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Zeros of homothetic vector fields in 4-dimensional manifolds of neutral signature

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ABSTRACT

This paper explores the situation regarding the zeros of a proper homothetic vector field X on a 4-dimensional manifold admitting a metric of neutral signature. The types of such zeros are described in terms of the algebraic types of the Ricci tensor and Weyl tensor at the said zero together with a geometrical description of the set of such zeros. A comparison is made between the situation occurring here and that for positive definite and Lorentz signatures. Examples are given to show that many of the theoretical possibilities derived for such zeros actually exist.

1. Introduction

Let M be a 4-dimensional, connected, Hausdorff, paracompact manifold admitting a global, smooth metric g of neutral signature $(+,+,-,-)$. If T_mM denotes the tangent space to M at $m \in M$ and $u, v \in T_mM$, $u \cdot v \equiv g(m)(u, v)$ denotes their inner product at m. A non-zero member $u \in T_mM$ is called *spacelike* if $u \cdot u > 0$, *timelike* if $u \cdot u < 0$ and *null* if $u \cdot u = 0$ and the 1-dimensional subspace (*direction*) spanned by u is called, respectively, *spacelike*, *timelike* and *null*. A 2-dimensional subspace (*2-space*) V of T_mM is called *spacelike* if each non-zero member of V is spacelike, or each non-zero member of V is timelike, *timelike* if V contains exactly two, distinct, null directions, *null* if V contains exactly one null direction and *totally null* if each non-zero member of V is null. Thus a totally null 2-space consists, apart from the zero vector, entirely of null vectors any two of which are orthogonal. This exhausts all possibilities for 2-spaces. For later use a 2-dimensional submanifold $N \subset M$ is called *totally null* if its tangent space is a totally null 2-space at each point of N. The Levi-Civita connection associated with g is denoted by ∇.

Geometrical pairs such as (M, g) above have been studied in physics in connection with the twistor approach to string theory and through the idea of a null Kähler structure, that is, a pair (M, g) admitting a real spinor field which is parallel (covariantly constant) with respect to ∇. Definitions and further details may be found in [3,7,10,8]. In the present paper a study of the zeros of homothetic vector fields on such structures will be made. Such a study, in the case when g has *Lorentz signature*, proved interesting in the characterisation of generalised plane waves in general relativity [2,11,12] which are precisely those cases where such zeros are non-isolated. In the case of neutral signature the study of such zeros is more complicated and a richer, more interesting geometrical structure evolves. This will be dealt with here.

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A (global, smooth) vector field X with associated local flows denoted by ϕ_t is called *homothetic* if each of the smooth maps $\phi_t : U \to \phi_t(U)$, where U is an open subset in M, satisfies the (local) homothety condition $\phi_t^* g = ag$ for some constant $a \in \mathbb{R}$, where an upper asterisk denotes the pullback map. These conditions are equivalent to $\mathcal{L}_X g = 2cg$ for some constant $c \in \mathbb{R}$ (where $\mathcal{L}$ denotes a Lie derivative) and also to

$$X_{a;b} = cg_{ab} + E_{ab}, \tag{1}$$

where a semi-colon denotes a covariant derivative with respect to ∇ and E is the *homothetic bivector field* of X with components $E_{ab} = -E_{ba}$. If c is non-zero, X is called *proper homothetic* whilst if $c = 0$ it is a *Killing vector field* (and then E is the *Killing bivector field* of X). The collection of Killing and homothetic vector fields on M are each Lie algebras of vector fields under the usual Lie bracket operation and are denoted, respectively, by $K(M)$ and $H(M)$. Thus $K(M) \subset H(M)$ and, as is well known, $\dim K(M) \leq 10$ and $\dim H(M) \leq 1 + \dim K(M)$ (since if $X, Y \in H(M)$ are linear independent some linear combination of them is Killing). If $X \in H(M)$ and $m \in M$ with $X(m) = 0$, m is called a *zero* of X (and is a *fixed point* of any local flow ϕ_t of X whose domain of definition contains m since then $\phi_t(m) = m$). Then if U is a coordinate neighbourhood of m with coordinate functions x^a, chosen as normal coordinates, the vector space isomorphism $\phi_{t*} : T_m M \to T_m M$, with ϕ_{t*} being the usual pushforward linear map, is represented in the basis $(\frac{\partial}{\partial x^a})_m$ by the transpose of the matrix $\exp(tA)$ where A is the matrix $(\frac{\partial X^b}{\partial x^a})_m$ (called the *linearisation* of X at m) and exp is the usual exponential map at m [1]. Here, since X is an affine vector field the maps ϕ_t preserve geodesics and their affine parameters and this means (see, for example, [12]) that the integral curves of X satisfy, in normal coordinates about m and in a standard notation, $\frac{dx^a(t)}{dt} = A^a{}_b x^b$. It follows from (1) that, since X vanishes at m, $A^a{}_b = (c\delta^a_b + E^a{}_b)(m)$. Thus the components of a homothetic vector field (in fact, any affine vector field) satisfy the condition that a coordinate system exists in which they are *linear* functions of the coordinates and the vector field is then said to be *linearisable*. (This trivially follows for *any* vector field X satisfying $X(m) \neq 0$ by the "straightening out lemma" which says that, in this case, a coordinate neighbourhood about m may be chosen in which, say, $X^1 = 1$ and all other components of X are zero.) This linearisability property is very useful for investigating the zeros of affine (including homothetic) vector fields. In particular, if X is a Killing vector field and $X(m) = 0$ any other zeros of X have coordinates x^a satisfying $A^a{}_b x^b = (E^a{}_b)_m x^b = 0$ (using an abuse of notation which identifies the coordinates of a zero of X with a member of $T_m M$ through the normal coordinate system). Since E is skew-symmetric its rank is an even number, either 4 or 2, and hence, in the first case, m is an *isolated zero* of X (meaning there exists an open neighbourhood of m in which m is the only zero) and, in the second case, the zeros of X constitute a 2-dimensional regular submanifold N of M (regular meaning that the manifold topology of N equals its induced topology from the manifold topology of M). These last two results for Killing vector fields are taken from [12] but the first author discovered they had been given earlier in [6].

If X is proper homothetic with a non-isolated zero at m, the zeros of X have coordinates which satisfy, in these normal coordinates, $(c\delta^a_b + E^a{}_b)_m x^b = 0$, that is, $x^a = -\frac{1}{c}(E^a{}_b)_m x^b$ for $c \neq 0$. An obvious contraction of this equation shows that these zeros lie along a null geodesic through m. For positive definite signature such a zero is thus necessarily isolated whilst for Lorentz signature any such zero is either isolated or the zeros constitute (part of) a null geodesic through m [2,12]. For neutral signature it will be seen later that any such zero is either isolated, or the zeros constitute (part of) a null geodesic through m or they comprise a 2-dimensional totally null regular submanifold of M constructed from null geodesics through m. [Intuitively, one may expect results like this since if m_1 and m_2 are zeros of X in some normal coordinate domain about, say, m_1 the local flow maps satisfy $\phi_t(m_1) = m_1$ and $\phi_t(m_2) = m_2$ but the "distance" between $\phi_t(m_1)$ and $\phi_t(m_2)$ is a multiple ($\neq 1$) of the "distance" between m_1 and m_2 (by the homothetic condition) and this can only be if this latter "distance" is zero.]

Henceforth one restricts to the situation when $\dim M = 4$ and g has neutral signature $(+, +, -, -)$. One may choose a pseudo-orthonormal basis x, y, s, t at $m \in M$ with $x \cdot x = y \cdot y = -s \cdot s = -t \cdot t = 1$ and an associated *null* basis of (null) vectors l, n, L, N at m given by $\sqrt{2}l = x + t$, $\sqrt{2}n = x - t$, $\sqrt{2}L = y + s$ and $\sqrt{2}N = y - s$ so that $l \cdot n = L \cdot N = 1$ and all other such inner products between basis members are zero. The associated completeness relations are $g_{ab} = x_a x_b + y_a y_b - s_a s_b - t_a t_b = l_a n_b + n_a l_b + L_a N_b + N_a L_b$.

The collection of all 2-forms (bivectors) at $m \in M$ is denoted by $\Lambda_m M$ and each has even matrix rank. If $B \in \Lambda_m M$ and this rank is 2, B is called *simple* and if 4, it is called *non-simple*. If B is simple it may be written as $B^{ab} = u^a v^b - v^a u^b$ for $u, v \in T_m M$ and the 2-space spanned by u and v is uniquely determined by B and called the *blade* of B (and then, unless more precision is required, B or its blade is written $u \wedge v$). A simple bivector is called *spacelike* (respectively, *timelike*, *null* or *totally null*) if its blade is *spacelike* (respectively, *timelike*, *null* or *totally null*). Now define the subspaces $\overset{+}{S}_m \equiv \{B \in \Lambda_m M : \overset{*}{B} = B\}$ and $\overset{-}{S}_m \equiv \{B \in \Lambda_m M : \overset{*}{B} = -B\}$ of $\Lambda_m M$ where $*$ denotes the Hodge duality operator. For neutral signature, $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are Lie algebras isomorphic to $o(1, 2)$. Because of the existence of the metric one may (with care and within the obvious rules) disregard the position of indices and then $\Lambda_m M = \overset{+}{S}_m \oplus \overset{-}{S}_m$ [18,16].

Denote the Riemann tensor from ∇ by *Riem* with components $R^a{}_{bcd}$ and with associated Ricci tensor *Ricc* with components $R_{ab} \equiv R^c{}_{acb}$ and Ricci scalar $R \equiv R_{ab} g^{ab}$. The Weyl conformal tensor is labelled C and given by

$$C(X,Y)Z = Riem(X,Y)Z + \frac{1}{2}\big(Ricc(X,Z)Y - Ricc(Y,Z)X + g(X,Z)QY - g(Y,Z)QX\big) + \frac{R}{6}\left(g(Y,Z)X - g(X,Z)Y\right)$$

where X, Y, Z are any (local) smooth vector fields on M and Q is the Ricci operator defined by $g(QX, Y) = Ricc(X, Y)$. In any local coordinate system x^a, $Q(\partial/\partial x^a) = R^d_a(\partial/\partial x^d)$ and then

$$C^a{}_{bcd} = R^a{}_{bcd} + \frac{1}{2}(\delta^a{}_d R_{bc} - \delta^a{}_c R_{bd} + g_{bc}R^a{}_d - g_{bd}R^a{}_c) + \frac{R}{6}(\delta^a{}_c g_{bd} - \delta^a{}_d g_{bc}). \tag{2}$$

2. Algebraic classification of the Ricci and Weyl (conformal) tensors and bivectors

In this section, a study of the algebraic possibilities for $Ricc$ and C at $m \in M$ for neutral signature can be summarised and specialised to the situation when m is a zero of a homothetic vector field. (The case when g has positive definite signature is trivial and the Lorentz signature case is discussed in [12]). For a non-zero Ricci tensor $Ricc(m)$ a study of the associated linear map on T_mM given in components by $k^a \to R^a{}_b k^b$ and with eigen equation $R_{ab}k^b = \alpha g_{ab}k^b = \alpha k_a$ for an eigenvector k in the complexification of T_mM and corresponding eigenvalue $\alpha \in \mathbb{C}$ shows that the possible Jordan forms for $Ricc(m)$, when written in Segre notation, are [16,13]; $\{1111\}$, $\{z\bar{z}11\}$, $\{z\bar{z}w\bar{w}\}$, $\{211\}$, $\{2z\bar{z}\}$, $\{22\}$ (over $\mathbb{R}$), $\{22\}$ (over $\mathbb{C}$), $\{31\}$ or $\{4\}$ or one of their possible degeneracies (which will be denoted by enclosing the relevant indices inside round brackets). Here the integers inside the Segre brackets refer to real elementary divisors except in one of the $\{22\}$ cases, where they are complex and the notation $z\bar{z}$ inside Segre brackets refers to a complex conjugate pair of eigenvectors (simple elementary divisors). (The term "complex", when applied to an eigenvector or eigenvalue, will always mean "complex but not real".) The trivial case when $Ricc(m) = 0$ is denoted type O. Any eigenvector associated with a non-simple elementary divisor is necessarily null and any real or complex *null* eigenvector is either associated with a non-simple elementary divisor or its eigenvalue is degenerate (that is, its associated eigenspace has dimension ≥ 2). These results are quite general for a (real) symmetric tensor [17,16,13]. It is remarked for later use that if $Ricc(m) \neq 0$ *but all its eigenvalues are zero* the type for $Ricc$ is $\{(211)\}$, $\{(31)\}$, $\{(22)\}$ or $\{4\}$ in each case with zero eigenvalue.

For a bivector F in neutral signature one studies the similar eigen problem $F_{ab}k^b = \alpha k_a$ [16,13]. The possible Jordan types which arise and again written in Segre notation are, first for simple bivectors, $\{z\bar{z}(11)\}$ (F spacelike and with eigenvalues $\pm i\alpha, 0, 0$ with $\alpha \in \mathbb{R}$), $\{11(11)\}$ (F timelike with eigenvalues $\pm\beta, 0, 0$, $\beta \in \mathbb{R}$), $\{(31)\}$ (F null) and $\{(22)\}$ (F totally null) (with eigenvalue zero in the last two cases). Second, for non-simple bivectors, these Segre types are $\{z\bar{z}w\bar{w}\}$, $\{(zz)(\bar{z}\bar{z})\}$, $\{22\}$ (with real eigenvalues), $\{22\}$ (with complex eigenvalues), $\{1111\}$ and $\{(11)(11)\}$. For a non-simple bivector either all eigenvalues are real or all are complex. No further degeneracies are possible in any of these cases. For later use it is remarked here that the search for (non-isolated) zeros of a homothetic vector field discussed in section 1 involved finding eigenvectors of the homothetic bivector with a specific non-zero eigenvalue, equal to the negative of the homothetic constant, $-c$. Such eigenvectors are necessarily null (since E is skew-symmetric) and the algebraic study of bivectors in Lorentz signature shows that at most one exists for a given value $-c$, up to a scaling. However, in neutral signature the eigenspace corresponding to the eigenvalue $-c$ may be 2-dimensional and hence a 2-dimensional submanifold of zeros may arise. Examples will be provided later.

For the Weyl tensor in neutral signature a classification has been considered in [14,16] and is based on the Petrov classification [17] and the associated Bel criteria [5], in Lorentz signature, this latter having proved rather useful in general relativity theory. The details are a little complicated but, fortunately, only some basic ideas from this are needed and these can be described here. First, in terms of a null basis l, n, L, N at m, define bivectors $F, G, H, \bar{F}, \bar{G}, \bar{H}$ at m by

$$F \equiv \frac{1}{2}(l \wedge n - L \wedge N), \qquad G \equiv \frac{1}{\sqrt{2}}(l \wedge N), \qquad H \equiv \frac{1}{\sqrt{2}}(n \wedge L),$$
$$\bar{F} \equiv \frac{1}{2}(l \wedge n + L \wedge N), \qquad \bar{G} \equiv \frac{1}{\sqrt{2}}(l \wedge L), \qquad \bar{H} \equiv \frac{1}{\sqrt{2}}(n \wedge N). \tag{3}$$

Then F, G and H are independent members of $\overset{+}{S}_m$ and $\bar{F}, \bar{G}$ and $\bar{H}$ are independent members of $\overset{-}{S}_m$. Thus $\overset{+}{S}_m$=Span(F, G, H) and $\overset{-}{S}_m$=Span$(\bar{F}, \bar{G}, \bar{H})$ so that each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ is 3-dimensional. Next define the *bivector metric* $\widetilde{G}$ with components $\widetilde{G}_{abcd} \equiv \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})$ and so that for any bivector Y, $Y_{ab} = \widetilde{G}_{abcd}Y^{cd}$ and for any other bivector Y' let $Y \cdot Y' \equiv Y_{ab}Y'^{ab} = \widetilde{G}_{abcd}Y^{ab}Y'^{cd}$ and $|Y| \equiv Y \cdot Y$, which should not cause any confusion with the notation for the inner product arising from the metric g. Then one has

$$|F| = |\bar{F}| = -1, \qquad |G| = |H| = |\bar{G}| = |\bar{H}| = 0,$$
$$F \cdot G = F \cdot H = \bar{F} \cdot \bar{G} = \bar{F} \cdot \bar{H} = 0, \qquad G \cdot H = \bar{G} \cdot \bar{H} = 1 \tag{4}$$

and a straightforward calculation shows that if $P \in \overset{+}{S}_m$ and $Q \in \overset{-}{S}_m$, necessarily $P \cdot Q = 0$.

The classification of C then consists of changing to the type $(0,4)$ tensor C' with components $C_{abcd} \equiv g_{ae}C^{e}{}_{bcd}$ and splitting C' into its dual and self-dual parts using the duality operator on either the second pair of indices (C^*) or the first pair ($^*C$) to obtain

$$C = \overset{+}{W} + \overset{-}{W}, \qquad \overset{+}{W} = \frac{1}{2}(C +^* C), \qquad \overset{-}{W} = \frac{1}{2}(C -^* C). \tag{5}$$

Then $\overset{+}{W}$ may be thought of as a linear map (again abusing index position) from $\overset{+}{S}_m$ to itself given by $Y^{ab} \to \overset{+}{W}_{abcd}Y^{cd}$, it being easy to check that $Y \in \overset{+}{S}_m$ implies that $\overset{+}{W}_{abcd}Y^{cd} \in \overset{+}{S}_m$ (and if $Y \in \overset{-}{S}_m$, $\overset{+}{W}_{abcd}Y^{cd} = 0$). Since $\overset{+}{S}_m$ can be thought of as a 3-dimensional vector space admitting the metric induced from the bivector metric above denoted by $\cdot$ and which then has Lorentz signature and $\overset{+}{W}_{abcd}$ is a symmetric tensor in the block index pairs ab and cd ($\overset{+}{W}_{abcd} = \overset{+}{W}_{cdab}$), the only possible Segre types for this map are (see, e.g., [16,14]) $\{111\}$, $\{1z\bar{z}\}$, $\{21\}$ and $\{3\}$ and their possible degeneracies. The associated canonical forms for $\overset{+}{W}$ in terms of some appropriately chosen basis F, G and H in $\overset{+}{S}_m$ may then be written out, up to possible scalings, as

$$\overset{+}{W}_{abcd} = \rho_1(G_{ab}H_{cd} + H_{ab}G_{cd} + 2F_{ab}F_{cd}) + \rho_2(G_{ab}G_{cd} \pm H_{ab}H_{cd}), \tag{6}$$

$$\overset{+}{W}_{abcd} = \rho_1(G_{ab}H_{cd} + H_{ab}G_{cd} + 2F_{ab}F_{cd}) \pm G_{ab}G_{cd}, \tag{7}$$

$$\overset{+}{W}_{abcd} = G_{ab}F_{cd} + F_{ab}G_{cd} \tag{8}$$

where $\rho_1, \rho_2 \in \mathbb{R}$ and where the $+$ sign (respectively the -sign) in (6) gives type $\{111\}$ (respectively, type $\{1z\bar{z}\}$), (7) gives type $\{21\}$ and (8) is type $\{3\}$. The types in (6), (7) and (8) are, in memory of the Petrov classification, labelled types **I**, **II** and **III**, respectively. One important special case is type $\{(21)\}$ which is (7) with $\rho_1 = 0$ and labelled type **N**. The type **I** may be subdivided but this is not important here. The type **N** case may be written in some basis as above, and up to a scaling, as

$$\overset{+}{W}_{abcd} = \pm G_{ab}G_{cd}. \tag{9}$$

If $C(m) = 0$ the type of $C(m)$ is labelled **O**. Similar comments apply to $\overset{-}{W}$ and $\overset{-}{S}_m$. Thus the type of $C(m)$ is written (**A**, **B**) where **A** (respectively, **B**) is the type for $\overset{+}{W}$ (respectively, $\overset{-}{W}$). The types (**A**, **B**) and (**B**, **A**) are identified. It follows that the only types for $\overset{+}{W} \neq 0$ *if all its eigenvalues are zero* are **N** and **III** and similarly for $\overset{-}{W}$. If one now defines a eigenvector/eigenvalue pair (Y, λ) for $C(m)$ by $C_{abcd}Y^{cd} = \lambda Y_{ab}$, with Y in the complexified version of $\Lambda_m M$ and $\lambda \in \mathbb{C}$, it follows easily from (5) that the only possibilities for $C(m) \neq 0$ if all its eigenvalues are zero are (**N**, **O**), (**III**, **O**), (**N**, **N**), (**III**, **III**) and (**N**, **III**) since any non-zero eigenvalue for $\overset{+}{W}(m)$ or $\overset{-}{W}(m)$ would give rise to a non-zero eigenvalue for $C(m)$. This last step will be important later.

An alternative way to view this classification is through the existence of certain "special" null directions at $m \in M$, as observed in the Lorentz case by Bel [5]. Thus at $m \in M$ consider the equations

$$k_{[e}C_{a]bc[d}k_{f]}k^b k^c = 0, \qquad C_{abcd}k^b k^c = k_a q_d + q_a k_d, \tag{10}$$

for a *null* vector $k \in T_m M$ and a 1-form q at m [14,16] (and note the typo in [14]). They are easily checked to be equivalent to each other and k is said to span a *principal null direction* (a pnd) for $C(m)$. In the special case that q is a multiple of k (possibly zero), k is said to span a *repeated principal null direction* (repeated pnd) for $C(m)$ and the first equation in (10) then yields the equivalent condition $k_{[e}C_{a]bcd}k^b k^d = 0$. A pnd which is not repeated is called *general*. For example, if $C(m)$ is of type (**N**, **N**) it can be shown that $\overset{+}{W}$ may be written in terms of G as in (9) and similarly for $\overset{-}{W}$ in terms of $\bar{G}$. There is exactly one repeated pnd for $C(m)$ spanned by l and no general pnds and l satisfies $C_{abcd}l^d = 0$ and spans the unique null direction with this property. If $C(m)$ is of type (**III**, **III**) it can be shown that $\overset{+}{W}$ may be written in terms of G and F as in (8) and similarly for $\overset{-}{W}$ in terms of $\bar{G}$ and $\bar{F}$. There is then exactly one repeated pnd, spanned by l and satisfying $C_{abcd}l^b l^d = 0$ (and spans the unique null direction with this property) and three independent general pnds spanned by n, L and N. For type (**N**, **III**) there is one repeated pnd spanned by l and one general pnd spanned by N. For type (**N**, **O**) there are infinitely many repeated pnds (and no other pnds) and are precisely those directions in the blade of G in (9) whilst for type (**III**, **O**) there are infinitely many repeated pnds comprising the blade of G (that is, $l \wedge N$) in (8) and infinitely many general pnds comprising the blade of H (that is, $n \wedge L$).

3. Algebraic restrictions at a homothetic zero

Whatever the dimension of M or the signature of g, the Ricci, Riemann and Weyl tensors at a zero of a Killing vector field exhibit certain obvious, natural symmetries amongst their eigenvectors. Restrictions also arise at the zeros of proper homothetic vector fields (see [12]). So with g of neutral signature let X be a *proper* homothetic vector field on M with a zero at m. Then $\mathcal{L}_X Ricc = 0$, $\mathcal{L}_X C = 0$ (since X is affine), $\mathcal{L}_X g = 2cg$, $\phi_t^* g = e^{2ct} g$ and $\phi_t(m) = m$. Thus regarding $Ricc$ as a map on pairs of vectors in $T_m M$ one sees that if $k \in T_m M$ is an eigenvector of $Ricc$ with eigenvalue $\alpha \in \mathbb{C}$, as above, this is equivalent to $Ricc(k, k') = \alpha g(k, k')$ for any $k' \in T_m M$. But then, since $\mathcal{L}_X Ricc = 0$, $\phi_t^* Ricc = Ricc$ and so (recalling $\phi_t(m) = m$)

$$\phi_t^* Ricc(k, k') = Ricc(k, k') = \alpha g(k, k') = \alpha e^{-2ct} \phi_t^* g(k, k')$$
$$\Rightarrow Ricc(\phi_{t*} k, \phi_{t*} k') = \alpha e^{-2ct} g(\phi_{t*} k, \phi_{t*} k') \tag{11}$$

(thus correcting an annoying typo in [12]). Thus, since ϕ_{t*} is an isomorphism on $T_m M$, and $k' \in T_m M$ is arbitrary, $\phi_{t*} k'$ runs through all of $T_m M$ and it follows that each $\phi_{t*} k$ is an eigenvector of $Ricc$ with eigenvalue αe^{-2ct}. However, if $\alpha \neq 0$ this would lead to infinitely many distinct eigenvalues for $Ricc(m)$ and a contradiction. So $\alpha = 0$ and hence all eigenvalues of $Ricc$ are zero at m. Thus if $Ricc(m) \neq 0$, it must, from the results of section 2, have type {(211)}, {(22)}, {(31)} or {4} and with zero eigenvalue in each case.

Regarding the Weyl tensor C as a linear map on $\Lambda_m M$, as was done earlier for $\overset{+}{W}$, a similar argument shows that if $\lambda \in \mathbb{C}$ is an eigenvalue of C at m then $\lambda = 0$. To see this note that if C is the Weyl tensor with components $C^a{}_{bcd}$ and X is a homothetic vector field on M then $\mathcal{L}_X C = 0$. Hence with C' denoting the type $(0, 4)$ Weyl tensor, as before, with components $C_{abcd} = g_{ae} C^e{}_{bcd}$, $\mathcal{L}_X C' = 2cC'$ and so $\phi_t^* C' = e^{2ct} C'$. Next, the bivector metric $\widetilde{G}$ satisfies $\mathcal{L}_X \widetilde{G} = 4c\widetilde{G}$ and so $\phi_t^* \widetilde{G} = e^{4ct} \widetilde{G}$. Viewing C' as a mapping $\Lambda_m M \otimes \Lambda_m M \to \mathbb{R}$ given by $(P, Q) \to C_{abcd} P^{ab} Q^{cd}$ for bivectors P, Q, and using a similar notation for $\widetilde{G}$, the equivalent condition that S is an eigenbivector of C (or C'), with eigenvalue $\lambda \in \mathbb{C}$ ($C_{abcd} S^{cd} = \lambda S_{ab}$) is then that $C'(S, Q) = \lambda \widetilde{G}(S, Q)$ for all bivectors $Q \in \Lambda_m M$. So

$$\phi_t^* C'(S, Q) = e^{2ct} C'(S, Q) = e^{2ct} \lambda \widetilde{G}(S, Q) = e^{2ct} \lambda e^{-4ct} \phi_t^* \widetilde{G}(S, Q)$$
$$\Rightarrow C'(\phi_{t*} S, \phi_{t*} Q) = \lambda e^{-2ct} \widetilde{G}(\phi_{t*} S, \phi_{t*} Q) \tag{12}$$

and since, again, a finite number of distinct eigenvalues for $C(m)$ must necessarily arise, $\lambda = 0$ follows as before and then a previous argument shows that all eigenvalues of C, $\overset{+}{W}$ and $\overset{-}{W}$ are zero at m. So if $C(m) \neq 0$ (that is, its type is not **O**) it follows that $C(m)$ is of type (**N**, **N**), (**III**, **N**), (**III**, **III**), (**N**, **O**) or (**III**, **O**).

Also useful are the equations $\mathcal{L}_X Ricc = 0$ and $\mathcal{L}_X C = 0$ written out in terms of the homothetic bivector E. These equations are, using (1),

$$2cR_{ab} + R_{ac} E^c{}_b + R_{cb} E^c{}_a = 0 \tag{13}$$

and

$$2cC^a{}_{bcd} + C^a{}_{bce} E^e{}_d + C^a{}_{bed} E^e{}_c + C^a{}_{ecd} E^e{}_b - C^e{}_{bcd} E^a{}_e = 0. \tag{14}$$

Finally if m is a zero of the proper homothetic vector field X and if $C(m)$ has type (**N**, **N**), (**III**, **III**) or (**N**, **III**) then, as described earlier, there are *finite* collections of pnds and repeated pnds for $C(m)$ at m. Since $\mathcal{L}_X C = 0$ ($\phi_t^* C = C$), a continuity argument shows that ϕ_{t*} preserves repeated pnds and also general pnds. Thus if $l \in T_m M$ spans a pnd or a repeated pnd of $C(m)$, $\phi_{t*} l$ is a multiple of l, that is, $\phi_{t*} l = \alpha l$, for some smooth function $\alpha(t)$ [12]. Then (section 1) $\exp(tA) l = \alpha l$ and setting $t = 0$ one sees that $\alpha(0) = 1$. Differentiating this last equation gives $A \exp(tA) l = \dot{\alpha} l$ where a dot denotes d/dt. This gives $\alpha A l = \dot{\alpha} l$ and so $\dot{\alpha} \alpha^{-1} = \mu =$constant, that is, $\alpha = e^{\mu t}$ with $\mu \in \mathbb{R}$ and $Al = \mu l$. It follows (section 1) that l is an eigenvector of the homothetic bivector $E(m)$. A similar argument shows that the same is true for the *unique* null eigendirection of $Ricc(m)$ if the latter is of Segre type {(211)}, {(31)} or {4} (with eigenvalues zero).

In the event that $C(m)$ has type (**N**, **O**) the pnds for $C(m)$ are all repeated and constitute the null directions in the totally null 2-space $l \wedge N$ (see (9) where $G = l \wedge N$) and so this collection is not discrete. In this case, as above, one sees that $\phi_{t*} l$ could be any member of $l \wedge N$ as could $\phi_{t*} N$. Thus $\exp(tA) l = \alpha l + \beta N$ and $\exp(tA) N = \gamma l + \delta N$ for α, β, γ and δ smooth functions of t. Putting $t = 0$ gives $\alpha(0) = 1, \beta(0) = 0, \gamma(0) = 0$ and $\delta(0) = 1$ and differentiating gives $A \exp(tA) l = \dot{\alpha} l + \dot{\beta} N$ and $A \exp(tA) N = \dot{\gamma} l + \dot{\delta} N$. So $A(\alpha l + \beta N) = \dot{\alpha} l + \dot{\beta} N$ and $A(\gamma l + \delta N) = \dot{\gamma} l + \dot{\delta} N$ and it is noted that α, δ and $\alpha\delta - \beta\gamma$ are positive on some open interval containing $t = 0$. It follows from linear algebra that Al and AN are linear combinations of l and N and hence that $l \wedge N$ is an invariant 2-space for the linear map represented by $A(m)$ and hence for that represented by $E(m)$. In the event that $C(m)$ has type (**III**, **O**) the repeated pnds constitute the null directions in the totally null 2-space $l \wedge N$ and the general pnds constitute the null directions in the totally null 2-space $n \wedge L$. Thus the 2-spaces $l \wedge N$ and $n \wedge L$ are invariant 2-spaces for the linear map represented by $A(m)$ and hence $E(m)$. If the Ricci tensor has type {(22)} with eigenvalue zero at m there is a 2-dimensional (zero-)eigenspace of $Ricc(m)$ and it is totally null. As above this leads to an invariant 2-space for $E(m)$.

If the zeros of X at m are not isolated they lie along geodesic paths through m whose tangents at m are null and are eigendirections of the bivector $E(m)$ with the same eigenvalue $-c$. A study of the algebra of bivectors (see, e.g. [16, 12] and section 2 above) shows that only isolated zeros are allowed in positive definite signature and at most one such tangent can arise in Lorentz signature in which case the zeros lie along a null geodesic through m. However, for neutral signature and since E is skew-symmetric, it follows from section 2 that exactly one independent such tangent *could* exist and that, should at least two independent such tangents $r, s \in T_mM$ exist, $E_{ab}r^b = -cr_a$ and $E_{ab}s^b = -cs_a$ and so, since $c \neq 0$, obvious contractions show that r and s are orthogonal null vectors. Now suppose that q is another null vector at m which is independent of and orthogonal to r and s. Then $W \equiv$Span(r, s, q) is a 3-dimensional subspace of T_mM and so, since dim$M = 4$, the orthogonal complement $W^\perp$ of W is 1-dimensional. However, it is clear that $W \subset W^\perp$ and a contradiction is obtained. It follows that the submanifold of zeros of a homothetic vector field in neutral signature has dimension ≤ 2 (and that the 2-dimensional option can only occur for neutral signature). One can summarise the work so far in the following theorems in which $m \in M$ is a zero of a proper homothetic vector field X.

Theorem 1. *$Ricc(m)$ necessarily has Segre type $\{(211)\}$, $\{(31)\}$, $\{(22)\}$, $\{4\}$ (with eigenvalue zero in each case) or O. In addition, when a null direction at m is the unique null eigendirection of $Ricc(m)$ it is an eigendirection of the homothetic bivector $E(m)$ for X. When a null direction is a non-unique null eigendirection of $Ricc(m)$ such null eigendirections give rise to a totally null 2-space at m and this 2-space is an invariant subspace for $E(m)$.*

Theorem 2. *The Weyl tensor $C(m)$ has type $(\mathbf{N}, \mathbf{N})$, $(\mathbf{III}, \mathbf{III})$, $(\mathbf{III}, \mathbf{N})$, $(\mathbf{N}, \mathbf{O})$, $(\mathbf{III}, \mathbf{O})$ or $\mathbf{O}$. In addition, if a null direction at m is a unique repeated pnd or a unique general pnd of $C(m)$ it is an eigendirection of the homothetic bivector $E(m)$ for X. When $C(m)$ has a non-unique repeated pnd or a non-unique general pnd such repeated pnds or general pnds give rise to totally null 2-space(s) at m and then these 2-spaces are invariant subspaces for $E(m)$.*

Theorem 3. *In neutral signature the set of zeros of X consists locally of either an isolated point m, (part of) a null geodesic through m or a 2-dimensional totally null submanifold through m. The 2-dimensional case here can only arise in neutral signature.*

4. Ricci types at a zero of *X*

Suppose that X is a proper homothetic vector field on M satisfying (1), let $m \in M$ with $X(m) = 0$ and let $Ricc(m)$ be of Segre type $\{(211)\}$ with zero eigenvalue. It then takes the form $R_{ab} = \lambda l_a l_b$ for $0 \neq \lambda \in \mathbb{R}$ and l is a null member of T_mM and spans the unique, null eigendirection of $Ricc(m)$ [16,13]. So, by Theorem 1, l is an eigenvector of E at m, $E_{ab}l^b = \mu l_a = -l^b E_{ba}$ for $\mu \in \mathbb{R}$, and (13) holds at m. Thus

$$2c\lambda l_a l_b + \lambda l_a l_c E^c{}_b + \lambda l_b l_c E^c{}_a = 0. \tag{15}$$

From this it easily follows that $c = \mu$ so that $\mu \neq 0$ and then $E_{ab}l^b = cl_a$. If E is simple then since it has a null eigenvector with non-zero eigenvalue it must be timelike and of the form $c(l \wedge q)$ with $q \in T_mM$ null and $l \cdot q = 1$. In this case $E_{ab}q^b = -cq_a$ and q is the only such solution of this equation up to a scaling. Hence, in this case, the zeros of X constitute a part of a null geodesic through m. Next suppose E is not simple and, since it has at least one real eigenvalue, all its eigenvalues are real. It then follows [16,13] that E may be written as the linear combination

$$E = c(l \wedge n) + \alpha(L \wedge N) + \beta(l \wedge N) + \gamma(l \wedge L) \tag{16}$$

with $\alpha, \beta, \gamma \in \mathbb{R}$. Since E is not simple, $\alpha \neq 0$ (otherwise E would be simple with blade containing l). If $c \neq \alpha$ one may change the basis l, n, L, N to $l' = l$, $N' = N$, $n' = n + \kappa N$ and $L' = L - \kappa l$ ($\kappa \in \mathbb{R}$) to set β to zero. Similarly, if $c \neq -\alpha$ a basis change $l' = l$, $L' = L$, $n' = n + \kappa L$, $N' = N - \kappa l$ can be used to set $\gamma = 0$. Thus either one has (16) with $c \neq \pm\alpha$ and $\beta = \gamma = 0$ or, $c = \alpha (\Rightarrow c \neq -\alpha)$ and $\gamma = 0$ and so $E_{ab}N^b = -cN_a$ (the case $c = -\alpha (\Rightarrow c \neq \alpha)$ and $\beta = 0$ and hence $E_{ab}L^b = -cL_a$ is similar.) In the first of these cases one has (16) with $\beta = \gamma = 0$ and $c \neq \pm\alpha$ and E is of Segre type $\{1111\}$ with a 1-dimensional zero set since, then, $E_{ab}n^b = -cn_a$. The other cases lead to either the Segre type $\{(11)(11)\}$ for E (if β and γ both vanish and either $c = \alpha$ or $c = -\alpha$) and a 2-dimensional zero set spanned by n and either L or N, or to Segre type $\{22\}$ (over $\mathbb{R}$) for E (which arises when $c = \pm\alpha$ and exactly one of β and γ is zero) and a 1-dimensional set of zeros. Thus m is never isolated in this case.

Now suppose that $Ricc(m)$ is of Segre type $\{(31)\}$ with zero eigenvalue. Then a null basis may be chosen at m so that $R_{ab} = \lambda(l_a y_b + y_a l_b)$ where $0 \neq \lambda \in \mathbb{R}$, $l, y \in T_mM$ with l null, y spacelike or timelike and $l.y = 0$ [16,13] and with l spanning the unique, null eigendirection of $Ricc(m)$. Then (1) holds and $E_{ab}l^b = \mu l_a$. Then a substitution into (13) and a contraction with y^a immediately gives $\mu = 2c \neq 0$ and then $E_{ab}y^b = 0$. It follows that E is necessarily simple and timelike and a null basis l, n, L, N may be constructed so that $E_{ab} = 2c(l_a n_b - n_a l_b)$. Thus there are no null solutions for k of $E_{ab}k^b = -ck_a$ and the zero m is isolated.

Next suppose that $Ricc(m)$ has Segre type {(22)} with eigenvalue zero. There are two possibilities for the canonical form for $Ricc(m)$ here [16,13]. The first is $R_{ab} = \mu l_a l_b + \nu L_a L_b$ for $\mu, \nu \in \mathbb{R}$ and $\mu\nu > 0$ and where l and L are part of a null basis l, n, L, N at m. The 2-space $l \wedge L$ is a totally null 2-space at m which is the zero eigenspace for $Ricc(m)$ and hence, from Theorem 1, an invariant 2-space for the linear map represented by E. Thus $l^a E_{ab} = \alpha l_b + \beta L_b$, $L^a E_{ab} = \gamma l_b + \delta L_b$ ($\alpha, \beta, \gamma, \delta \in \mathbb{R}$). A substitution into (13) gives $\alpha = \delta = -c(\neq 0)$ and $\mu\beta + \nu\gamma = 0$. So either $\alpha = \delta = -c$ and $\beta = \gamma = 0$ or $\alpha = \delta = -c$ and $\mu\beta = -\nu\gamma \neq 0$. In the second case (and scaling l and L so that $\mu = \nu = \pm 1$), $l \pm iL$ are eigenvectors of E with eigenvalues $-c \pm i\gamma$ and so E has Segre type $\{z\bar{z}w\bar{w}\}$ and is non-simple (since $c \neq 0$–see section 2) with no real eigenvalues and so the zero m is isolated. In the first case expanding E in terms of the basis bivectors $l \wedge n$, $L \wedge N$, $l \wedge N$, $l \wedge L$, $n \wedge N$ and $n \wedge L$, then since $lE = -cl$ and $LE = -cL$ (which is easily checked to imply that E is non-simple) one gets $E = c(l \wedge n + L \wedge N)$ (a term in $l \wedge L$ may be removed by a change of basis similar to that given in a previous case) and so E has Segre type {(11)(11)} and is non-simple. [The apparent bias seen in the solution for E being a member of $\overset{-}{S}_m$ is explained by the choice of eigenspace $l \wedge L \in \overset{-}{S}_m$ for $Ricc(m)$.] In this case the zeros of X constitute a 2-dimensional submanifold of M since n and N satisfy the equation $E_{ab}k^b = -ck_a$. The homothetic bivector is non-simple.

The other possibility for $Ricc(m)$ of type {(22)} and eigenvalue zero is given, in a null basis l, n, L, N at m by $R_{ab} = \lambda(l_a L_b + L_a l_b)$ ($0 \neq \lambda \in \mathbb{R}$ and $l \wedge L$ is the totally null zero eigenspace of $Ricc(m)$) and again $l^a E_{ab} = \alpha l_b + \beta L_b$, $L^a E_{ab} = \gamma l_b + \delta L_b$. A substitution into (13) gives $\beta = \gamma = 0$ and $2c + \alpha + \delta = 0$. So $l^b E_{ba} = \alpha l_a$, $L^b E_{ba} = \delta L_a$ and $\alpha + \delta = -2c$. It follows that if one of α and δ is zero E is simple whilst if they are both non-zero *and* if E is simple, a consideration of the blade of E shows that it is a multiple of $l \wedge L$ and then one achieves the contradiction $l^b E_{ba} = L^b E_{ba} = 0$. Since $\alpha = \delta = 0 \Rightarrow c = 0$ and hence a contradiction, E is simple $\Leftrightarrow$ exactly one of α and δ is zero (and then E is necessarily timelike). If, say, $\alpha = 0 \neq \delta$ then $\delta = -2c$ and expanding E out in terms of a bivector basis shows that there exists a null basis $\hat{l}, \hat{n}, L, \hat{N}$ in which $E_{ab} = 2c(L_a \hat{N}_b - \hat{N}_a L_b)$ and the zero m is isolated (and similarly if $\alpha \neq 0 = \delta$). If E is not simple, $\alpha\delta \neq 0$ (and $\alpha + \delta \neq 0$ because $c \neq 0$), one may write in the original basis, $E = -\alpha(l \wedge n) - \delta(L \wedge N) + \rho(l \wedge L)$ for $\rho \in \mathbb{R}$. Then since $\alpha \neq -\delta$ one may, by a base change of the form $l' = l$, $L' = L$, $n' = n + \kappa L$, $N' = N - \kappa l$, set $\rho = 0$ and E is then of Segre type {1111}. The possibilities are then either $\alpha = -c (\Leftrightarrow \delta = -c \Leftrightarrow \alpha = \delta)$ with E of Segre type {(11)(11)} or $\alpha = c, \delta = -3c$ or $\delta = c, \alpha = -3c$ or $\alpha \neq c \neq \delta \neq \alpha$. These give, respectively, a 2-dimensional zero set, a 1-dimensional zero set, a 1-dimensional zero set and an isolated zero.

Now suppose $Ricc(m)$ has Segre type {4} with eigenvalue zero. Then [16,13] in some null basis at m,

$$R_{ab} = l_a L_b + L_a l_b + \nu N_a N_b \tag{17}$$

for $0 \neq \nu \in \mathbb{R}$. The null eigenvector for $Ricc(m)$ is l and is unique up to a scaling. Thus $l^b E_{ba} = \lambda l_a$ ($\lambda \in \mathbb{R}$) and one can substitute into (13). After a lengthy calculation one gets that E is diagonal and $E = 3c(l \wedge n) - c(L \wedge N)$ and is non-simple of Segre type {1111}. The zero m is not isolated being part of a 1-dimensional geodesic of zeros arising from L.

It is remarked that $Ricc$ may be identically zero on M (the *Ricci flat* case) or $Ricc$ may not be identically zero on M but may vanish at some or all of the zeros of X.

5. Weyl types at a zero of X

In this section the Weyl type at m is one of (**N**, **O**), (**III**, **O**), (**N**, **N**), (**III**, **III**) and (**N**, **III**) and (14) holds.

If the type of $C(m)$ is (**N**, **N**) there is a unique repeated pnd spanned by $l \in T_m M$ and satisfying $C_{abcd} l^d = 0$ (see section 2) and one may choose a null basis l, n, L, N in which $C(m)$ may be written in the notation of section 2 as

$$C_{abcd} = A G_{ab} G_{cd} + B \bar{G}_{ab} \bar{G}_{cd} \tag{18}$$

with $A, B \in \mathbb{R}$, $A \neq 0 \neq B$. To see this one writes the Weyl tensor as $C = \overset{+}{W} + \overset{-}{W}$ and then decomposes (in an obvious abbreviated notation) $\overset{+}{W}$ as the most general linear combination of FF, GG, HH, $FG + GF$, $FH + HF$ and $GH + HG$ and similarly for $\overset{-}{W}$ in terms of $\bar{F}\bar{F}$, etc. A contraction with l and use of the above pnd condition then shows that only the terms GG and $\bar{G}\bar{G}$ remain. One also sees that $\phi_{t*} l$ is a multiple of l and so, from (1) and Theorem 2, $E_{ab} l^b = \mu l_a$. [A substitution of (18) into (14) actually gives this latter result again, by the uniqueness of the pnd.] It follows that E is not spacelike (since a spacelike bivector has no null eigenvectors). To see that E cannot be null or totally null note that if it was (and since l is an eigenvector of E) it must take either the form $E = l \wedge y$ or $E = l \wedge s$ for the null case, where l belongs to the above null basis used in (18), y is spacelike, s timelike and $l \cdot y = l \cdot s = 0$, or $l \wedge P$ in the totally null case where $P \in T_m M$ is null and orthogonal to l (and which implies that one may choose $P = L$ or $P = N$ again in the null basis used in (18)). A substitution of any of these forms into (14) and a contraction with N^d reveals in each case one of the contradictions that $A = 0$ or $B = 0$. It follows that E is timelike or non-simple and since $E_{ab} l^b = \mu l_a$, a general expression for E can be written in terms of the basis $F \pm \bar{F}$, G, H, $\bar{G}$ and $\bar{H}$ in (3) as

$$E_{ab} = \mu(l_a n_b - n_a l_b) + \nu(L_a N_b - N_a L_b) + \omega(l_a N_b - N_a l_b) + \rho(l_a L_b - L_a l_b) \tag{19}$$

for $\nu, \omega, \rho \in \mathbb{R}$. On substituting into (14) and contracting first with N^d and then with L^d one finds $c = \mu + \nu$ and then $c = \mu - \nu$. Thus $\nu = 0$ and $\mu = c \neq 0$ and so, from (19) E is simple timelike and may be written as $E = c(l \wedge n')$ for n' null and $l \cdot n' = 1$. Then $E_{ab}n'^b = -cn'_a$ and a 1-dimensional submanifold of zeros arises.

Now suppose that $C(m)$ is of type (**III**, **III**). In this case (see [16,14]) $C = \overset{+}{W} + \overset{-}{W}$ at m and one may assume that a basis l, n, L, N is chosen where $\overset{+}{W}$ takes the form (8) and $\overset{-}{W}$ is a general linear combination of $\bar{F}\bar{F}$, $\bar{G}\bar{G}$, $\bar{H}\bar{H}$, $\bar{F}\bar{G} + \bar{G}\bar{F}$, $\bar{F}\bar{H} + \bar{H}\bar{F}$ and $\bar{G}\bar{H} + \bar{H}\bar{G}$ and with l a repeated pnd and n, L and N general pnds for $C(m)$. Thus one requires $C_{abcd}l^b l^d = 0$ whilst $C_{abcd}k^b k^c = k_a q_d + q_a k_d$ holds for $k = n, L, N$ and with q (dependent on k), non-zero and not parallel to k (see (10)). These conditions together with the tracefree condition $C^c{}_{acb} = 0$ then show that the only terms remaining are

$$C_{abcd} = \beta(F_{ab}G_{cd} + G_{ab}F_{cd}) + \bar{\beta}(\bar{F}_{ab}\bar{G}_{cd} + \bar{G}_{ab}\bar{F}_{cd}) \tag{20}$$

with $\beta, \bar{\beta} \in \mathbb{R}$ and $\beta \neq 0 \neq \bar{\beta}$. Since ϕ_{t*} preserves each repeated pnd and each non-repeated pnd, and recalling $l \cdot n = L \cdot N = 1$ and $E_{ab} = -E_{ba}$, one has $E_{ab}l^b = \mu l_a$, $E_{ab}n^b = -\mu n_a$, $E_{ab}L^b = \nu L_a$ and $E_{ab}N^b = -\nu N_a$ ($\mu, \nu \in \mathbb{R}$). Then a substitution into (14) and a contraction with $N^b N^d n^c$ gives $-2c + \mu + \nu = 0$ whilst a contraction with $L^b L^d n^c$ gives $-2c + \mu - \nu = 0$. So $\nu = 0$ and $\mu = 2c \neq 0$ and E is simple and timelike, $E_{ab} = 2c(l_a n_b - n_a l_b)$. It follows that the zero m is isolated.

If $C(m)$ is of type (**N**, **O**), (**III**, **O**) or (**III**, **N**) the calculations are similar but complicated and are omitted. Suffice it to say that if the Weyl type is (**III**, **N**), E cannot be simple and can only be Segre type {1111} and the zero must be isolated. If the Weyl type is (**N**, **O**) or (**III**, **O**), E can be simple or non-simple and the zero need not be isolated. An example of the first of this latter pair will be given later.

The Weyl tensor may also vanish identically on M or at some or all of the zeros of X. Examples of many of the possibilities allowed by the above restrictions are collected together in the final section. However, it is not claimed that all of the possibilities for $Ricc(m)$ and $C(m)$ and the associated zeros and homothetic bivectors given above can actually arise, only what would transpire should they arise. The examples in section 6 throw some light on this. In summary, one has the following theorems in which m is a zero of a proper homothetic vector field X with homothetic bivector $E(m)$.

Theorem 4. *If $Ricc(m)$ is of type {(211)} with eigenvalue zero the homothetic bivector $E(m)$ may be simple or non-simple and there is a 1- or a 2-dimensional submanifold of zeros of X. If $Ricc(m)$ is of type {(31)} with eigenvalue zero the homothetic bivector $E(m)$ is simple and timelike and the zero m is isolated. If $Ricc(m)$ is of type {(22)} with eigenvalue zero $E(m)$ may be simple or non-simple (and if non-simple, possibly with complex eigenvalues), the zero is either isolated or they constitute a 1- or a 2-dimensional submanifold of M. If $Ricc(m)$ has type {4} with eigenvalue zero the homothetic bivector $E(m)$ is non-simple and a 1-dimensional submanifold of zeros arises.*

*If $C(m)$ is of type (**N**, **N**) the homothetic bivector $E(m)$ is simple and timelike and a 1-dimensional submanifold of zeros arises. If $C(m)$ is of type (**III**, **III**), the homothetic bivector is again simple and timelike but the zero is isolated. [But see example E in section 6.]*

It thus follows that if $C(m)$ is of type (**N**, **N**), $Ricc(m)$ is necessarily of type {(211)} with eigenvalue zero, or O, whilst if $C(m)$ is of type (**III**, **III**), $Ricc(m)$ is necessarily of type {(31)}, or of (the second case of) type {(22)} (each with eigenvalue zero), or type O.

Theorem 5. *If $Ricc(m)$ is of type {(211)} with zero eigenvalue and $C(m)$ is of type (**N**, **N**), the unique null eigendirection of $Ricc(m)$ coincides with the unique repeated principal null direction of $C(m)$ and lies in the blade of the (necessarily simple) bivector $E(m)$. If $C(m)$ is of type (**III**, **III**) and $Ricc(m)$ is of type {(31)} (respectively, of (the second case of) type {(22)}) (each with zero eigenvalue), the unique repeated null direction of $C(m)$ coincides with the unique null eigendirection of $Ricc(m)$ (respectively, with one of the null eigendirections of $Ricc(m)$) and in each case the blade of the (necessarily simple) homothetic bivector $E(m)$ is spanned by the unique repeated pnd of $C(m)$ and the (unique) general pnd of $C(m)$ not orthogonal to the repeated pnd.*

The results of this theorem follow from those in sections 4 and 5. It also turns out that for type (**III**, **N**) the zero is isolated and the Ricci tensor vanishes at m whilst for type (**N**, **O**) a 1-dimensional submanifold of zeros arises and $Ricc(m)$ is of type {(211)} with zero eigenvalue, or O.

A comparison of the situation when g has Lorentz signature and M admits a proper homothetic vector field X with a zero at $m \in M$ is instructive. In this latter case the results of section 3 still hold and $Ricc(m)$ has Segre type {(211)}, {(31)} (with zero eigenvalue and a unique null eigendirection in each case) or O whilst the Weyl tensor, which may now be described in terms of the well-known Petrov types [17], must have type **N**, **III** or **O** at m. One then has (see [12,11]) that if $Ricc(m)$ is of type {(211)} with zero eigenvalue, the homothetic bivector is timelike (simple) or non-simple at m and in each case the zeros of X constitute a 1-dimensional submanifold which is part of a null geodesic, whilst if it is {(31)} with eigenvalue zero the homothetic bivector is timelike (simple) at m and the zero is isolated. If the Weyl tensor has Petrov type **N** the homothetic bivector is timelike (simple) at m, the zeros of X constitute part of a null geodesic and $Ricc$ has type {(211)} with eigenvalue zero or O at any of these zeros, whilst if it is of Petrov type **III** the homothetic bivector is timelike (simple) at m, m is isolated and $Ricc(m)$ is of type {(31)} or O. In the event that $Ricc(m) \neq 0$ and $C(m) \neq 0$, the (unique) null eigendirection of $Ricc(m)$ coincides with the (unique, repeated) principal null direction of $C(m)$ and also with one of the null directions in the blade of the (necessarily simple, timelike) homothetic bivector at m. If $C(m)$ is of Petrov type **III** the two null directions in the blade of the homothetic bivector at m coincide with the (unique) repeated and the (unique) general (non-repeated) null directions of $C(m)$. In each signature case interesting possibilities arise also when $Ricc(m)=0$ or $C(m)=0$. In the Lorentz case it was shown in [2] (see also [11]) that (M,g) admits a 1-dimensional geodesic null line of zeros if and only if it is (locally) a *plane wave* (for definitions see [9] and [15]), these being very important in Einstein's general theory of relativity. The similarities between the two signatures are clear.

6. Examples

In these examples $M=\mathbb{R}^4$, the coordinate ordering is u,v,y,s and $m=(0,0,0,0)$ unless otherwise stated. The calculations were mostly done using MAPLE. The homothetic bivector at a zero m of the homothetic vector field X is easily computed from the expression for X from (1) recalling that $X(m)=0$. Many of the possibilities arrived at theoretically for a homothetic zero in sections 3, 4 and 5 are achieved in these examples.

6.1. Example A

Here the metric g is given by

$$y^2du^2+2dudv+dy^2-ds^2. \tag{21}$$

In this example (M,g) admits the proper homothetic vector field X with components $(0,2v,y,s)$ whose zeros constitute the null geodesic line $(u,0,0,0)$. The Ricci tensor can be checked to be everywhere of Segre type $\{(211)\}$ with zero eigenvalue, being a multiple of l_al_b where l is the (Killing) vector field $\partial/\partial v$. The Weyl tensor type can be checked to be type $(\mathbf{N},\mathbf{N})$ everywhere with repeated pnd spanned by l. The homothetic bivector at m is simple and timelike. It is calculated from the expression for X by computing $\partial X^a/\partial x^b$, then writing $g_{ac}\partial X^c/\partial x^b=E_{ab}+cg_{ab}$ (since $X(m)=0$) and finally imposing the condition $E_{ab}=-E_{ba}$ at the point m to find that $c=1$ and that E is timelike with blade $\partial/\partial u\wedge\partial/\partial v$.

6.2. Example B

In this case the metric g is given by

$$u(y^2-s^2)du^2+2dudv+dy^2-ds^2. \tag{22}$$

Here C vanishes identically on M so that (M,g) is conformally flat. The Ricci tensor is of Segre type $\{(211)\}$ (in fact, a multiple of ul_al_b) with eigenvalue zero and unique null eigendirection spanned by $l\equiv\partial/\partial v$ everywhere except on the hypersurface $u=0$ where $Ricc$ vanishes. Again X, with components $(0,2v,y,s)$, is proper homothetic and its zeros lie along a null geodesic through m. Its bivector at m, computed as above from the expression for X, is timelike with blade $\partial/\partial u\wedge\partial/\partial v$. However, the vector field $Y=s\partial/\partial y+y\partial/\partial s$ is a Killing vector field which vanishes on the 2-dimensional submanifold $s=y=0$ and, hence, at m (see section 1) and whose Killing bivector at m is timelike with blade the orthogonal complement of that of X, $\partial/\partial y\wedge\partial/\partial s$. Thus the combination $X+\lambda Y$ $(\lambda\in\mathbb{R})$ with components $(0,2v,y+\lambda s,s+\lambda y)$ is a proper homothetic vector field and if $\lambda\neq\pm1$ its zeros lie precisely on the same 1-dimensional submanifold as those of X, namely, $v=y=s=0$, whilst if $\lambda=\pm1$ the zeros constitute a 2-dimensional totally null submanifold $v=0=y\pm s$ (with immersion (for $\lambda=1$) $(u,y)\to(u,0,y,-y)$ and orthogonal, null tangents $(1,0,0,0)$ and $(0,0,1,-1)$). The homothetic bivector of $X+\lambda Y$ is then non-simple, of Segre type $\{1111\}$ for $\lambda\neq\pm1$ and $\{(11)(11)\}$ for $\lambda=\pm1$ (as expected in this last case since a 2-dimensional submanifold of zeros requires the bivector to have a 2-dimensional eigenspace with eigenvalue $-c$ and in this case, $c=1$). This example shows that C may be zero at a fixed point of a proper homothetic vector field and also that the type of $Ricc$ may differ at different zeros of X. It also shows that independent proper homothetic vector fields may exist on the same manifold with the same set of zeros or with different sets of zeros, and with different homothetic bivectors.

6.3. Example C

Here the metric g is given by

$$u(y^2+s^2)du^2+2dudv+dy^2-ds^2. \tag{23}$$

For this metric the vector field $(0,2v,y,s)$ is proper homothetic, vanishes along the null geodesic $v=y=s=0$, as before, and its bivector at m is timelike. The Weyl type is $(\mathbf{N},\mathbf{N})$ with repeated pnd spanned by $\partial/\partial v$ except when $u=0$ where C vanishes. The Ricci tensor vanishes identically on M. This example shows that $Ricc$ may vanish at a zero of a proper homothetic vector field and that the type of C may differ at different zeros of X.

6.4. Example D

Here the metric g is an adaptation of one given in [4] and of one given in [12] and is

$$e^{uy}dudv + dy^2 - ds^2. \tag{24}$$

The vector field with components $X = (u, -3v, -y, -s)$ is proper homothetic with an isolated zero at m. The homothetic bivector at m is timelike, the Ricci tensor is of type {31} on M and {(31)} with zero eigenvalue at points where $u = 0$ and, in particular, at m, and with null eigendirection spanned by $\partial/\partial v$. The Weyl type is (**III**, **III**) everywhere on M with $\partial/\partial v$ spanning the repeated pnd and with the three general pnds given by the span of $\partial/\partial u$ and the two null directions in the $(\partial/\partial s \wedge \partial/\partial y)$ 2-space at m.

6.5. Example E

Here the metric g is given by

$$(y+s)^2du^2 + 2dudv + dy^2 - ds^2. \tag{25}$$

In this case the vector field $l = \partial/\partial v$ is null as is the vector field $k = \partial/\partial y - \partial/\partial s$. The vector field $X = (0, 2v, y, s)$ is proper homothetic and vanishes along the null geodesic line $v = y = s = 0$ and its bivector is timelike at m, as before. The Weyl tensor is not zero and satisfies $C_{abcd}l^d = C_{abcd}k^d = 0$ on M and has type (**N**, **O**) everywhere on M. The Ricci tensor is identically zero on M. This metric admits the Killing vector field Y with components $(u, -v, -s, -y)$ which vanishes at the isolated zero m and has a non-simple homothetic bivector there. The vector field $X + 2Y$ with components $(2u, 0, y - 2s, s - 2y)$ is then proper homothetic, vanishes on the null geodesic $u = s = y = 0$ and has non-simple homothetic bivector at m of Segre type {1111} (cf. the end of Theorem 4).

6.6. Example F

Here the manifold is restricted by $y > 0 < v$ and the metric g is given by

$$2dudv + 2yvdyds. \tag{26}$$

In this case the Weyl tensor is identically zero on M, the Ricci tensor is of Segre type {(211)} with zero eigenvalue everywhere on M (and note that the vector fields $\partial/\partial u$, $\partial/\partial v$, $\partial/\partial y$ and $\partial/\partial s$ are each null with $\partial/\partial u$ and $\partial/\partial v$ each orthogonal to each of $\partial/\partial y$ and $\partial/\partial s$ and $\partial/\partial u$ spans the null eigendirection of the Ricci tensor. The vector field $(u, 0, 0, s)$ is proper homothetic and vanishes on the 2-dimensional submanifold $u = s = 0$ $(y > 0 < v)$, which is totally null. The homothetic bivector is non-simple and of Segre type {(11)(11)} at each zero.

6.7. Example G

Here the metric is given by

$$2dudv + usdv^2 + dy^2 - ds^2. \tag{27}$$

The Weyl type here is (**III**, **III**) and the Ricci tensor has Segre type {(31)} with zero eigenvalue everywhere and null eigendirection spanned by $\partial/\partial u$. The vector field $(-3u, v, -y, -s)$ is proper homothetic and has an isolated zero at m. Its bivector at m is timelike with blade $\partial/\partial u \wedge \partial/\partial v$.

6.8. Example H

Consider the flat metric for this dimension and signature given variously by

$$dx^2 + dy^2 - dt^2 - ds^2, \qquad 2dudv + dy^2 - ds^2, \qquad 2dudv + 2dwdz \tag{28}$$

in coordinates labelled, respectively, (x, y, t, s), (u, v, y, s) and (u, v, w, z), where $\sqrt{2}u = x - t$, $\sqrt{2}v = x + t$, $\sqrt{2}w = y + s$ and $\sqrt{2}z = y - s$. These three metrics are, of course, the same metric and the respective vector fields given in components by (x, y, t, s), $(2u, 0, y, s)$ and $(u, 0, 0, z)$, in each of the above charts, are proper homothetic. The zeros of these constitute, respectively, the isolated zero m, a 1-dimensional null line $u = y = s = 0$ and a 2-dimensional totally null submanifold $u = z = 0$. The homothetic bivectors are, in each case, easily calculated. Thus all the possibilities for the zero set may occur for homothetic vector fields for the same metric.

These examples show that many of the possibilities arising in the previous sections for the algebraic types of $Ricc$ and C and the homothetic bivector at a zero of a proper homothetic vector field actually occur.

Data availability

No data was used for the research described in the article.

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