ON THE NONLINEAR STABILITY AND THE EXISTENCE OF SELECTIVE DECAY STATES OF 3D-QUASI-GEOSTROPHIC POTENTIAL VORTICITY EQUATION

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ABSTRACT. In this article, we study the dynamics of large scale motion in atmosphere and ocean governed by the 3D-quasi-geostrophic potential vorticity (QGPV) equation with a constant stratification. It is shown that for a Kolmogorov forcing on the first energy shell, there exist a family of exact solutions which are dissipative Rossby waves. The nonlinear stability of these exact solutions are analyzed based on the assumptions on the growth rate of the forcing. In the absence of forcing, we show the existence of selective decay states for the 3D-QGPV equation. The selective decay states are the 3D-Rossby waves traveling horizontally at a constant speed. All these results can be regarded as the expansion of that of the 2D-QGPV system and in the case of 3D-QGPV system with isotropic viscosity. Finally, we present a geometric foundation for the model as a general equation for non-equilibrium reversible-irreversible coupling.

1. INTRODUCTION

Large-scale structures that exhibit persistent spatial and temporal coherence are ubiquitous in geophysical flows, particularly in motions of atmosphere and ocean. Examples of such coherent structures include macroturbulent kolks, surface boils, the big red spot on Jupiter and so on. Mathematically, problems pertaining to large-scale coherent structures in geophysical flows are concerned about long time selective decay, the effect of large-scale forcing, non-linear stability as well as chaotic dynamics, cf. [1]. It is clear that the non-linear dynamics involved in the geophysical flows in the atmosphere and the ocean are rather complex. In this article we investigate the large-scale coherent structures from the perspective of non-linear stability and long-time selective decay based on the following continuously stratified quasi-geostrophic potential vorticity equation (QGPV) with dissipation [2–4]

$$\partial_t \Delta\psi + J(\psi, \Delta\psi) + \beta\partial_x\psi = D(\Delta)\psi + F(t, x, y, z), \tag{1}$$

where β is the beta-plane coefficient in the mid-latitude, $\Delta = \partial_{xx} + \partial_{yy} + S\partial_{zz}$ with S a measure of the stratification relative to rotation, $D(\Delta)$ is the general dissipation operator encompassing Ekman drag, Newtonian viscosity and hyper-viscosity [5]

$$D(\Delta)\psi = \sum_{i=0}^{N}(-1)^i\mu_i\Delta^i\psi, \quad \mu_i \geq 0, \text{ for } i \geq 1, \tag{2}$$

F is the external forcing, and $J(\psi, \Delta\psi)$ is $2D$ Jacobian determinant

$$J(\phi, \psi) = \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x}. \tag{3}$$

We consider (1) in a channel, that is we fix the spatial domain as a box given by

$$\Omega = (0, L_x) \times (0, L_y) \times (0, H), \tag{4}$$

and choose the supplemented boundary conditions as

$$\begin{cases} \psi(t, 0, y, z) = \psi(t, L_x, y, z), \\ \psi(t, x, 0, z) = \psi(t, x, L_y, z), \\ \psi(t, x, y, 0) = \psi(t, x, y, H) = 0, \\ \frac{\partial^2\psi}{\partial z^2}(t, x, y, 0) = \frac{\partial^2\psi}{\partial z^2}(t, x, y, H) = 0. \end{cases} \tag{5}$$

Furthermore, since the stream function is determined up to a constant which we take as zero, we assume the zero average condition for the stream function

$$\int_\Omega \psi \, d\Omega = 0. \tag{6}$$

Key words and phrases. quasi-geostrophic equation, nonlinear stability, selective decay states.

The continuously stratified QGPV-model can be derived from the rotating Boussinesq equations as the asymptotic limit of fast rotation and strong stratification, cf. [2, 6]. It is a widely used model in oceanography and meteorology to describe large-scale oceanic and atmospheric flows [7–10], to simulate global warming, see Simonnet et al. [11, 12], Berloff and Meacham [13], and Ghil [14] and references therein.

There have been many works on the stability of solutions to the QGPV type equations with or without dissipation. Exact solutions to the QGPV equation without the dissipative term are constructed corresponding to Baroclinic Rossby waves in [15–17] and to Barotropic Rossby waves in [4, 16]. The problems of baroclinic and barotropic instability is investigated in [13, 18, 19]; see [20–22] for the study of viscous instability associated with the QGPV equation. For the nonlinear stability of the QGPV equation, Majda and Wang in [1] constructed a family of exact solutions, including time-independent solutions such as simple shear flow, swirling eddies and Taylor vortices, and more general time-dependent solutions in presence of general Kolmogorov forcing and topography. The nonlinear stability of these exact solutions and the effect of rotation, topography and dissipation on the stability is also studied. Although both the atmospheric and oceanic flows are confined in a thin layer, the flows are still 3-dimensional, and baroclinic. It is natural then to study the similar problem for the simplest 3D baroclinic model, namely the 3D-QGPV model (1), and to generalize the nonlinear stability results of the 2d-case (S=0) to the more general case with constant $S > 0$.

Another aspect of the long-time nonlinear dynamics of the 3D-QGPV equation that we address in this article is the long-time selective decay property for the free flow. It is well known that the energy and enstrophy of 2D-Navier-Stokes equations decrease exponentially. In [23], Foias and Saut showed that the ratio of the enstrophy to energy also decreases in time with a non-zero asymptotical value which is one of eigenvalues of the Stokes operator. Numerical simulation of the evolution of coherent structures for freely decaying two dimensional Navier-Stokes flows also indicates that the enstrophy decays much more rapidly than the energy [24, 25]. These results lead physicists to hypothesize that there exists an intermediate time scale over which the state of Navier-Stokes equations minimizes the enstrophy with constant energy–termed as Physicist's Selective Decay Principle. For 2D Navier-stokes equations the selective decay state is the steady-state generalized Taylor vortices [23]. Subsequently in [5], Majda-Shim-Wang generalize these results to the 2D-geophysical flow case governed by the QGPV equation without stratification, i.e. $S = 0$. It is found that the selective decay states for 2D-QGPV equation with the beta-effect are the superposition of generalized 2D-Taylor vortices and those 2D-Rossby waves. The case of continuously stratified 3D-QGPV equation is also investigated in [5] albeit with the classical viscous dissipation, i.e., $D(\Delta)\psi = \frac{1}{Re}\Delta_3(\partial_{xx}\psi + \partial_{yy}\psi + \frac{1}{Pr}F^2\partial_{zz}\psi)$ with Re the Reynolds number, Pr the Prandtl number, Δ_3 the 3D Laplacian and F the ratio of Froude number over Rossby number, cf. [5] for details. They establish that the Selective Decay Principle only holds in the case of isotropic viscosity ($Pr = F = 1$), while they construct a counterexample when the viscosity is anisotropic.

In this article, we borrow and generalize the techniques from [5] to study the nonlinear stability and the Selective Decay Principle of solutions to the QGPV equation with the general dissipation operator (2) that includes the Ekman drag, the Newtonian viscosity and the hyper-viscosity. Specifically we show that there exist a family of exact solutions which represent the 3D dissipative Rossby waves, and are nonlinear stable if the L^2 norm of the Kolmogorov forcing does not grow exponentially in time, i.e.,

$$\|F\| \leq ce^{\alpha t},$$

for some $0 < \alpha < D(-\Lambda_1)$ where

$$\Lambda_1 = 4\pi^2 \min\left\{\frac{1}{L_x^2}, \frac{1}{L_y^2}\right\} + \frac{S^2\pi^2}{H^2}.$$

Furthermore, it is shown that there exists a weak selective decay state which is an eigenvector of Δ, in the absence of external forcing and $\mu_0 \leq 0$. These selective states are the superposition of generalized 3D-Taylor vortices and 3D-Rossby waves, but traveling horizontally. In particular, when $D(\Delta)\psi = \mu_2\Delta^2\psi$, the corresponding 3D dissipative Rossby waves of form

$$\sum_{|\mathbf{k}|^2=\Lambda_{k_o}} A_{\mathbf{k}}(0)e^{\frac{i2\pi k_x \beta t}{L_x \Lambda_{k_o}}} e_{\mathbf{k}},$$

are the strong selective decay states [5] describing the large scale coherent structure observed in the atmosphere.

Finally, we investigate the geometrical foundations of the 3D-QG equations. In particular, we exhibit the Hamiltonian character of the model by presenting the dynamical equations in terms of a metriplectic bracket and a Hamiltonian function.

The rest of this paper is organized as follows. Some exact solutions to the 3D-QGPV equations are constructed, and some useful lemmas are proved in Section 2. In Section 3, we show the nonlinear stability of the exact solutions. The existence of selective decay states of the 3D-QGPV equations are proved in Section 4. The geometric foundations of the model is presented in Section 5.

2. The basic definitions

In this section, we present the notation and the basic definitions that will be used throughout the paper.

Letting

$$\mathcal{Z} = \left\{ \mathbf{k} = (k_x, k_y, k_z) \in \mathbb{Z}^3,\ k_x^2 + k_y^2 \neq 0,\ k_z \geq 1 \right\},$$

we define

$$|\mathbf{k}|^2 = \left(\frac{2\pi k_x}{L_x}\right)^2 + \left(\frac{2\pi k_y}{L_y}\right)^2 + \left(\frac{\pi S k_z}{H}\right)^2, \qquad \mathbf{k} \in \mathcal{Z},$$

and

$$e_{\mathbf{k}}(x,y,z) = \left(\frac{2}{|\Omega|}\right)^{1/2} e^{2\pi i (x k_x/L_x + y k_y/L_y)} \sin\left(\frac{k_z \pi z}{H}\right), \tag{7}$$

with

$$|\Omega| = L_x L_y H.$$

Then $e_{\mathbf{k}}$ are the eigenfunctions of $-\Delta$ with the given boundary conditions, i.e.

$$-\Delta e_{\mathbf{k}} = |\mathbf{k}|^2 e_{\mathbf{k}}.$$

Let us define the functional spaces of real valued functions,

$$\begin{aligned} X &= \left\{ \psi \in L^2(\Omega, \mathbb{R}) : \psi \text{ satisfies (6) and the first three conditions in (5)} \right\}, \\ X_1 &= X \cap H^2(\Omega, \mathbb{R}) \end{aligned} \tag{8}$$

Then the family $\{e_{\mathbf{k}}\}_{\mathbf{k}\in\mathcal{Z}}$ forms an orthonormal basis for the space X. That is we can expand $\psi(t) \in X$

$$\psi(x,y,z,t) = \sum_{\mathbf{k}\in\mathcal{Z}} \hat{\psi}_{\mathbf{k}}(t) e_{\mathbf{k}}(x,y,z), \quad \overline{\hat{\psi}_{\mathbf{k}}(t)} = \hat{\psi}_{\mathbf{k}^*}(t), \quad \mathbf{k}^* = (-k_x, -k_y, k_z),$$

and have the orthogonality relation

$$\int_\Omega e_{\mathbf{k}}(x,y,z) e_{\mathbf{l}}^*(x,y,z) d\Omega = \delta_{\mathbf{k},\mathbf{l}}, \tag{9}$$

Let us denote the eigenvalues of $-\Delta$ by Λ_k, $k \in \mathbb{N}$ counting multiplicity and sorted so that $\Lambda_i < \Lambda_j$ if $i < j$. That is for $k \in \mathbb{N}$, there exists $\mathbf{k} \in \mathcal{Z}$ s.t.

$$\Lambda_k = |\mathbf{k}|^2.$$

In particular, we note that

$$\Lambda_1 = 4\pi^2 \min\left\{\frac{1}{L_x^2}, \frac{1}{L_y^2}\right\} + \frac{S^2\pi^2}{H^2}. \tag{10}$$

We also remark that for sufficiently smooth $\psi \in X$, we have

$$(-\Delta)^s \psi = \sum_{\mathbf{k}\in\mathcal{Z}} |\mathbf{k}|^{2s} \hat{\psi}_{\mathbf{k}} e_{\mathbf{k}}, \qquad s \geq 0 \tag{11}$$

$$\int_\Omega (-\Delta)^{s_1}\psi (-\Delta)^{s_2}\psi d\Omega = \sum_{\mathbf{k}\in\mathcal{Z}} |\mathbf{k}|^{2(s_1+s_2)} |\hat{\psi}_{\mathbf{k}}|^2, \tag{12}$$

$$\|\psi\|^2 = \int_\Omega |\psi|^2 = \sum_{\mathbf{k}\in\mathcal{Z}} |\hat{\psi}_{\mathbf{k}}|^2,$$

$$\|(-\Delta)^s\psi\|^2 = \sum_{\mathbf{k}\in\mathcal{Z}} |\mathbf{k}|^{4s} |\hat{\psi}_{\mathbf{k}}|^2, \qquad s \geq 0. \tag{13}$$

For $\psi \in X_1$, the relative vorticity is defined as,

$$\omega = \Delta\psi = \sum_{\mathbf{k}\in\mathcal{Z}} \hat{\omega}_{\mathbf{k}}(t) e_{\mathbf{k}}, \qquad \hat{\omega}_{\mathbf{k}} = -|\mathbf{k}|^2 \hat{\psi}_{\mathbf{k}}. \tag{14}$$

A forcing $F(t) \in X$ is called *(generalized) p-Kolmogorov forcing* if it is of the form

$$F(x,y,z,t) = \sum_{|\mathbf{k}|^2 = \Lambda_p} F_{\mathbf{k}}(t) e_{\mathbf{k}}(x,y,z). \tag{15}$$

for some $p > 1$, that is if F is an eigenmode of the Laplacian, $-\Delta F = \Lambda_p F$. When $p = 1$ in (15), F is called a *Kolmogorov forcing*.

3. Non-linear Stability of the Exact Solution

For a given generalized Kolmogorov forcing on the Λ_p energy shell, the equation (1) admits an exact solution of the type

$$\psi_{\text{exact}}(x, y, z, t) = \sum_{|\mathbf{k}|^2 = \Lambda_p} \hat{\psi}_{\text{exact},\mathbf{k}}(t) e^{i\omega(\mathbf{k})t} e_{\mathbf{k}}(x, y, z), \qquad \omega(\mathbf{k}) = \frac{2\pi k_x \beta}{L_x |\mathbf{k}|^2}, \tag{16}$$

whose coefficients can be determined from the following ODE obtained by plugging the ansatz (16) into (1)

$$\frac{d\hat{\psi}_{\text{exact},\mathbf{k}}(t)}{dt} = -\frac{D(-\Lambda_p)}{\Lambda_p} \hat{\psi}_{\text{exact},\mathbf{k}}(t) - \frac{1}{\Lambda_p} \hat{F}_{\mathbf{k}}(t) e^{-i\omega(\mathbf{k})t}, \tag{17}$$

which can be solved exactly. In general, for each initial condition $\hat{\psi}_{\text{exact},\mathbf{k}}(0)$, a different exact solution will be obtained. But as (17) is linearly dissipated, the effect of the initial condition will vanish exponentially and the $\hat{\psi}_{\text{exact},\mathbf{k}}(t)$ will be determined solely by the forcing as can be seen later in (31).

The solution (16) represents a dispersive wave known as a Rossby wave which travels horizontally at the speed of $\omega(\mathbf{k})$.

The following theorem shows that when the system is damped and driven at large scales, a large coherent structure emerges. In particular, in the case of a Kolmogorov forcing, the exact solution on the first energy shell is nonlinearly stable if the energy of the forcing grows at most sub-exponentially in time with growth rate not exceeding a constant depending on the dissipation of the system.

Theorem 1. *Assume that the dissipation operator (2) satisfies $\mu_0 \geq 0$ and $\mu_j > 0$ for some $1 \leq j \leq N$ so that*

$$\bar{D}(-\Lambda_1) = \sum_{j=1}^{N} \mu_j \Lambda_1^{j-1} > 0.$$

Assume there is a Kolmogorov forcing, $-\Delta F = \Lambda_1 F$ so that there is an exact solution (16) of the main equation (1) with boundary conditions (5) on the first energy shell $p = 1$. Also assume that the energy of the forcing satisfies

$$\|F(t)\| \leq c e^{\alpha t}, \qquad \forall t \geq 0, \tag{18}$$

for some $0 < \alpha < \bar{D}(-\Lambda_1)$ and for some $c > 0$. Then for any solution $\omega = \Delta\psi$ of the IVP, there exists $c_1 > 0$, $c_2 > 0$,

$$\|\omega(t) - \omega_{exact}(t)\| \leq c_1 e^{-c_2 t}, \qquad \forall t \geq 0,$$

where $\omega_{exact} = \Delta\psi_{exact}$, that is the exact solution is globally exponentially stable.

Before proving Theorem 1, we first need to give some basic results that will be used throughout the proof. For sufficiently smooth $\psi \in X$, we define the energy,

$$E(\psi) = \frac{1}{2}\|\mathbf{v}\|^2 = \frac{1}{2}\|\nabla\psi\|^2 = \frac{1}{2}\sum_{\mathbf{k}\in\mathcal{Z}} |\mathbf{k}|^2 |\hat{\psi}_k|^2 = \frac{1}{2}\sum_{\mathbf{k}\in\mathcal{Z}} |\mathbf{k}|^{-2} |\hat{\omega}_k|^2, \tag{19}$$

the enstrophy,

$$\mathcal{E}(\psi) = \frac{1}{2}\|\omega\|^2 = \frac{1}{2}\|\Delta\psi\|^2 = \frac{1}{2}\sum_{\mathbf{k}\in\mathcal{Z}} |\mathbf{k}|^4 |\hat{\psi}_k|^2 = \frac{1}{2}\sum_{\mathbf{k}\in\mathcal{Z}} |\hat{\omega}_k|^2, \tag{20}$$

the functional

$$W_p(\psi) = \mathcal{E}(\psi) - \Lambda_p E(\psi) = \frac{1}{2}\sum_{|\mathbf{k}|^2 \geq \Lambda_1} |\hat{\psi}_{\mathbf{k}}|^2 |\mathbf{k}|^2 (|\mathbf{k}|^2 - \Lambda_p), \tag{21}$$

and the small-scale enstrophy

$$\mathcal{E}_2(\psi) = \mathcal{E}((1 - P_1)\omega) = \frac{1}{2}\sum_{|\mathbf{k}|^2 \geq \Lambda_2} |\hat{\psi}_{\mathbf{k}}|^2 |\mathbf{k}|^4,$$

where P_1 is the projection operator onto the first energy shell Λ_1. When $\psi(t)$ is fixed, we will denote $E(t) = E(\psi(t))$, $\mathcal{E}(t) = \mathcal{E}(\psi(t))$, etc.

Lemma 1. *For any $\psi \in X_1$, we have*

$$\left(1 - \frac{\Lambda_1}{\Lambda_2}\right)\mathcal{E}_2(t) \le W_1(t) \le \mathcal{E}_2(t), \qquad \forall t \ge 0. \tag{22}$$

As a consequence W_1 is positive definite and the small scale enstrophy $\mathcal{E}_2$ stays uniformly bounded for all $t > 0$.

Proof. By definition $W_1(t) \le \mathcal{E}_2(t)$ for all $t \ge 0$ and since

$$W_1 = \frac{1}{2}\sum_{|\mathbf{k}|^2 \ge \Lambda_2} |\hat{\psi}_{\mathbf{k}}|^2 |\mathbf{k}|^2 (|\mathbf{k}|^2 - \Lambda_1),$$

(22) is a consequence of the fact that $|\mathbf{k}|^2 - \Lambda_1 \ge |\mathbf{k}|^2\left(1 - \frac{\Lambda_1}{\Lambda_2}\right)$ whenever $|\mathbf{k}|^2 \ge \Lambda_2$.

Let $c = \left(1 - \frac{\Lambda_1}{\Lambda_2}\right)$, then by (22) we have

$$\begin{aligned} c\mathcal{E}_2(t) \le W_1(t) = W_1(0) &\le \mathcal{E}_2(0) \le c^{-1}W_1(0) \\ &= c^{-1}W_1(t) \le c^{-1}\mathcal{E}_2(t). \end{aligned}$$

or in other words

$$c^2\mathcal{E}_2(0) \le c\mathcal{E}_2(t) \le \mathcal{E}_2(0),$$

which proves that $\mathcal{E}(t)$ stays uniformly bounded in time. □

Lemma 2. *Assume a generalized Kolmogorov forcing on the p-th shell, i.e., $-\Delta F = \Lambda_p F$. Then for a solution $\psi(t) \in X_1$ of 3D-QGPV, the following statements hold true.*

(1) Without dissipation ($\mu_j = 0$, $\forall j = 0, \dots, N$), W_p is conserved, i.e., $W_p(t) = W_p(0)$ for all $t \ge 0$.
(2) If the lowest order term of the dissipation operator (2) satisfies

$$\mu_0 = 0, \qquad \text{for } p \ge 2,$$

or

$$\mu_0 \ge 0, \qquad \text{for } p = 1$$

then we have

$$\frac{dW_p(t)}{dt} \le -2\bar{D}(-\Lambda_p)W_p(t), \qquad \bar{D}(-\Lambda_p) := \sum_{j=1}^{N} \mu_j \Lambda_p^{j-1} \ge 0. \tag{23}$$

Proof. Since $\int_\Omega \beta\psi\phi = \int_\Omega J(\psi,\omega)\phi = 0$ for $\phi = \psi$ or $\phi = \omega$, the β term and the nonlinear term do not contribute to the rate of change of energy and enstrophy. Thanks to (12) and the definition (2) of the dissipation operator, we have

$$\int_\Omega \psi D(\Delta)\psi = \sum_{j=0}^{N} \mu_j \int_\Omega \psi(-\Delta)^j \psi d\Omega = \sum_{\mathbf{k}\in\mathcal{Z}}\sum_{j=0}^{N} \mu_j |\mathbf{k}|^{2j} |\hat{\psi}_{\mathbf{k}}|^2, \tag{24}$$

and

$$\int_\Omega \Delta\psi D(\Delta)\psi = \sum_{j=0}^{N} \mu_j \int_\Omega \Delta\psi(-\Delta)^j \psi d\Omega = -\sum_{\mathbf{k}\in\mathcal{Z}}\sum_{j=0}^{N} \mu_j |\mathbf{k}|^{2j+2} |\hat{\psi}_{\mathbf{k}}|^2.$$

By taking the inner product of the main equation (1) by ψ and $\Delta\psi$, using the above results, and noting that the integral with β term cancels out,

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2}\frac{d}{dt}\|\nabla\psi\|^2 = -\int_\Omega \psi D(\Delta)\psi - \int_\Omega F\psi = -\sum_{\mathbf{k}\in\mathcal{Z}}\sum_{j=1}^{N} \mu_j |\mathbf{k}|^{2j}|\hat{\psi}_{\mathbf{k}}|^2 - \int_\Omega F\psi, \\ \frac{d\mathcal{E}}{dt} &= \frac{1}{2}\frac{d}{dt}\|\Delta\psi\|^2 = \int_\Omega \Delta\psi D(\Delta)\psi + \int_\Omega F\Delta\psi = -\sum_{\mathbf{k}\in\mathcal{Z}}\sum_{j=0}^{N} \mu_j |\mathbf{k}|^{2j+2}|\hat{\psi}_{\mathbf{k}}|^2 + \int_\Omega F\Delta\psi. \end{aligned} \tag{25}$$

Now using the fact,

$$\Lambda_p \int_\Omega F\psi + \int_\Omega F\Delta\psi = 0,$$

we obtain the rate of change of W_p given by

$$\begin{aligned}\frac{dW_p}{dt} &= \Lambda_p \int_\Omega \psi D(\Delta)\psi + \int_\Omega \Delta\psi D(\Delta)\psi \\ &= -\sum_{\mathbf{k}\in\mathcal{Z}}\sum_{j=1}^{N} \mu_j |\mathbf{k}|^{2j}(|\mathbf{k}|^2 - \Lambda_p)|\hat{\psi}_{\mathbf{k}}|^2.\end{aligned} \tag{26}$$

From (26), we also have

$$\frac{dW_p}{dt} = \sum_{j=0}^{N} \mu_j h_j, \qquad h_j = \Lambda_p\|\nabla^j\psi\|^2 - \|\nabla^{j+1}\psi\|^2,$$

where $\nabla = \Delta^{1/2}$. For $j \geq 1$,

$$\begin{aligned}h_j =& \Lambda_p\|\nabla^j\psi\|^2 - \Big(\|\left(\nabla^{j+1} + \Lambda_p\nabla^{j-1}\right)\psi\|^2 \\ & - 2\langle\nabla^{j+1}\psi, \Lambda_p\nabla^{j-1}\psi\rangle - \Lambda_p^2\|\nabla^{j-1}\psi\|^2\Big) \\ =& \Lambda_p\|\nabla^j\psi\|^2 - \|\left(\nabla^{j+1} + \Lambda_p\nabla^{j-1}\right)\psi\|^2 \\ & - 2\Lambda_p\|\nabla^j\psi\|^2 + \Lambda_p^2\|\nabla^{j-1}\psi\|^2 \\ \leq& \Lambda_p h_{j-1}\end{aligned}$$

where we omit $\|\left(\nabla^{j+1} + \Lambda_p\nabla^{j-1}\right)\psi\|^2$ term in the last inequality. Hence by induction,

$$h_j \leq \Lambda_p^{j-1} h_1.$$

We also have by definition that

$$h_1 = -2W_p.$$

Also note that if $p = 1$, then $h_0 = \Lambda_1\|\psi\|^2 - \|\nabla\psi\|^2 \leq 0$ by Poincaré inequality. Hence under the given assumptions on μ_0,

$$\frac{dW_p}{dt} = \mu_0 h_0 + \sum_{j=1}^{N}\mu_j h_j \leq \sum_{j=1}^{N}\mu_j h_j.$$

from which the result follows.

□

Remark. *The biggest hurdle for proving the nonlinear stability in the case of a generalized Kolmogorov forcing is that W_p is in general not positive definite for $p > 1$.*

The following lemma will play a crucial role in the proof of nonlinear stability result.

Lemma 3. *Assume a Kolmogorov forcing and a genuine dissipation $\bar{D}(-\Lambda_1) > 0$ and $\mu_0 \geq 0$. Then for any solution $\psi(t) \in X_1$ of the 3D-QGPV, the small scale enstrophy decays exponentially in time, i.e.*

$$\mathcal{E}_2(t) = \frac{1}{2}\sum_{|\mathbf{k}|^2\geq\Lambda_2}|\hat{\omega}_{\mathbf{k}}|^2 \leq \left(1 - \frac{\Lambda_1}{\Lambda_2}\right)^{-1} e^{-2\bar{D}(-\Lambda_1)t}\mathcal{E}_2(0). \tag{27}$$

Proof. By (23), we have

$$\frac{dW_1(t)}{dt} \leq -2\bar{D}(-\Lambda_1)W_1(t)$$

Using the fact that $W_1(t) \geq 0$ by Lemma 1, the Gronwall's inequality, and (22),

$$\left(1 - \frac{\Lambda_1}{\Lambda_2}\right)\mathcal{E}_2(t) \leq W_1(t) \leq e^{-2\bar{D}(-\Lambda_1)t}W_1(0) \leq e^{-2\bar{D}(-\Lambda_1)t}\mathcal{E}_2(0),$$

and the result follows. □

Lemma 4. *Assume F is a generalized p-Kolmogorov forcing which satisfies,*

$$\|F(t)\| \leq ce^{\alpha t}, \qquad \forall t \geq 0 \tag{28}$$

for some $c > 0$, $\alpha > 0$ and assume also that $D(-\Lambda_p) \geq 0$. Then for the exact solution (16) on the pth shell, there exists $c_0 > 0$ such that

$$\|\omega_{exact}(t)\| \leq c_0 e^{\alpha t}, \qquad \forall t \geq 0. \tag{29}$$

Proof. The linear PDE satisfied by the exact solution is

$$\partial_t \omega_{\text{exact}} + \beta \partial_x \psi_{\text{exact}} = -\frac{D(-\Lambda_p)}{\Lambda_p} \omega_{\text{exact}} + F(t, x, y, z), \tag{30}$$

since the nonlinearity vanishes

$$J(\psi_{\text{exact}}, \omega_{\text{exact}}) = -\Lambda_p J(\psi_{\text{exact}}, \psi_{\text{exact}}) = 0.$$

Taking the inner product of (30) with $\omega_{\text{exact}} = -\Lambda_p \psi_{\text{exact}}$, we find that

$$\begin{aligned} \frac{1}{2} \frac{d\|\omega_{\text{exact}}\|^2}{dt} &\leq -\frac{D(-\Lambda_p)}{\Lambda_p} \|\omega_{\text{exact}}\|^2 + \|F\| \|\omega_{\text{exact}}\| \\ &\leq -\frac{D(-\Lambda_p)}{2\Lambda_p} \|\omega_{\text{exact}}\|^2 + \frac{\Lambda_p}{2D(-\Lambda_p)} \|F\|^2, \end{aligned}$$

which implies by the Gronwall's inequality that

$$\|\omega_{\text{exact}}\|^2 \leq \|\omega_{\text{exact}}(0)\|^2 e^{-\frac{D(-\Lambda_p)}{\Lambda_p} t} + \frac{\Lambda_p}{D(-\Lambda_p)} \int_0^t e^{\frac{D(-\Lambda_p)}{\Lambda_p}(s-t)} \|F(s)\|^2 ds. \tag{31}$$

Now (29) is a consequence of the above estimate. □

Note that the effect of the initial condition decays exponentially (when $D(-\Lambda_p) > 0$) for the exact solution on the pth shell and the growth rate of this exact solution is bounded by the growth rate of the forcing.

Now we are ready for the proof of Theorem 1.

Proof of Theorem 1. Since

$$\|\omega(t) - \omega_{\text{exact}}(t)\| \leq \|P_1 \omega(t) - \omega_{\text{exact}}(t)\| + \|(I - P_1)\omega(t)\|$$

and $\mathcal{E}_2(t)^{1/2} = \|(I - P_1)\omega(t)\|$ decays exponentially by (27), it remains to show that

$$\|P_1 \omega(t) - \omega_{\text{exact}}(t)\|$$

also decays exponentially. Using the fact that $P_1 F = F$, the PDE satisfied by the projection $P_1 \omega$ becomes

$$\partial_t \Delta P_1 \psi + P_1 J(\psi, \omega) + \beta \partial_x P_1 \psi = D(\Delta) P_1 \psi + F(t, x, y, z). \tag{32}$$

Let

$$\delta\omega(t) = P_1 \omega(t) - \omega_{\text{exact}}(t).$$

Since $P_1 \omega_{\text{exact}} = \omega_{\text{exact}}$, we have $P_1 \delta\omega = \delta\omega$ and

$$D(\Delta)\delta\psi = D(-\Lambda_1)\delta\psi = -\kappa \delta\omega,$$

where

$$\kappa = \frac{D(-\Lambda_1)}{\Lambda_1} > 0.$$

Hence by (30) and (32), the linear PDE satisfied by $\delta\omega$ is

$$\partial_t \delta\omega + \beta \partial_x \delta\psi = D(\Delta)\delta\psi - P_1 J(\psi, \omega),$$

Testing the above equation with $\delta\omega$, we get

$$\frac{1}{2} \frac{d}{dt} \|\delta\omega\|^2 = -\kappa \|\delta\omega\|^2 - (P_1 J(\psi, \omega), \delta\omega). \tag{33}$$

We will prove later that for any γ such that,

$$0 < \gamma < \bar{D}(-\Lambda_1) - \alpha, \tag{34}$$

where $\alpha < \bar{D}(-\Lambda_1)$ is as given in the assumptions, we have

$$\|P_1 J(\psi, \omega)\| \leq c_3 e^{-\gamma t} + c_4 e^{-\bar{D}(-\Lambda_1)t} \|\delta\omega\|. \tag{35}$$

In addition to (34), let us also choose $\gamma > 0$ small enough so that

$$\gamma < \kappa. \tag{36}$$

Then using the Cauchy-Schwarz inequality in (33) and (35), we obtain

$$\frac{1}{2} \frac{d}{dt} \|\delta\omega\|^2 \leq -\kappa \|\delta\omega\|^2 + c_3 e^{-\gamma t} \|\delta\omega\| + c_4 e^{-\bar{D}(-\Lambda_1)t} \|\delta\omega\|^2.$$

Since $\|\delta\omega\| \le 1 + \|\delta\omega\|^2$,

$$\frac{d}{dt}\|\delta\omega\|^2 \le 2(-\kappa + c_3 e^{-\gamma t} + c_4 e^{-\bar{D}(-\Lambda_1)t})\|\delta\omega\|^2 + 2c_3 e^{-\gamma t}.$$

Now we choose T large enough so that

$$2(-\kappa + c_3 e^{-\gamma t} + c_4 e^{-\bar{D}(-\Lambda_1)t}) \le -\kappa < 0, \qquad \forall t \ge T.$$

Hence

$$\frac{d}{dt}\|\delta\omega\|^2 \le -\kappa\|\delta\omega\|^2 + 2c_3 e^{-\gamma t}, \qquad \forall t \ge T.$$

This implies by Gronwall's inequality that

$$\|\delta\omega(t)\|^2 \le e^{-\kappa(t-s)}\|\delta\omega(0)\|^2 + \frac{2c_3}{\kappa-\gamma}(1 - e^{(\kappa-\gamma)(s-t)})e^{-\gamma t}, \qquad \forall t \ge s \ge T.$$

Since $\gamma < \kappa$ by (36), the above estimate shows that $\|P_1\omega(t) - \omega_{\text{exact}}(t)\|$ indeed decays exponentially. The proof is finished.

Proof of the claim (35). Let us denote

$$\psi = \sum_{|\mathbf{l}|^2 \ge \Lambda_1} \frac{\hat{\omega}_{\mathbf{l}}}{-|\mathbf{l}|^2} e_{\mathbf{l}}, \qquad \omega = \sum_{|\mathbf{m}|^2 \ge \Lambda_1} \hat{\omega}_{\mathbf{m}} e_{\mathbf{m}}.$$

Then

$$J(\psi,\omega) = \sum_{|\mathbf{l}|^2 \ge \Lambda_1, |\mathbf{m}|^2 \ge \Lambda_1} \frac{\hat{\omega}_{\mathbf{l}}}{-|\mathbf{l}|^2} \hat{\omega}_{\mathbf{m}} J(e_{\mathbf{l}}, e_{\mathbf{m}}),$$

and

$$P_1 J(\psi,\omega) = \sum_{|\mathbf{k}|^2 = \Lambda_1} g_{\mathbf{k}}(t) e_{\mathbf{k}}(\mathbf{x}), \tag{37}$$

where

$$g_{\mathbf{k}} = (J(\psi,\omega), e_{\mathbf{k}}). \tag{38}$$

Let us denote

$$\mathbf{k} = \left(\frac{2k_x\pi}{L_x}, \frac{2k_y\pi}{L_y}, \frac{k_z S\pi}{H}\right), \qquad \mathbf{k}' = (\frac{2k_x\pi}{L_x}, \frac{2k_y\pi}{L_y}), \qquad \mathbf{k}^\perp = (-\frac{2k_y\pi}{L_y}, \frac{2k_x\pi}{L_x}),$$

and

$$\int_\Omega e_{\mathbf{l}} e_{\mathbf{m}} e_{\mathbf{k}}^* = \delta_{\mathbf{l}'+\mathbf{m}',\mathbf{k}'} c_{l_z,m_z,k_z}, \tag{39}$$

$$c_{l,m,k} = \frac{2\sqrt{2}}{|\Omega|^{1/2}\pi} \int_0^\pi \sin lz \sin mz \sin kz dz, \qquad l \ge 1,\ m \ge 1,\ k \ge 1 \tag{40}$$

so that

$$(J(e_{\mathbf{l}}, e_{\mathbf{m}}), e_{\mathbf{k}}) = \mathbf{l}^\perp \cdot \mathbf{m}' \int_\Omega e_{\mathbf{l}} e_{\mathbf{m}} e_k^*, \tag{41}$$

By (39), (37) and (41),

$$g_{\mathbf{k}} = \sum_{\mathbf{l}'+\mathbf{m}'=\mathbf{k}'} \frac{\mathbf{l}^\perp \cdot \mathbf{m}'}{|\mathbf{l}|^2} \hat{\omega}_{\mathbf{l}} \hat{\omega}_{\mathbf{m}} c_{l_z,m_z,k_z}, \qquad |\mathbf{k}|^2 = \Lambda_1. \tag{42}$$

Let Λ_1' be the first eigenvalue of the horizontal Laplacian, i.e.

$$\Lambda_1' = \Lambda_1 - \frac{\pi^2 S^2}{H^2} = 4\pi^2 \min\left\{\frac{1}{L_x^2}, \frac{1}{L_y^2}\right\}. \tag{43}$$

Then by (10) and (43), for $\mathbf{k} = \left(\frac{2k_x\pi}{L_x}, \frac{2k_y L_y}{\pi}, \frac{k_z S\pi}{H}\right)$, we have

$$|\mathbf{k}|^2 = \Lambda_1 \implies k_z = 1 \text{ and } |\mathbf{k}'|^2 = \Lambda_1'. \tag{44}$$

By (44), (42) becomes

$$g_{\mathbf{k}} = \sum_{\mathbf{l}'+\mathbf{m}'=\mathbf{k}'} d_{\mathbf{l},\mathbf{m}}, \qquad |\mathbf{k}|^2 = \Lambda_1, \tag{45}$$

where

$$d_{\mathbf{l},\mathbf{m}} = \frac{\mathbf{l}^\perp \cdot \mathbf{m}'}{|\mathbf{l}|^2} \hat{\omega}_{\mathbf{l}} \hat{\omega}_{\mathbf{m}} c_{l_z,m_z,1}.$$

Notice that

$$\sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}'\\|\mathbf{l}|=|\mathbf{m}|}} d_{\mathbf{l},\mathbf{m}} = 0,$$

since for $|\mathbf{l}| = |\mathbf{m}|$, $d_{\mathbf{l},\mathbf{m}} = -d_{\mathbf{m},\mathbf{l}}$ because $\mathbf{l}^\perp \cdot \mathbf{m}' = -\mathbf{m}^\perp \cdot \mathbf{l}'$ and $c_{l_z,m_z,1} = c_{m_z,l_z,1}$. Hence

$$g_{\mathbf{k}} = \sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}',\\|\mathbf{l}|\neq|\mathbf{m}|}} d_{\mathbf{l},\mathbf{m}}, \qquad |\mathbf{k}|^2 = \Lambda_1. \tag{46}$$

The sum above can be separated into three parts

$$g_{\mathbf{k}} = \sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}',\\|\mathbf{l}|^2=\Lambda_1,|\mathbf{m}|^2\geq\Lambda_2}} d_{\mathbf{l},\mathbf{m}} + \sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}',\\|\mathbf{l}|^2\geq\Lambda_2,|\mathbf{m}|^2=\Lambda_1}} d_{\mathbf{l},\mathbf{m}} + \sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}',\\|\mathbf{l}|^2\geq\Lambda_2,|\mathbf{m}|^2\geq\Lambda_2\\|\mathbf{l}|\neq|\mathbf{m}|}} d_{\mathbf{l},\mathbf{m}} \tag{47}$$

By swapping $\mathbf{m}$ and $\mathbf{l}$ in the second sum, the first two sums in the right hand side of (47) can be combined as

$$\sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}',\\|\mathbf{l}|^2=\Lambda_1,|\mathbf{m}|^2\geq\Lambda_2}} d_{\mathbf{l},\mathbf{m}} + \sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}',\\|\mathbf{l}|^2\geq\Lambda_2,|\mathbf{m}|^2=\Lambda_1}} d_{\mathbf{l},\mathbf{m}} = \sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}',\\|\mathbf{l}|^2=\Lambda_1,|\mathbf{m}|^2\geq\Lambda_2}} C_{\mathbf{k},\mathbf{l}}, \tag{48}$$

where we define

$$C_{\mathbf{k},\mathbf{l}} = \sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}',\\|\mathbf{m}|^2\geq\Lambda_2}} \mathbf{l}^\perp \cdot \mathbf{m}' \left(\frac{1}{\Lambda_1} - \frac{1}{|\mathbf{m}|^2}\right) \hat{\omega}_{\mathbf{m}} c_{1,m_z,1}, \qquad |\mathbf{k}|^2 = |\mathbf{l}|^2 = \Lambda_1. \tag{49}$$

If we also define

$$H_{\mathbf{k}} = \sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}',\\|\mathbf{l}|^2\geq\Lambda_2,|\mathbf{m}|^2\geq\Lambda_2\\|\mathbf{l}|\neq|\mathbf{m}|}} \frac{\mathbf{l}^\perp \cdot \mathbf{m}'}{|\mathbf{l}|^2} \hat{\omega}_{\mathbf{l}} \hat{\omega}_{\mathbf{m}} c_{l_z,m_z,1}, \qquad |\mathbf{k}|^2 = \Lambda_1, \tag{50}$$

then by (47), (48), (49) and (50) we can write

$$g_{\mathbf{k}} = H_{\mathbf{k}} + \sum_{|\mathbf{l}|^2=\Lambda_1} C_{\mathbf{k},\mathbf{l}} \hat{\omega}_{\mathbf{l}}. \tag{51}$$

We will prove later that for some $c_5 > 0$ depending on $|\Omega|$ and $\|\omega(0)\|$,

$$\sum_{|\mathbf{k}|^2=\Lambda_1} \left| \sum_{|\mathbf{l}|^2=\Lambda_1} C_{\mathbf{k},\mathbf{l}} \hat{\omega}_{\mathbf{l}} \right|^2 \leq c_5 e^{-2\bar{D}(-\Lambda_1)t} \left(\|\delta\omega\|^2 + \|\omega_{\text{exact}}\|^2\right), \tag{52}$$

and for some $c_6 > 0$ also depending on $|\Omega|$ and $\|\omega(0)\|$ that

$$\sum_{|\mathbf{k}|^2=\Lambda_1} |H_{\mathbf{k}}|^2 \leq c_6 e^{-4\bar{D}(-\Lambda_1)t}. \tag{53}$$

By (37) and the orthonormality (9) of the basis vectors $e_{\mathbf{k}}$, we have

$$\|P_1 J(\psi, \omega)\|^2 = \sum_{|\mathbf{k}|^2=\Lambda_1} |g_{\mathbf{k}}|^2. \tag{54}$$

Now by using the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$, (51), (52), (53) and (54),

$$\begin{aligned} \|P_1 J(\psi, \omega)\|^2 &\leq 2 \sum_{|\mathbf{k}|^2=\Lambda_1} |H_{\mathbf{k}}|^2 + 2 \sum_{|\mathbf{k}|^2=\Lambda_1} \left| \sum_{|\mathbf{l}|^2=\Lambda_1} C_{\mathbf{k},\mathbf{l}} \hat{\omega}_{\mathbf{l}} \right|^2 \\ &\leq 2c_6 e^{-4\bar{D}(-\Lambda_1)t} + 2c_5 e^{-2\bar{D}(-\Lambda_1)t} \left(\|\delta\omega\|^2 + \|\omega_{\text{exact}}\|^2\right). \end{aligned} \tag{55}$$

By the assumption (18), the estimate (29) on the growth of the L^2-norm of the exact solution is valid, i.e. for any $0 < \alpha < \bar{D}(-\Lambda_1)$, there exists $c_0 > 0$ such that

$$\|\omega_{\text{exact}}(t)\| \leq c_0 e^{\alpha t}, \qquad \forall t \geq 0. \tag{56}$$

By (55), (56) and the elementary inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a \geq 0$, $b \geq 0$,

$$\|P_1 J(\psi, \omega)\| \leq \sqrt{2c_6} e^{-2\bar{D}(-\Lambda_1)t} + \sqrt{2c_5} e^{-\bar{D}(-\Lambda_1)t} \|\delta\omega\| + \sqrt{2c_5} c_0 e^{-\bar{D}(-\Lambda_1)t} e^{\alpha t} \qquad \forall t \geq 0.$$

Thus the claim (35) is proved by choosing $c_3 = \max\left\{\sqrt{2c_6}, \sqrt{2c_5}c_0\right\}$, $c_4 = \sqrt{2c_5}$, and $0 < \gamma < \bar{D}(-\Lambda_1) - \alpha$ in the above . □

Proof of the Claim (52). First note that, from (40), the following sum can be exactly computed as

$$\sum_{m_z \geq 1} |c_{1,m_z,1}| = \frac{2\sqrt{2}}{|\Omega|^{1/2}\pi} \sum_{m_z \geq 1} |2\frac{(-1+(-1)^{m_z})}{m_z(m_z^2-4)}| = \frac{2\sqrt{2}}{|\Omega|^{1/2}\pi}\frac{5}{3}. \tag{57}$$

Next, the condition

$$\mathbf{l}' + \mathbf{m}' = \mathbf{k}', \quad |\mathbf{k}'|^2 = \Lambda_1', \quad |\mathbf{l}|^2 = \Lambda_1, \quad |\mathbf{m}|^2 \geq \Lambda_2,$$

implies

$$|\mathbf{m}'| = |\mathbf{k}' - \mathbf{l}'| \leq |\mathbf{k}'| + |\mathbf{l}'| = 2\sqrt{\Lambda_1'},$$

and

$$|\mathbf{l}^\perp \cdot \mathbf{m}'| \leq 2\Lambda_1' \leq 2\Lambda_1, \tag{58}$$

since $|\mathbf{l}^\perp|^2 = \Lambda_1'$. Recalling the estimate (27) on the decay of the small scale enstrophy, we have for all $t \geq 0$,

$$|\hat{\omega}_{\mathbf{m}}(t)| \leq \mathcal{E}_2(t)^{1/2} \leq \left(1 - \frac{\Lambda_1}{\Lambda_2}\right)^{-1/2} e^{-\bar{D}(-\Lambda_1)t}\|\omega(0)\|, \qquad |\mathbf{m}|^2 \geq \Lambda_2. \tag{59}$$

Combining (57), (58) and (59) in (49) yields

$$|C_{\mathbf{k},\mathbf{l}}(t)| \leq \sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}',\\ |\mathbf{m}|^2 \geq \Lambda_2}} |\mathbf{l}^\perp \cdot \mathbf{m}'|\frac{1}{\Lambda_1}|\hat{\omega}_{\mathbf{m}}||c_{1,m_z,1}|, \qquad |\mathbf{k}|^2 = |\mathbf{l}|^2 = \Lambda_1,$$

we get

$$|C_{\mathbf{k},\mathbf{l}}|^2 \leq c_7 e^{-2\bar{D}(-\Lambda_1)t}, \qquad |\mathbf{k}|^2 = |\mathbf{l}|^2 = \Lambda_1,$$

where

$$c_7 = \frac{800}{9\pi^2}\left(\frac{\Lambda_2}{\Lambda_2 - \Lambda_1}\right)\frac{1}{|\Omega|}\|\omega(0)\|^2.$$

Hence,

$$\sum_{|\mathbf{l}|^2 = \Lambda_1} |C_{\mathbf{k},\mathbf{l}}|^2 \leq c_8 c_7 e^{-2\bar{D}(-\Lambda_1)t}, \qquad |\mathbf{k}|^2 = \Lambda_1, \tag{60}$$

where

$$c_8 = \sum_{|\mathbf{l}|^2 = \Lambda_1} 1 = \#\left\{|\mathbf{l}|^2 = \Lambda_1\right\}, \tag{61}$$

is basically the multiplicity of the first eigenvalue and is finite. By noting that

$$\sum_{|\mathbf{l}|^2 = \Lambda_1} |\hat{\omega}_{\mathbf{l}}|^2 = \|P_1\omega\|^2 = \|\delta\omega + \omega_{\text{exact}}\|^2 \leq 2\|\delta\omega\|^2 + 2\|\omega_{\text{exact}}\|^2, \tag{62}$$

and using the Cauchy-Schwarz inequality

$$\sum_{|\mathbf{k}|^2 = \Lambda_1} \left| \sum_{|\mathbf{l}|^2 = \Lambda_1} C_{\mathbf{k},\mathbf{l}}\hat{\omega}_{\mathbf{l}} \right|^2 \leq \sum_{|\mathbf{k}|^2 = \Lambda_1} \left(\sum_{|\mathbf{l}|^2 = \Lambda_1} |C_{\mathbf{k},\mathbf{l}}|^2\right)\left(\sum_{|\mathbf{l}|^2 = \Lambda_1} |\hat{\omega}_{\mathbf{l}}|^2\right),$$

(52) follows by combining (60), (62) and letting $c_5 = 2c_7c_8^2$.

□

Proof of the Claim (53). Since $|\mathbf{l}^\perp| = |\mathbf{l}'| \geq \sqrt{\Lambda_1'}$, $|\mathbf{k}'| = \sqrt{\Lambda_1'}$, we have

$$|\mathbf{m}'| = |\mathbf{k}' - \mathbf{l}'| \leq |\mathbf{l}'| + \sqrt{\Lambda_1'} \leq 2|\mathbf{l}'|,$$

and

$$\left|\frac{\mathbf{l}^\perp \cdot \mathbf{m}'}{|\mathbf{l}|^2}\right| \leq \frac{|\mathbf{l}'||\mathbf{m}'|}{|\mathbf{l}|^2} \leq 2, \tag{63}$$

From (40), we can obtain by direct computation that

$$|c_{l_z,m_z,1}|\frac{|\Omega|^{1/2}\pi}{2\sqrt{2}} = |\frac{2(1+(-1)^{l_z+m_z})l_z m_z}{(l_z^2 - m_z^2)^2 - 2(l_z^2 + m_z^2) + 1}| \leq \frac{4l_z m_z}{2l_z^2 + 2m_z^2} = 1. \tag{64}$$

Using (63) and (64) in (50), we obtain

$$|H_{\mathbf{k}}| \leq \sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}',\\ |\mathbf{l}|^2\geq\Lambda_2, |\mathbf{m}|^2\geq\Lambda_2\\ |\mathbf{l}|\neq|\mathbf{m}|}} \left|\frac{\mathbf{l}^{\perp}\cdot\mathbf{m}'}{|\mathbf{l}|^2}\right| |\hat{\omega}_{\mathbf{l}}||\hat{\omega}_{\mathbf{m}}||c_{l_z,m_z,1}| \leq c_9 \sum_{\substack{\mathbf{l}'+\mathbf{m}'=\mathbf{k}',\\ |\mathbf{l}|^2\geq\Lambda_2, |\mathbf{m}|^2\geq\Lambda_2\\ |\mathbf{l}|\neq|\mathbf{m}|}} |\hat{\omega}_{\mathbf{l}}||\hat{\omega}_{\mathbf{m}}|, \tag{65}$$

where

$$c_9 = \frac{4\sqrt{2}}{|\Omega|^{1/2}\pi}.$$

Using the Cauchy-Schwarz inequality in the last sum in (65) and the estimate (27) for $\mathcal{E}_2$, we get

$$|H_{\mathbf{k}}| \leq c_9 \left(\sum_{\substack{|\mathbf{m}'|\geq\Lambda_2'\\ l_z\geq 1}} |\hat{\omega}_{\mathbf{k}'-\mathbf{m}',l_z}|^2\right)^{1/2} \left(\sum_{|\mathbf{m}|^2\geq\Lambda_2} |\hat{\omega}_{\mathbf{m}}|^2\right)^{1/2} \leq c_9 \sum_{|\mathbf{m}|^2\geq\Lambda_2} |\hat{\omega}_{\mathbf{m}}(t)|^2 = c_9\mathcal{E}_2(t) \leq c_9 c_{10} e^{-2\bar{D}(-\Lambda_1)t},$$

where

$$c_{10} = \left(1-\frac{\Lambda_1}{\Lambda_2}\right)^{-1} \|\omega(0)\|^2.$$

The claim (53) follows from the above estimate by defining

$$c_6 = c_8 c_9^2 c_{10}^2,$$

where c_8 is defined in (61). □

□

4. Existence of selective decay states

In the previous section, we have shown that the exact solution (16) to equation (1) is nonlinearly stable in the presence of a Kolmogorov external forcing which satisfies the constraint $\|F(t)\| \leq ce^{\alpha t}$ for some $0 < \alpha < D(-\Lambda_1)$, $c > 0$ and for all $t \geq 0$. Apparently, if $F = 0$ and in the absence of dissipative term, the exact solution reduces to

$$\psi_{\text{exact}} = \sum_{|\mathbf{k}|^2=\Lambda_p} \hat{\psi}_{\text{exact},\mathbf{k}}(t) e_{\mathbf{k}}(x,y,z) e^{i\omega(\mathbf{k})t}, \qquad \omega(\mathbf{k}) = \frac{2\pi k_x \beta}{L_x|\mathbf{k}|^2},$$

where

$$\frac{d\hat{\psi}_{\text{exact},\mathbf{k}}(t)}{dt} = \frac{D(-\Lambda_p)}{-\Lambda_p}\hat{\psi}_{\text{exact},\mathbf{k}}$$

and the effect of initial condition vanishes exponentially when $D(-\Lambda_p) > 0$. Thus there exists infinitely many exact solutions on each energy shell $|\mathbf{k}|^2 = \Lambda_p$. In the case $\beta = 0$, these exact solutions are time-independent generalized Taylor vortices, while for the case $\beta \neq 0$, the solutions are Rossby waves.

In this section, following the idea in [5], we show that in this situation the normalized exact solution is a selective state for very large t. Roughly speaking, a selective state is a solution which is approached by solutions of the 3D-quasi geostrophic equations after a long time, and the state minimize the enstrophy for a given energy. For more detail definition of selective state, we refer reader to the reference [5].

The existence of selective state based on the numerical results [24,25] which suggest consistently more rapid decay of the enstrophy over the energy, this allows us to introduce the quotient of the enstrophy over energy. That is, we define the functional

$$R(t) = \frac{\mathcal{E}(t)}{E(t)} = \frac{\sum_{\mathbf{k}\in\mathcal{Z}}|\mathbf{k}|^4|\hat{\psi}_{\mathbf{k}}|^2}{\sum_{\mathbf{k}\in\mathcal{Z}}||\mathbf{k}|^2\hat{\psi}_{\mathbf{k}}|^2},$$

where E and $\mathcal{E}$ are the energy and enstrophy functionals defined by (19) and (20). In what follows, we will analyze the asymptotical property of the function R by which one can search for the selective state.

Lemma 5. *Assume there is no forcing and the dissipation satisfies $\mu_j > 0$ for some $j \geq 2$ and $\mu_0 \leq 0$. Then we have*

$$\frac{dR(t)}{dt} = \frac{d}{dt}\left(\frac{\mathcal{E}(t)}{E(t)}\right) \leq 0.$$

Proof. Recalling that the rate of change of energy and enstrophy is given by (25) and taking $F = 0$, we can easily obtain the rate of change in R with time t as

$$\begin{aligned}\frac{dR}{dt} &= \frac{\mathcal{E}'(t)E(t) - \mathcal{E}(t)E'(t)}{E(t)^2}\\ &= \frac{-1}{E(t)}\sum_{\mathbf{k}\in\mathcal{Z}}\mu_0|\hat{\psi}_{\mathbf{k}}|^2\left(|\mathbf{k}|^2 - R(t)\right) + \frac{-1}{E(t)}\sum_{i=1}^{N}\sum_{\mathbf{k}\in\mathcal{Z}}\mu_i|\hat{\psi}_{\mathbf{k}}|^2|\mathbf{k}|^{2i}\left(|\mathbf{k}|^2 - R(t)\right)\\ &= I_1 + I_2.\end{aligned} \tag{66}$$

Note that for any integer $i \geq 2$,

$$\begin{aligned}|\mathbf{k}|^{2i}\left(|\mathbf{k}|^2 - R(t)\right) &= |\mathbf{k}|^{2i+2} - 2R|\mathbf{k}|^{2i} + R^2|\mathbf{k}|^{2i-2} - R^2|\mathbf{k}|^{2i-2} + R|\mathbf{k}|^{2i}\\ &= \left(|\mathbf{k}|^{i+1} - R|\mathbf{k}|^{i-1}\right)^2 + R|\mathbf{k}|^{2i-2}\left(|\mathbf{k}|^2 - R\right)\\ &= |\mathbf{k}|^{2(i-1)}(|\mathbf{k}|^2 - R)^2 + R|\mathbf{k}|^{2(i-2)}(|\mathbf{k}|^2 - R)^2 + R^2\left(|\mathbf{k}|^{2i-2} - R|\mathbf{k}|^{2i-4}\right)\end{aligned} \tag{67}$$

Hence by induction on (67), for any integer $i \geq 2$, we have

$$|\mathbf{k}|^{2i}\left(|\mathbf{k}|^2 - R(t)\right) = R^{i-1}\left(|\mathbf{k}|^4 - R|\mathbf{k}|^2\right) + \sum_{s=0}^{i-2}R^s|\mathbf{k}|^{2(i-1-s)}\left(|\mathbf{k}|^2 - R\right)^2. \tag{68}$$

Note also that

$$\sum_{\mathbf{k}\in\mathcal{Z}}|\hat{\psi}_{\mathbf{k}}|^2\left(|\mathbf{k}|^4 - R|\mathbf{k}|^2\right) = \mathcal{E} - RE = 0. \tag{69}$$

Plugging (68) into the definition of I_2 given in (66) and using (69) twice

$$\begin{aligned}I_2 &= \frac{-1}{E(t)}\sum_{i=1}^{N}\sum_{\mathbf{k}\in\mathcal{Z}}\mu_i|\hat{\psi}_{\mathbf{k}}|^2|\mathbf{k}|^{2i}\left(|\mathbf{k}|^2 - R(t)\right)\\ &= \frac{-1}{E(t)}\sum_{i=2}^{N}\sum_{\mathbf{k}\in\mathcal{Z}}\mu_i|\hat{\psi}_{\mathbf{k}}|^2|\mathbf{k}|^{2i}\left(|\mathbf{k}|^2 - R(t)\right)\\ &= \frac{-1}{E(t)}\sum_{i=2}^{N}\sum_{\mathbf{k}\in\mathcal{Z}}\mu_i|\hat{\psi}_{\mathbf{k}}|^2\left(R^{i-1}\left(|\mathbf{k}|^4 - R|\mathbf{k}|^2\right) + \sum_{s=0}^{i-2}R^s|\mathbf{k}|^{2(i-1-s)}\left(|\mathbf{k}|^2 - R\right)^2\right)\\ &= \frac{-1}{E(t)}\sum_{i=2}^{N}\sum_{\mathbf{k}\in\mathcal{Z}}\mu_i|\hat{\psi}_{\mathbf{k}}|^2\sum_{s=0}^{i-2}R^s|\mathbf{k}|^{2(i-1-s)}\left(|\mathbf{k}|^2 - R\right)^2 \leq 0\end{aligned}$$

Since $\mu_0 \leq 0$ and $|\mathbf{k}|^2 \geq \Lambda_1$ for $\mathbf{k} \in \mathcal{Z}$, by (69) we have

$$I_1 = \frac{-\mu_0}{E(t)}\sum_{\mathbf{k}\in\mathcal{Z}}|\hat{\psi}_{\mathbf{k}}|^2|\mathbf{k}|^{-2}(|\mathbf{k}|^4 - R|\mathbf{k}|^2) \leq \frac{-\mu_0}{E(t)}\sum_{\mathbf{k}\in\mathcal{Z}}|\hat{\psi}_{\mathbf{k}}|^2\frac{1}{\Lambda_1}(|\mathbf{k}|^4 - R|\mathbf{k}|^2) = 0.$$

□

Lemma 6. *Under the conditions of Lemma 5, the limit*

$$\lim_{t\to+\infty} R(t) = R^*, \tag{70}$$

must exist. Moreover, R^* is an eigenvalue of the operator

$$\Delta = \partial_{xx} + \partial_{yy} + S\partial_{zz}.$$

Proof. Lemma 5 shows that $R(t)$ is a monotone non-increasing function which is bounded from below and the limit (70) must exist. In fact

$$\Lambda_1 \leq R(t) \leq R(0), \qquad \forall t \geq 0. \tag{71}$$

Let us define

$$\alpha = \min_{\mathbf{k}\in\mathcal{Z}}\left||\mathbf{k}|^2 - R(t)\right|.$$

To show that R^* is an eigenvalue of the Laplacian, it suffices to show that

$$\lim_{t\to+\infty} \alpha(t) = 0. \tag{72}$$

By assumption $\mu_j > 0$ for some $j \geq 2$. From the proof of previous lemma,

$$+\infty > R(0) - R^* = \int_0^\infty -\frac{dR}{dt}dt \geq \int_0^\infty \frac{\sum_{\mathbf{k}\in\mathcal{Z}} \mu_j |\hat{\psi}_{\mathbf{k}}|^2 |\mathbf{k}|^{2j-2}(|\mathbf{k}|^2 - R)^2}{\sum_{\mathbf{k}\in\mathcal{Z}} |\mathbf{k}|^2 |\hat{\psi}_k|^2} dt > \mu_j \Lambda_p^{j-2} \int_0^\infty \alpha^2(t)dt,$$

which shows that (72) must hold true. □

Following the same proof as in Section 3.6 of [1], the following theorem can be proved. The theorem states that the energy-normalized solution of the unforced and dissipated system will come sufficiently close to an eigenvector of the Laplacian infinitely many times in any L^p-norm $1 \leq p < 6$ as $t \to \infty$.

Theorem 2. *For any initial condition, in the absence of an external forcing and $\mu_j > 0$ for some $j \geq 2$ and $\mu_0 \leq 0$, then there exists a eigenvalue value Λ_{k_o} of the Laplacian such that*

$$\lim_{t\to+\infty} \frac{\mathcal{E}(t)}{E(t)} = \Lambda_{k_o}.$$

Furthermore, for any increasing sequence of times $\{t_j\}_{j=1}^{+\infty}$ such that $t_j \to +\infty$, there exists a subsequence $\{t_{j_k}\}_{k=1}^{+\infty}$ and a selective decay state ϕ^* such that the solution $\psi(t, x, y, z)$ to (1) satisfying

$$\lim_{k\to+\infty} \|\nabla(\psi(t_{j_k}, x, y, z)/\sqrt{E(t_{j_k})}) - \nabla\phi^*(x, y, z)\|_{L^p} = 0$$

where $\phi^*(x, y, z)$ is the corresponding eigenvector of the Laplacian and $1 \leq p < 6$.

We can prove a statement stronger than the previous result on the asymptotic behavior. In fact, the below theorem shows that any energy-normalized solution of the unforced and dissipated system will tend in time and in the L^2 norm to an eigenvector of the Laplace operator with corresponding eigenvalue given by the limit of the functional $R(t)$.

Theorem 3. *For any initial condition, in the absence of an external forcing and under the same conditions as in the preceding theorem, there exists an eigenvalue value Λ_{k_o} of the Laplacian such that*

$$\lim_{t\to+\infty} \frac{\mathcal{E}(t)}{E(t)} = \Lambda_{k_o}.$$

Furthermore, there exists an exact solution ϕ_{k_o} to (1), whose expression is

$$\phi = \sum_{|\mathbf{k}|^2=\Lambda_{k_o}} A_{\mathbf{k}}(0) e^{\frac{i2\pi k_x \beta t}{L_x \Lambda_{k_o}}} e_{\mathbf{k}},$$

such that

$$\lim_{t\to+\infty} \|\nabla(\psi(t, x, y, z)/\sqrt{E(t)}) - \nabla\phi\| = 0.$$

Proof. Let us define

$$X_1 = \left\{\psi \in W^{2,2}(\Omega) | \psi \text{ satisfies } (5), (6)\right\}. \tag{73}$$

and the subspace E_{k_o} of X_1 as

$$E_{k_o} = \{ \sum_{|\mathbf{k}|^2=\Lambda_{k_o}} a_{\mathbf{k}} e_{\mathbf{k}}(x, y, z) \mid \overline{a_{k_x,k_y,k_z}(t)} = a_{-k_x,-k_y,k_z}(t)\}.$$

And, let P be the mapping such that

$$P : X_1 \to E_{k_o}^\perp, \qquad P\psi = \psi - \sum_{|\mathbf{k}|^2=\Lambda_{k_o}} \psi_{\mathbf{k}} e_{\mathbf{k}} = \sum_{|\mathbf{k}|^2\neq\Lambda_{k_o}} \psi_{\mathbf{k}} e_{\mathbf{k}}$$

Then, to show the theorem, at first, we need to show that

$$P\left(\nabla \frac{\psi(t, x, y, z)}{\sqrt{E(t)}}\right) = P\nabla\psi^* \to 0 \text{ in } L^2(\Omega), \qquad \text{as } t \to \infty.$$

For this purpose, let us derive some estimates first. Without loss of generality we assume $\mu_2 > 0$ as the argument below can be extended to the case $\mu_j > 0$ for some $j \geq 2$. Then, from (25), we have

$$\begin{aligned}\frac{dE}{dt} &= -\sum_{i=1}^{N}\sum_{\mathbf{k}\in\mathcal{Z}} \mu_i |\hat{\psi}_{\mathbf{k}}|^2 |\mathbf{k}|^{2i} \\ &= -\sum_{\mathbf{k}\in\mathcal{Z}} \mu_2 |\hat{\psi}_{\mathbf{k}}|^2 |\mathbf{k}|^4 \leq -\mu_2 \frac{\sum_{\mathbf{k}\in\mathcal{Z}} |\hat{\psi}_{\mathbf{k}}|^2 |\mathbf{k}|^4}{\sum_{\mathbf{k}\in\mathcal{Z}} |\hat{\psi}_{\mathbf{k}}|^2 |\mathbf{k}|^2} \sum_{\mathbf{k}\in\mathcal{Z}} |\hat{\psi}_{\mathbf{k}}|^2 |\mathbf{k}|^2 \\ &= -\mu_2 R(t) \sum_{\mathbf{k}\in\mathcal{Z}} |\hat{\psi}_{\mathbf{k}}|^2 |\mathbf{k}|^2 \leq -2\mu_2 \Lambda_{k_o} E(t),\end{aligned}$$

from which we deduce that

$$E(t) \leq E(0) e^{-2\mu_2 \Lambda_{k_o} t}. \tag{74}$$

Note that

$$\mathcal{E}(t) = R(t)E(t),$$

which yields by (71) and (74) that

$$\mathcal{E}(t) = R(t)E(t) \leq R(0)E(t) \leq R(0)E(0)e^{-\mu_2 \Lambda_{k_o} t} = \mathcal{E}(0)e^{-2\mu_2 \Lambda_{k_o} t}. \tag{75}$$

Next, we estimate

$$\left| \int J(\psi, \Delta\psi)\Delta^2\psi \, dxdydz \right|.$$

Because the energy is bounded, $E(t) < E(0) < +\infty$, we can easily obtain that

$$\begin{aligned}\left| \int J(\psi, \Delta\psi)\Delta^2\psi \, dxdydz \right| &\leq \|\nabla\psi\|_{L^\infty} \|\nabla\Delta\psi\| \|\Delta^2\psi\| \\ &\leq C\|\nabla\psi\|^{\frac{1}{4}} \|\nabla\Delta\psi\|^{\frac{7}{4}} \|\Delta^2\psi\| \\ &\leq C\|\nabla\psi\|^{\frac{1}{4}} \|\Delta\psi\|^{7/8} \|\Delta^2\psi\|^{15/8} \\ &\leq \frac{\mu_2}{2}\|\Delta^2\psi\|^2 + C_1\|\nabla\psi\|^4 \|\Delta\psi\|^{14}\end{aligned} \tag{76}$$

where we have used the Agmon's inequality and interpolation inequality

$$\begin{aligned}\|\nabla\psi\|_{L^\infty} &\leq C\|\nabla\psi\|^{\frac{1}{4}} \|\nabla\Delta\psi\|^{\frac{3}{4}}, \\ \|\nabla\Delta\psi\|^{\frac{7}{4}} &\leq \|\Delta\psi\|^{7/8} \|\Delta^2\psi\|^{7/8}.\end{aligned}$$

Taking the inner product of equation (1) with $\Delta^2\psi$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\Delta\psi\|^2 = \int_\Omega J(\psi, \Delta\psi)\Delta^2\psi \, dxdydz - \int_\Omega D(\Delta)\psi\Delta^2\psi \, dxdyxz.$$

By using (76), we deduce that

$$\frac{d}{dt}\|\nabla\Delta\psi\|^2 + \mu_2\|\Delta^2\psi\|^2 \leq C_1\|\nabla\psi\|^4\|\Delta\psi\|^{14} \leq C_2 E(0)^2 \mathcal{E}(0)^7 e^{-18\mu_2\Lambda_{k_o} t}. \tag{77}$$

In view of

$$\Lambda_1\|\nabla\Delta\psi\|^2 \leq \|\Delta^2\psi\|^2,$$

and applying the Gronwall inequality, we arrive at

$$\|\nabla\Delta\psi\|^2 \leq e^{-\Lambda_1\mu_2 t}\|\nabla\Delta\psi(0, x, y, z)\|^2 + C_2 E(0)^2 \mathcal{E}(0)^7 e^{-\mu_2\Lambda_1 t}. \tag{78}$$

Next, we derive the equation for the $\psi^* = \psi(t,x,y,z)/\sqrt{E(t)}$. Upon performing the straightforward calculation, we get

$$\begin{aligned}
\frac{d\Delta\psi^*}{dt} &= \frac{d}{dt}\left(\frac{\Delta\psi}{\sqrt{E(t)}}\right)\\
&= \frac{1}{\sqrt{E}}\frac{d\Delta\psi}{dt} - \frac{\Delta\psi}{2E^{\frac{3}{2}}}\frac{dE}{dt}\\
&= \frac{1}{\sqrt{E}}\left(\frac{d\Delta\psi}{dt} - \frac{\Delta\psi}{2E}\frac{dE}{dt}\right)\\
&= \frac{1}{\sqrt{E}}\Big(-\beta\frac{\partial\psi}{\partial x} - J(\psi,\Delta\psi) + \mu_2\Delta^2\psi + \frac{\Delta\psi}{2E}\mu_2\int_\Omega \psi\Delta^2\psi\\
&\quad + \sum_{j\geq 0, j\neq 2}^{N} \mu_j(-\Delta)^j\psi + \frac{\Delta\psi}{2E}\sum_{j\geq 0, j\neq 2}^{N}\mu_j\int_\Omega \psi(-\Delta)^j\psi\Big)\\
&= -\beta\frac{\partial\psi^*}{\partial x} - \sqrt{E}J(\psi^*,\Delta\psi^*) + \mu_2\Delta^2\psi^* + \mu_2 R(t)\Delta\psi^*\\
&\quad + \sum_{j\geq 0, j\neq 2}^{N} \mu_j(-\Delta)^j\psi^* + \frac{\Delta\psi^*}{2}\sum_{j\geq 0, j\neq 2}^{N}\mu_j\int_\Omega \psi^*(-\Delta)^j\psi^*.
\end{aligned} \tag{79}$$

from which it deduces that

$$\begin{aligned}
\frac{d\Delta\psi^*}{dt} &= -\beta\frac{\partial\psi^*}{\partial x} - \sqrt{E}J(\psi^*,\Delta\psi^*) + \mu_2\Delta^2\psi^* + \mu_2 R(t)\Delta\psi^*\\
&\quad + \sum_{j\geq 0, j\neq 2}^{N} \mu_j(-\Delta)^j\psi^* + \frac{\Delta\psi^*}{2}\sum_{j\geq 0, j\neq 2}^{N}\mu_j\int_\Omega \psi^*(-\Delta)^j\psi^*.
\end{aligned} \tag{80}$$

Now, we show that

$$P\left(\psi(t,x,y,z)/\sqrt{E(t)}\right) = P\psi^* \to 0 \text{ in } L^2(\Omega),$$

by making use of the equation (80). For this purpose, we first estimate

$$\int_\Omega J(\psi^*,\Delta\psi^*)P\psi^*.$$

Upon performing straightforward calculation we arrive at

$$\begin{aligned}
|\int_\Omega J(\psi^*,\Delta\psi^*)P\psi^*| &= |\int_\Omega J(\psi^*,P\psi^*)\Delta\psi^*|\\
&\leq \|\nabla\psi^*\|_{L^4}^2\|\Delta\psi^*\|\\
&\leq C\|\nabla\psi^*\|^{\frac{1}{2}}\|\Delta\psi^*\|^{\frac{5}{2}}\\
&= CR^{\frac{5}{4}}(t).
\end{aligned} \tag{81}$$

where we have used the interpolation inequality and identity

$$\|\nabla\psi^*\|_{L^4} \leq C\|\nabla\psi^*\|^{\frac{1}{4}}\|\Delta\psi^*\|^{\frac{3}{4}}, \quad \|\nabla\psi^*\| = \sqrt{2}.$$

By virtue of the decomposition

$$\begin{aligned}
&P\psi = P_1\psi + P_2\psi = \sum_{|\mathbf{k}|^2>k_o}\hat{\psi_{\mathbf{k}}}e_{\mathbf{k}} + \sum_{|\mathbf{k}|^2<k_o}\hat{\psi_{\mathbf{k}}}e_{\mathbf{k}},\\
&P_1\psi = \sum_{|\mathbf{k}|^2>k_o}\hat{\psi_{\mathbf{k}}}e_{\mathbf{k}}, \quad P_2\psi = \sum_{|\mathbf{k}|^2<k_o}\hat{\psi_{\mathbf{k}}}e_{\mathbf{k}}.
\end{aligned} \tag{82}$$

and from (80), it yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla P_s\psi^*\|^2+\mu_2\|\Delta P_s\psi^*\|^2-\mu_2 R(t)\|\nabla P_s\psi^*\|^2\\&=\sqrt{E}\int_\Omega J(\psi^*,\Delta\psi^*)P_s\psi^*-\sum_{j\ge0,j\ne2}^N\mu_j\int_\Omega\psi^*(-\Delta)^jP_s\psi^*\\&\quad+\frac{\|\nabla P_s\psi^*\|^2}{2}\sum_{j\ge0,j\ne2}^N\mu_j\int_\Omega\psi^*(-\Delta)^j\psi^*,\quad s=1,2.\end{aligned}\tag{83}$$

Direct calculation gives

$$\begin{aligned}&\left(\sum_{j\ge0,j\ne2}^N\sum_{|\mathbf{k}|^2>k_o}\mu_j|\mathbf{k}|^{2j}|\hat{\psi_\mathbf{k}}^*|^2\right)\left(\sum_{|\mathbf{k}|^2\le k_o}|\mathbf{k}|^2|\hat{\psi_\mathbf{k}}^*|^2\right)-\\&\left(\sum_{j\ge0,j\ne2}^N\sum_{|\mathbf{k}|^2\le k_o}\mu_j|\mathbf{k}|^{2j}|\hat{\psi_\mathbf{k}}^*|^2\right)\left(\sum_{|\mathbf{k}|^2>k_o}|\mathbf{k}|^2|\hat{\psi_\mathbf{k}}^*|^2\right)\\&=\sum_{j\ge0,j\ne2}^N\sum_{|\mathbf{k}|^2>k_o}\sum_{|\mathbf{k}'|^2\le k_o}\mu_j|\mathbf{k}|^{2j}|\mathbf{k}'|^2|\hat{\psi_\mathbf{k}}^*|^2|\hat{\psi_{\mathbf{k}'}}^*|^2\\&\quad-\sum_{j\ge0,j\ne2}^N\sum_{|\mathbf{k}|^2>k_o}\sum_{|\mathbf{k}'|^2\le k_o}\mu_j|\mathbf{k}'|^{2j}|\mathbf{k}|^2|\hat{\psi_\mathbf{k}}^*|^2|\hat{\psi_{\mathbf{k}'}}^*|^2\\&=\sum_{j\ge1,j\ne2}^N\sum_{|\mathbf{k}|^2>k_o}\sum_{|\mathbf{k}'|^2\le k_o}\mu_j|\hat{\psi_\mathbf{k}}^*|^2|\hat{\psi_{\mathbf{k}'}}^*|^2|\mathbf{k}||\mathbf{k}'|\left(|\mathbf{k}|^{j-1}-|\mathbf{k}'|^{j-1}\right)\left(|\mathbf{k}|^j|\mathbf{k}'|+|\mathbf{k}||\mathbf{k}'|^j\right)\\&\quad+\sum_{|\mathbf{k}|^2>k_o}\sum_{|\mathbf{k}'|^2\le k_o}\mu_0|\mathbf{k}'|^2|\hat{\psi_\mathbf{k}}^*|^2|\hat{\psi_{\mathbf{k}'}}^*|^2-\sum_{|\mathbf{k}|^2>k_o}\sum_{|\mathbf{k}'|^2\le k_o}\mu_0|\mathbf{k}|^2|\hat{\psi_\mathbf{k}}^*|^2|\hat{\psi_{\mathbf{k}'}}^*|^2>0\end{aligned}$$

which means that

$$\frac{2\sum_{j\ge0,j\ne2}^N\mu_j\sum_{|\mathbf{k}|^2>k_o}|\mathbf{k}|^{2j}|\hat{\psi_\mathbf{k}}^*|^2}{\sum_{j\ge0,j\ne2}^N\mu_j\sum_{|\mathbf{k}|^2\ge\Lambda_1}|\mathbf{k}|^{2j}|\hat{\psi_\mathbf{k}}^*|^2}>\sum_{|\mathbf{k}|^2>k_o}|\mathbf{k}|^2|\hat{\psi_\mathbf{k}}^*|^2,\qquad\sum_{|\mathbf{k}|^2\ge\Lambda_1}|\mathbf{k}|^2|\hat{\psi_\mathbf{k}}^*|^2=2.$$

That is,

$$-\sum_{j\ge0,j\ne2}^N\mu_j\int_\Omega\psi^*(-\Delta)^jP_1\psi^*+\frac{\|\nabla P_1\psi^*\|^2}{2}\sum_{j\ge0,j\ne2}^N\mu_j\int_\Omega\psi^*(-\Delta)^j\psi^*<0.$$

Let S_m and S_M be given by

$$S_m=\min\left\{|\mathbf{k}|^2\big||\mathbf{k}|^2>\Lambda_{k_o}\right\},\qquad S_M=\max\left\{|\mathbf{k}|^2\big||\mathbf{k}|^2<\Lambda_{k_o}\right\},\tag{84}$$

then, for larger enough t, we have

$$\|\Delta P_1\psi^*\|^2-R(t)\|\nabla P_1\psi^*\|^2\ge\left(S_m-R(t)\right)\|\nabla P_1\psi^*\|^2>0$$

which implies

$$\frac{1}{2}\frac{d}{dt}\|\nabla P_1\psi^*\|^2+\mu_2\left(R(t)-S_m\right)\|\nabla P_1\psi^*\|^2\le\sqrt{E}CR^{\frac{5}{4}}(t)=C_1e^{-\mu_2\Lambda_{k_o}t}.\tag{85}$$

where the estimation (74) has been used. After applying the Gronwall-inequality, the preceding inequality implies that for large t

$$\|\nabla P_1\psi^*\|^2\le e^{-\mu_2(S_m-\Lambda_{k_o})t}\|\nabla P_1\psi^*\|^2+Ce^{-\mu_2(S_m-\Lambda_{k_o})t}+Ce^{-\mu_2\Lambda_{k_o}t},\tag{86}$$

which implies

$$P_1\left(\nabla\frac{\psi(t,x,y,z)}{\sqrt{E(t)}}\right)=P_1\nabla\psi^*\to0\text{ in }L^2(\Omega).\tag{87}$$

Next, we show

$$P_2\left(\nabla\frac{\psi(t,x,y,z)}{\sqrt{E(t)}}\right)=P_2\nabla\psi^*\to0\text{ in }L^2(\Omega).\tag{88}$$

also holds true.

Note that for larger enough t, we have

$$\|\Delta P_2\psi^*\|^2 - R(t)\|\nabla P_2\psi^*\|^2 \le (S_M - \Lambda_{k_o})\,\|\nabla P_2\psi^*\|^2.$$

Similarly, we have

$$-\sum_{j\ge 0, j\ne 2}^{N}\mu_j\int_\Omega \psi^*(-\Delta)^j P_2\psi^* + \frac{\|\nabla P_2\psi^*\|^2}{2}\sum_{j\ge 0, j\ne 2}^{N}\mu_j\int_\Omega \psi^*(-\Delta)^j\psi^* > 0.$$

by which we arrive at

$$\frac{1}{2}\frac{d}{dt}\|\nabla P_2\psi^*\|^2 + \mu_2\,(S_M - \Lambda_{k_o})\,\|\nabla P_2\psi^*\|^2 \ge -\sqrt{E}CR^{\frac{5}{4}}(t) = -C_1 e^{-\mu_2\Lambda_{k_o}t} \tag{89}$$

which is equivalent to

$$\frac{d}{dt}\left(e^{2\mu_2(S_M-\Lambda_{k_o})t}\|\nabla P_2\psi^*\|^2\right) \ge -C_1 e^{\mu_2(2S_M-3\Lambda_{k_o})t}, \tag{90}$$

integrating which from larger time t_0 to time $t > t_0$, we deduce

$$e^{2\mu_2(S_M-\Lambda_{k_o})(t-t_0)}\|\nabla P_2\psi^*\|^2 \ge \|\nabla P_2\psi^*\|^2 - C_1 e^{\mu_2(2S_M-3\Lambda_{k_o})t}.. \tag{91}$$

The preceding inequality guarantee (88).

Finally, we show that the solution ψ^* to the equation

$$\begin{aligned}\frac{d\Delta\psi^*}{dt} + \beta\frac{\partial\psi^*}{\partial x} &= -\sqrt{E}J(\psi^*,\Delta\psi^*) + \mu_2\Delta^2\psi^* + \mu_2R(t)\Delta\psi^*\\ &\quad+\sum_{j\ge 0, j\ne 2}^{N}\mu_j(-\Delta)^j\psi^* + \frac{\Delta\psi^*}{2E}\sum_{j\ge 0, j\ne 2}^{N}\mu_j\int_\Omega\psi^*(-\Delta)^j\psi^*\end{aligned} \tag{92}$$

converges to a solution ϕ^* of the equation

$$\frac{d(I-P)\Delta\phi}{dt} + \beta\frac{\partial(I-P)\phi}{\partial x} = 0. \tag{93}$$

with respect to the norm of H^1 as time goes to infinity.

By the definition of P, we see that

$$(I-P)\phi = \sum_{|\mathbf{k}|^2=\Lambda_{k_o}}\hat{\phi_{\mathbf{k}}}(t)e_{\mathbf{k}}$$

by inserting which into (93) one can find $\hat{\phi_{\mathbf{k}}}(t)$ solving

$$\Lambda_{k_o}\frac{d\phi_{\mathbf{k}}}{dt} = i\frac{2\pi k_x\beta}{L_x}\phi_{\mathbf{k}}(t).$$

from which one can exactly get

$$\phi_{\mathbf{k}} = A_{\mathbf{k}}(0)e^{\frac{i2\pi k_x\beta}{L_x\Lambda_{k_o}}t}.$$

It is known that

$$\psi^* = P\psi^* + (I-P)\psi^*, \quad \|\nabla\psi^*(t)\| = \sqrt{2},$$

and

$$\|\nabla P\psi^*(t)\| \to 0, \text{ as } t\to+\infty.$$

Hence, we only need to show that the solution ψ^* to the equation

$$\begin{aligned}\frac{d\Delta(I-P)\psi^*}{dt} + \beta\frac{\partial(I-P)\psi^*}{\partial x} &= -\sqrt{E}(I-P)J(\psi^*,\Delta\psi^*)\\ &\quad+\mu_2\Delta^2(I-P)\psi^* + \mu_2R(t)\Delta(I-P)\psi^*\\ &\quad+\sum_{j\ge 0, j\ne 2}^{N}\mu_j(-\Delta)^j(I-P)\psi^*\\ &\quad+\frac{\Delta(I-P)\psi^*}{2E}\sum_{j\ge 0, j\ne 2}^{N}\mu_j\int_\Omega\psi^*(-\Delta)^j\psi^*\end{aligned} \tag{94}$$

converges to the solution ϕ^* of the equation

$$\frac{d(I-P)\Delta\phi}{dt} + \beta\frac{\partial(I-P)\phi}{\partial x} = 0. \tag{95}$$

Denote $(I-P)\psi^* = \psi^*_{k_o} \in E_{k_o}$. Then, $\psi^*_{k_o}$ solving

$$\begin{aligned}\frac{d\psi^*_{k_o}}{dt} - \frac{\beta}{\Lambda_{k_o}}\frac{\partial\psi^*_{k_o}}{\partial x} &= \frac{\sqrt{E}}{\Lambda_{k_o}}(I-P)J(\psi^*,\Delta\psi^*) - \mu_2\Lambda_{k_o}\psi^*_{k_o} + \mu_2 R(t)\psi^*_{k_o} \\ &- \sum_{j\geq 0, j\neq 2}^{N} \mu_j \Lambda^{j-1}_{k_o}\psi^*_{k_o} + \frac{\psi^*_{k_o}}{2}\sum_{j\geq 0, j\neq 2}^{N} \mu_j \int_\Omega \psi^*(-\Delta)^j\psi^*\end{aligned} \tag{96}$$

Let's denote

$$f = \frac{\sqrt{E}}{\Lambda_{k_o}}(I-P)J(\psi^*,\Delta\psi^*)$$

and let

$$\psi^*_{k_o} = \sum_{|\mathbf{k}|^2=\Lambda_{k_o}} \psi^*_{\mathbf{k}}(t)e_{\mathbf{k}}, \quad f = \sum_{|\mathbf{k}|^2=\Lambda_{k_o}} f_{\mathbf{k}}(t)e_{\mathbf{k}},$$

then $\psi^*_{\mathbf{k}}(t)$ solves

$$\begin{aligned}\frac{d\psi^*_{\mathbf{k}}(t)}{dt} - \frac{2\pi k_x\beta i}{L_x\Lambda_{k_o}}\psi^*_{\mathbf{k}}(t) &= f_{\mathbf{k}}(t) - \mu_2\Lambda_{k_o}\psi^*_{\mathbf{k}}(t) + \mu_2 R(t)\psi^*_{\mathbf{k}}(t) \\ &- \sum_{j\geq 0, j\neq 2}^{N} \mu_j \Lambda^{j-1}_{k_o}\psi^*_{k_o} + \frac{\psi^*_{k_o}}{2}\sum_{j\geq 0, j\neq 2}^{N} \mu_j \int_\Omega \psi^*(-\Delta)^j\psi^*\end{aligned} \tag{97}$$

Furthermore, integral the preceding equation on both sides and let

$$\begin{aligned}F(t) = \int_{t_0}^{t} e^{-\frac{2\pi k_x\beta i}{L_x\Lambda_{k_o}}\tau}\Bigg(& f_{\mathbf{k}}(\tau) - \mu_2\Lambda_{k_o}\psi^*_{\mathbf{k}}(\tau) + \mu_2 R(\tau)\psi^*_{\mathbf{k}}(\tau) \\ &- \sum_{j\geq 0, j\neq 2}^{N} \mu_j \Lambda^{j-1}_{k_o}\psi^*_{k_o} + \frac{\psi^*_{k_o}}{2}\sum_{j\geq 0, j\neq 2}^{N} \mu_j \int_\Omega \psi^*(-\Delta)^j\psi^*\Bigg)\,d\tau,\end{aligned}$$

we then have the exact expression for $\psi^*_{\mathbf{k}}(t)$, which is

$$\psi^*_{\mathbf{k}}(t) = e^{\frac{2\pi k_x\beta i}{L_x\Lambda_{k_o}}t}\left(F(t) + \psi^*_{\mathbf{k}}(t_0)e^{-\frac{2\pi k_x\beta i}{L_x\Lambda_{k_o}}t_0}\right).$$

Using the (74) and (81), one can see that the improper integral

$$\int_{t_0}^{+\infty} e^{-\frac{2\pi k_x\beta i}{L_x\Lambda_{k_o}}\tau} f_{\mathbf{k}}(\tau)\,d\tau$$

converges. The decreasing of $(R(t) - \Lambda_{k_o})$ with the increasing of t guarantees the convergence of the improper integral

$$\int_{t_0}^{+\infty} e^{-\frac{2\pi k_x\beta i}{L_x\Lambda_{k_o}}\tau}\left(R(\tau) - \Lambda_{k_o}\right)\,d\tau.$$

Then $\|\nabla\psi^*\| = \sqrt{2}$, (87) and (88) together deduce that

$$\lim_{t\to+\infty} \Lambda_{k_o}(\psi^*_{k_o})^2 = 2.$$

and for very large t the function

$$\left|-\sum_{j\geq 0, j\neq 2}^{N} \mu_j \Lambda^{j-1}_{k_o} + \frac{1}{2}\sum_{j\geq 0, j\neq 2}^{N} \mu_j \int_\Omega \psi^*(-\Delta)^j\psi^*\right|$$

is decreasing and approaching zero. Hence, the improper integral

$$\int_{t_0}^{\infty} e^{-\frac{2\pi k_x\beta i}{L_x\Lambda_{k_o}}\tau}\left(-\sum_{j\geq 0, j\neq 2}^{N} \mu_j \Lambda^{j-1}_{k_o}\psi^*_{k_o} + \frac{\psi^*_{k_o}}{2}\sum_{j\geq 0, j\neq 2}^{N} \mu_j \int_\Omega \psi^*(-\Delta)^j\psi^*\right)\,d\tau$$

converges. Therefore,

$$F(t) \to A, \text{ as } t \to +\infty.$$

Now, let's denote

$$\phi(t) = \sum_{|\mathbf{k}|^2=\Lambda_{k_o}} (A + \psi_{\mathbf{k}}^*(t_0))\, e^{\frac{2\pi k_x \beta i}{L_x \Lambda_{k_o}} t} e_{\mathbf{k}}$$

which solves (95), and

$$\lim_{t\to+\infty} \|\nabla\psi^*(t) - \nabla\phi(t)\| = 0.$$

□

5. A Note on the Geometric Foundations of the 3D-QG model

In this section, we introduce a geometric framework for the 3D-QG model (1). In more explicit terms, we shall write the equations (1) with zero external force as a GENERIC (General Equation for Non-Equilibrium Reversible-Irreversible Coupling) which is the sum of a Hamiltonian and a gradient dynamics [26, 27]. GENERIC is a geometric framework permits one to determine geometric underlyings of both reversible and reversible parts of a physical system in a complete picture [28]. In this picture, reversible part of the dynamics is governed by a Hamiltonian system, [29] whereas irreversible part is governed by a gradient system. We cite a recent study on the gradient systems and the entropy maximization [30]. In literature, this coupling is also called as metriplectic [31, 32]. We cite [33] for gradient flows and triple bracket generated metriplectic systems. See also [34] for a discussion on the potential descending principle and the irreversibility. Let us first recall the basics of the Hamiltonian and the gradient dynamics then present how these geometric frameworks fit the present discussions done on 3D-QG model.

A dynamical system is said to be Hamiltonian if the dynamics of an observable F, that is a real valued function(al) on the space of state variables, can be written in the Hamiltonian form

$$F_t = \{F, H\} \tag{98}$$

where H is being the Hamiltonian function(al) [35, 36]. Here, the Poisson bracket $\{\bullet, \bullet\}$ is a skew-symmetric bi-linear operation on the space of smooth function(al)s depending on the state variables, and it satisfies both

- the Leibniz identity $\{FG, H\} = F\{G, H\} + \{F, H\}G$, and
- the Jacobi identity $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$,

for any three function(al)s F, G and H. Skew-symmetry of the Poisson bracket manifests that the Hamiltonian function is a conserved quantity of the dynamics. So that, this framework is proper for the reversible systems.

To be more concrete on the Hamiltonian systems, let us recall the three dimensional bounded region Ω presented in (4), and consider the space $\mathcal{F}(\Omega)$ of smooth functions on Ω satisfying the boundary conditions (5). The periodicity of the functions are important here both for the theoretical and practical issues. For the practical sense, it permits us to omit total divergence terms in the integrals. Fix the Euclidean volume form $dV = dxdydz$, and then identify the linear algebraic dual of $\mathcal{F}(\Omega)$ with itself by the means of the L^2-pairing. Introduce now a Poisson bracket of two functionals F and H defined on $\mathcal{F}(\Omega)$ as follows

$$\{F, H\}(\omega) = \left\langle \omega + \beta y, J\left(\frac{\delta F}{\delta\omega}, \frac{\delta H}{\delta\omega}\right)\right\rangle = \int_\Omega (\omega + \beta y) J\left(\frac{\delta F}{\delta\omega}, \frac{\delta H}{\delta\omega}\right) dV \tag{99}$$

where J is the two dimensional Jacobian determinant in (3), and $\delta F/\delta\omega$ is the Fréchet derivative of F with respect to $\omega \in \mathcal{F}(\Omega)$. Instead of checking that this bracket satisfies both the Leibniz and the Jacobi identities, we point that the bracket is in actually a form, so called, the Lie-Poisson bracket [29] which automatically satisfies these identities. It is immediate to see this by realizing that the Jacobian J is a Lie algebra bracket on $\mathcal{F}(\Omega)$. The dynamics of the function ω is computed by employing the chain rule

$$F_t = \left\langle \frac{\delta F}{\delta\omega}, \omega_t \right\rangle. \tag{100}$$

For the right hand side of (99), we employ the by-parts technique in the lights of the periodicity of functions and their first order partial derivatives. This results with

$$\int_\Omega (\omega + \beta y) J\left(\frac{\delta F}{\delta\omega}, \frac{\delta H}{\delta\omega}\right) dV = \int_\Omega \frac{\delta F}{\delta\omega} J(\frac{\delta H}{\delta\omega}, \omega + \beta y) dV = \left\langle \frac{\delta F}{\delta\omega}, J(\frac{\delta H}{\delta\omega}, \omega + \beta y)\right\rangle. \tag{101}$$

So that, by assuming the arbitrariness of $\delta F/\delta\omega$, we arrive at the following dynamical equation

$$\omega_t = J(\frac{\delta H}{\delta\omega}, \omega + \beta y). \tag{102}$$

Take particularly that ω is the vorticity of the continuum and ψ is the stream function satisfying the relation $\omega = \Delta\psi$. We further define a Hamiltonian functional

$$H(\omega) := -\frac{1}{2}\int_\Omega \omega\psi dV, \tag{103}$$

which is exactly the kinetic energy E in (19). Notice that $\delta H/\delta\omega$ is $-\psi$, and then, after a direct calculation, the equation (102) takes the particular form

$$\partial_t\Delta\psi + J(\psi, \Delta\psi) + \beta\partial_x\psi = 0, \tag{104}$$

which is the reversible part of the dynamics in (1), namely the Rossby equation. See also [37] for the Hamiltonian analysis of the Rossby equation.

The skew-symmetry of the Poisson bracket gives that the Hamiltonian function (103) is a conserved quantity for the reversible dynamics since $H_t = \{H, H\} = 0$. Since the bracket is degenerate, there are more conserved quantities of the system. For an example, consider the potential enstrophy

$$S(\omega) := \frac{1}{2}\int_\Omega (\omega + \beta y)^2 dV. \tag{105}$$

Being the Casimir function of the Poisson bracket (99), that is $\{S, H\} = 0$, we have that S is constant along the motion. Here, $\omega + \beta y$ is called the potential vorticity [1]. We refer [38] for the use of the Casimirs as the generator of the irreversible part of the dynamics.

For the irreversible part of the dynamics, a gradient dynamics must be coupled to this reversible Hamiltonian system. In accordance with this, let us now introduce a symmetric bracket

$$(F, H)_{\mathcal{J}} = \left\langle \frac{\delta F}{\delta\omega}, \mathcal{J}\frac{\delta H}{\delta\omega} \right\rangle = \int_\Omega \frac{\delta F}{\delta\omega}\mathcal{J}\frac{\delta H}{\delta\omega} dV \tag{106}$$

where $\mathcal{J}$ is a symmetric differential operator. In order to describe the dissipative part of the 3D-QG model (1), we choose the symmetric operator $\mathcal{J}$ as the differential operator $D(\Delta)$ presented in (2). So that, we have the following symmetric bracket

$$(F, H)_D = -\left\langle \frac{\delta F}{\delta\omega}, D(\Delta)\frac{\delta H}{\delta\omega} \right\rangle = -\int_\Omega \frac{\delta F}{\delta\omega} D(\Delta)\frac{\delta H}{\delta\omega} dV \tag{107}$$

where the differential operator $D(\Delta)$ is the one given in (2).

The dynamics of the vorticity generated by a functional H with respect to the symmetric bracket (107) is computed to be

$$\omega_t = -D(\Delta)\frac{\delta H}{\delta\omega} \tag{108}$$

We recall the Hamiltonian function H in (103) as the generator of the dynamics, then the gradient dynamics (108) becomes the irreversible part of the 3D-QG model (1) given by

$$\partial_t\Delta\psi = D(\Delta)\psi \tag{109}$$

with the external force is considered to be zero.

Let us now define a GENERIC bracket as a sum of the Poisson bracket in (99) and the symmetric bracket in (107) given by

$$[F, H] = \{F, H\} + (F, H)_D, \tag{110}$$

see [39]. In this case the dynamics of an observable is computed to be

$$F_t = \{F, H\} + (F, H)_D \tag{111}$$

involving both the reversible and irreversible parts altogether. More explicitly, by recalling the Hamiltonian function H exhibited in (103), we compute the dynamics of the stream function ψ as follows

$$\partial_t\Delta\psi + J(\psi, \Delta\psi) + \beta\partial_x\psi = D(\Delta)\psi. \tag{112}$$

This is the 3D-QG model (1) with zero external force. Notice that, the Hamiltonian can not be conserved along the motion due to the computation

$$H_t = [H, H] = \{H, H\} + (H, H)_D = -\int_\Omega \psi D(\Delta)\psi dV. \tag{113}$$

We refer [40] for some similar discussions.

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References

[1] Andrew Majda and Xiaoming Wang. *Nonlinear dynamics and statistical theories for basic geophysical flows*. Cambridge University Press, 2006.
[2] J. Pedlosky. *Geophysical fluid dynamics*. Springer, 1987.
[3] M. Ghil and S. Childress. *Topics in geophysical fluid dynamics: Atmospheric dynamics, dynamo theory, and climate dynamics*, volume 60 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1987.
[4] A. Majda. *Introduction to P.D.E.s and waves for the atmosphere and ocean, Courant Institute Lecture Notes Series*, volume 9. Amer. Math. Soc., 2003.
[5] Andrew Majda, Sang-Yeun Shim, and Xiaoming Wang. Selective decay for geophysical flows. *Methods Appl. Anal.*, 7(3):511–553, 2000.
[6] Pedro F. Embid and Andrew J. Majda. Low Froude number limiting dynamics for stably stratified flow with small or finite Rossby numbers. *Geophys. Astrophys. Fluid Dynam.*, 87(1-2):1–50, 1998.
[7] Walter H. Munk. On the wind-driven ocean circulation. *J. Meteor.*, 7(2):79–93, 1950.
[8] G. R. Ierley. On the onset of inertial recirculation in barotropic general circulation models. *J. Phys. Oceanogr.*, 17(12):2366–2374, 1987.
[9] B. Legras and M. Ghil. Persistent anomalies, blocking and variations in atmospheric predictability. *J. Atmos. Sci.*, 42:433–471, 1985.
[10] J. Shukla J. G. Charney and K. C. Mo. Comparison of a barotropic blocking theory with observation. *J. Atmos. Sci.*, 38:762–779, 1981.
[11] E. Simonnet, M. Ghil, K. Ide, R. Temam, and S. Wang. Low-frequency variability in shallow-water models of the wind-driven ocean circulation. part i: Steady-state solution. *J. Phys. Oceanogr.*, 33:712–728, 2003.
[12] E. Simonnet, M. Ghil, K. Ide, R. Temam, and S. Wang. Low-frequency variability in shallow-water models of the wind-driven ocean circulation. part ii: Time-dependent solutions. *J. Phys. Oceanogr.*, 33:729–752, 2003.
[13] P Berloff and SP Meacham. On the stability of the wind-driven circulation. *J. Mar. Res.*, 56(5):937–993, 1998.
[14] Michael Ghil. The wind-driven ocean circulation: applying dynamical systems theory to a climate problem. *Discrete Contin. Dyn. Syst.-A*, 37(1):189–228, 2017.
[15] J. H. LaCasce. Baroclinic Rossby waves in a square basin. *J. Phys. Oceanogr.*, 30(12):3161–3178, 2000.
[16] Joseph A. Biello and Andrew J. Majda. The effect of meridional and vertical shear on the interaction of equatorial baroclinic and barotropic Rossby waves. *Stud. Appl. Math.*, 112(4):341–390, 2004.
[17] W. K. Dewar. On "Too fast" baroclinic planetary waves in the general circulation. *J. Phys. Oceanogr.*, 28:1739–1758, 1998.
[18] Aviv Solodoch, Andrew L. Stewart, and C. McWilliams, James. Baroclinic instability of axially symmetric flow over sloping bathymetry. *J. Fluid Mech.*, 799:265–296, 2016.
[19] Henk Dijkstra, Taylan Sengul, Jie Shen, and Shouhong Wang. Dynamic transitions of quasi-geostrophic channel flow. *SIAM J. Appl. Math.*, 75(5):2361–2378, 2015.
[20] G. R. Ierley and W. R. Young. Viscous instabilities in the western boundary layer. *J. Phys. Oceanogr.*, 21(9):1323–1332, 1991.
[21] Chanh kieu, Taylan Sengul, Quan Wang, and Dongming Yan. On the Hopf (double Hopf) bifurcations and transitions of two-layer western boundary currents. *Commun. Nonlinear Sci. Numer. Simul.*, 65:196–215, 2018.
[22] Hernandez Marco Han, Daozhi and Quan Wang. On the instabilities and transitions of the western boundary current. *Communications in Computational Physics*, accepted, 2018.
[23] C. Foias and J.-C. Saut. Asymptotic behavior, as $t \to +\infty$, of solutions of Navier-Stokes equations and nonlinear spectral manifolds. *Indiana Univ. Math. J.*, 33(3):459–477, 1984.
[24] W. H. Matthaeus, W.T. Stribling, D. Martinez, S. Oughton, and D. Montgomery. Decaying two-dimensional Navier-Stokes turbulence at very long times. *Physica D*, 51:531–538, 1991.
[25] D. Montgomery, X. Shan, and W. H. Matthaeus. Navier-Stokes relaxation to sinh-Poisson states at finite Reynolds numbers. *Phys. Fluids A*, 5:2207–2216, 1993.
[26] Miroslav Grmela. Bracket formulation of dissipative fluid mechanics equations. *Physics Letters A*, 102:355–358, 1984.
[27] Miroslav Grmela and Hans Christian Öttinger. Dynamics and thermodynamics of complex fluids. i. development of a general formalism. *Physical Review E*, 56(6):6620, 1997.
[28] Michal Pavelka, Václav Klika, and Miroslav Grmela. *Multiscale thermo-dynamics: Introduction to GENERIC*. De Gruyter, 2018.
[29] Jerrold E Marsden and Tudor S Ratiu. *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems*, volume 17. Springer Science & Business Media, 2013.
[30] Adam Janečka and Michal Pavelka. Gradient dynamics and entropy production maximization. *Journal of Non-Equilibrium Thermodynamics*, 43(1):1–19, 2018.
[31] Philip J Morrison. Bracket formulation for irreversible classical fields. *Physics Letters A*, 100(8):423–427, 1984.
[32] Philip J Morrison. A paradigm for joined Hamiltonian and dissipative systems. *Physica D: Nonlinear Phenomena*, 18(1-3):410–419, 1986.
[33] Anthony M Bloch, Philip J Morrison, and Tudor S Ratiu. Gradient flows in the normal and Kähler metrics and triple bracket generated metriplectic systems. *Recent Trends in Dynamical Systems*, pages 371–415. Springer, 2013.
[34] Tian Ma and Shouhong Wang. Dynamic law of physical motion and potential-descending principle. preprint, July 2017.

[35] Vladimir I Arnold and Boris A Khesin. *Topological methods in hydrodynamics*, volume 125. Springer Science & Business Media, 1999.
[36] Manuel De León and Paulo R Rodrigues. *Methods of differential geometry in analytical mechanics*, volume 158. Elsevier, 2011.
[37] LI Piterbarg. Hamiltonian formalism for Rossby waves. *Translations of the American Mathematical Society-Series 2*, 182:131–166, 1997.
[38] François Gay-Balmaz and Darryl D Holm. Selective decay by Casimir dissipation in inviscid fluids. *Nonlinearity*, 26(2):495, 2013.
[39] Partha Guha. Metriplectic structure, Leibniz dynamics and dissipative systems. *Journal of Mathematical Analysis and Applications*, 326(1):121–136, 2007.
[40] Vinicio Pelino and Antonello Pasini. Dissipation in Lie–Poisson systems and the Lorenz-84 model. *Physics Letters A*, 291(6):389–396, 2001.

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