\tilde{\lambda}_1\right)e_i \cdot \Phi_1 - \frac{(1-pm)m^2}{(m-1)(1-p)}\tilde{\lambda}_1 \left(\frac{e_i}{m} - \frac{\eta(e_i)\xi}{m}\right) \cdot \Phi_1 \\ &= -\left(p\lambda_1 + \frac{m}{m-1}\tilde{\lambda}_1\right)e_i \cdot \Phi_1 - \frac{m(1-pm)}{(m-1)(1-p)}\tilde{\lambda}_1 \eta(e_i)\xi \cdot \Phi_1. \end{aligned} \quad (2.12)$$

Accordingly, in limiting case, eta-Killing spinor with Killing pair is described as

$$(u_1, v_1) = \left(\frac{p(m-1)\lambda_1 + m\tilde{\lambda}_1}{m-1}, \frac{m(1-pm)\tilde{\lambda}_1}{(m-1)(1-p)}\right). \quad (2.13)$$

In case $n \geq 2$, eta-Killing spinor is characterized with eta-Killing pair given in (2.20). Let M be a 3-dimensional Sasakian spin manifold. Then, by using Proposition 3.2 and Proposition 3.3 given in Kim,⁵ eta-Killing pair can be rewritten in terms of the scalar curvature of M as follows:

$$(u_1, v_1) = \left(\frac{-2 + \sqrt{4 + 2Scal}}{4}, \frac{4 - \sqrt{4 + 2Scal}}{4}\right). \quad (2.14)$$

This means

$$\left(\frac{p(m-1)\lambda_1 + mp\lambda_1}{m-1}, \frac{m(1-pm)\tilde{\lambda}_1}{(m-1)(1-p)} \right) = \left(\frac{-2 + \sqrt{4 + 2Scal}}{4}, \frac{4 - \sqrt{4 + 2Scal}}{4} \right). \tag{2.15}$$

Accordingly, the first eigenvalue of D and D_B can be described as follows:

$$\begin{aligned} \lambda_1 &= \frac{(3p-1)\left(-2 + \sqrt{4 + 2Scal}\right) - p(1-p)\left(4 - \sqrt{4 + 2Scal}\right)}{4p(3p-1)}, \\ \tilde{\lambda}_1 &= \frac{(1-p)\left(4 - \sqrt{4 + 2Scal}\right)}{6(3p-1)}, \end{aligned} \tag{2.16}$$

where $p < \frac{1}{6}$. □

In the next theorem to consider application of Theorem 1.5 over Sasakian spin manifolds, we use same divergence-free symmetric tensor given in the proof of Theorem 1.1.

Theorem 2.2. *Let (M, ϕ, ξ, η) be an $2n + 1$ -dimensional a closed Sasakian spin manifold $n \geq 1$. Let $B^{-1} = \frac{I}{m} - \frac{1}{m}\xi \otimes \eta$, $\text{div}(B^{-1}) = 0$. Let λ_1 and $\tilde{\lambda}_1$ be the first eigenvalues of D and D_B , respectively, and denote $\kappa_1 = \frac{\tilde{\lambda}_1}{\lambda_1}$. Then, for any real-valued functions $q, r : M \rightarrow \mathbb{R}$ satisfying*

$$\frac{m^2}{2(m-1)} > q > 0 \text{ and } \frac{q(m^2 + q(m-1))}{(m-1)^2} > (2qr + mr^2), \tag{2.17}$$

we have

$$\lambda_1^2 \geq \sup_{\kappa(q,r,\kappa_1)} \inf_M \left(\frac{Scal}{4\kappa(q,r,\kappa_1)} \right), \tag{2.18}$$

where $\kappa(q, r, \kappa_1)$ are real-valued functions on M defined by

$$\begin{aligned} \kappa(q, r, \kappa_1) &= \left(\frac{(m-1)^2(m^2q - 2q^2(m-1))}{m^4(m^3 + q(m-1)(2m-1))^2} \left((m-1)^3(mq^3 - 2m^3qr \right. \right. \\ &\quad - 4m^2q^2r + m^2qr^2 + 2mq^2r - q^3 + 2m^2qr + 4mq^2r - mqr^2 \\ &\quad - 2q^2r + m^3r^2 + 2qm^2 + mq^3) + 3m^4q - qm^3 + 3m^4q^2 \\ &\quad \left. \left. - 6m^3 + 2m^2 \right) - \kappa_1^2 \right). \end{aligned} \tag{2.19}$$

The equality case of (2.18) is satisfied in two cases if and only if there exist an eta-Killing spinor Φ_1 with the followings Killing pairs:

1.

$$\left(\frac{1}{2}, -\frac{m}{4} + \frac{Scal}{4(m-1)} \right), \left(-\frac{1}{2}, \frac{m}{4} - \frac{Scal}{4(m-1)} \right), \tag{2.20}$$

where $m \geq 5$ and Φ_1 is the first eigenspinor of D_B .

2.

$$\left(\frac{-2 + \sqrt{4 + 2Scal}}{4}, \frac{4 - \sqrt{4 + 2Scal}}{4} \right), \tag{2.21}$$

where $m = 3$ and such Φ_1 is the first eigenspinor of D_B .

Proof. Let's define the nondegenerate symmetric tensor field B^{-1} on M by $B^{-1} = \frac{I}{m} - \frac{1}{m}\xi \otimes \eta$. So, the positive functions $\kappa(q, r, \kappa_1)$ can be rewritten as in (2.19). These give us desired inequality (2.18).

In equality case, in terms of the real functions p, q, r the spinorial Levi-Civita connection can be written as

$$\begin{aligned}\nabla_{e_i} \Phi_1 &= -p\lambda_1 e_i \cdot \Phi_1 - q\tilde{\lambda}_1 B^{-1}(e_i) \cdot \Phi_1 - r \left(\frac{m-1}{m} \right) \tilde{\lambda}_1 e_i \cdot \Phi_1 \\ &= -p\lambda_1 e_i \cdot \Phi_1 - q\tilde{\lambda}_1 \left(\frac{e_i}{m} - \frac{\eta(e_i)\xi}{m} \right) \cdot \Phi_1 - r \left(\frac{m-1}{m} \right) \tilde{\lambda}_1 e_i \cdot \Phi_1 \\ &= -\frac{1}{m} (mp\lambda_1 + q\tilde{\lambda}_1 + r(m-1)\tilde{\lambda}_1) e_i \cdot \Phi_1 + \frac{1}{m} q\tilde{\lambda}_1 \eta(e_i)\xi \cdot \Phi_1,\end{aligned}\tag{2.22}$$

where real-valued functions p is given by

$$p = \frac{m(m-1)(q-r(m-1))}{m^3 + (m-1)q(2m-1)}.\tag{2.23}$$

Accordingly, in equality case, eta-Killing spinor with Killing pair is described by

$$(u_1, v_1) = \left(-\frac{1}{m} (mp\lambda_1 + q\tilde{\lambda}_1 + r(m-1)\tilde{\lambda}_1), \frac{1}{m} q\tilde{\lambda}_1 \right).\tag{2.24}$$

In the case $n \geq 2$, eta-Killing spinor is characterized with Killing pair given in (2.20).

Let M be 3-dimensional Sasakian spin manifold. By using Proposition 3.2 and Proposition 3.3 given in Kim,⁵ (u_1, v_1) can be characterized with Killing pair given in (2.21). \square

ACKNOWLEDGEMENT

This study was supported by TUBITAK The Scientific and Technological Research Council of Turkey (Project Number: 120F109)

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How to cite this article: Eker S. Generalization of Kim's estimates in terms of the trace of divergence-free symmetric tensor. *Math Meth Appl Sci.* 2022;45(9):5471-5482. doi:10.1002/mma.8122