

Analysis of Second Order Cluster Consensus for Directed Networks with Time Delay^{*}

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Abstract: In this paper, we consider the cluster consensus problem in delayed directed networks where agents are described by second-order discrete-time dynamics. Unlike first order systems, delay is shown to lead to instability in second order systems if the controller parameters are not chosen properly. Furthermore, the numerical results reveal that as the amount of delay increases, the stability region in the controller parameter space shrinks.

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1. INTRODUCTION

The problem of agreement on a common value in a distributed context, referred to as *distributed consensus*, has become a very interesting topic in the control literature due to its applicability in various fields such as formation control of unmanned aerial vehicles (Ren et al., 2007), load balancing in computer networks (Amelina et al., 2015), clock synchronization in wireless sensor networks (Schenato and Gamba, 2007) and analysis of opinion dynamics (Lorenz, 2005).

There exists a vast amount of research on the traditional consensus problem that concerns the conditions for which all agents in the multi-agent network agree on a single value. The principal condition for reaching a single equilibrium consensus is the existence of a spanning tree in the underlying graph of the network (Ren and Beard, 2005). However, due to the limited connection ranges of the agents in most multi-agent systems, a spanning tree may not exist in the network graph. In this case, the network will naturally split into clusters where agents in a cluster will achieve consensus with all agents in the same cluster. The cluster consensus problem has been investigated under different names including multi-consensus, multi-equilibria consensus, and group consensus.

Olfati-Saber and Murray (2004) studied the traditional consensus problem in continuous time undirected networks both with and without time-delays. The average-consensus problem is investigated for continuous-time networks with switching topology in the presence of constant and time-varying delays (Lin and Jia, 2008). The authors utilize a Lyapunov-Krasovskii function in order to ensure average consensus when the topology of the network is arbitrarily

switching. While there are very important results for the effect of delay on first order consensus protocols, the literature on the effect of delay on second order consensus protocols started to grow recently as second order cluster-consensus problem became increasingly popular due to its practical applications.

Lin and Jia (2009) examined the single equilibrium consensus problem for second order networks and obtained sufficient conditions on the sampling time, controller parameters and the components of the Laplacian matrix so as to guarantee consensus. Lin and Jia (2010) presented a sufficient condition in terms of linear matrix inequalities for the stability of a second-order consensus algorithm in continuous-time delayed networks. The proposed algorithm, which does not take the position states of the neighboring agents into account, is shown to ensure consensus when the delay is bounded. Ni et al. (2017) investigated the leader-following lag consensus problem where the agents are modeled with second-order dynamics in continuous-time and with input delay. The authors proposed a sliding mode protocol to achieve fixed-time consensus. Zhu and Cheng (2010) considered a second order leader-following consensus protocol in continuous-time and studied the effect of delay on stability by deriving conditions on the controller parameters. Xie et al. (2016) investigated the same problem with the assumption that the agent updates are driven by a properly defined event which depends on the measurement error. Hu and Hong (2007) obtained necessary and sufficient conditions on the controller parameters in the existence of time-varying coupling delays. Hou et al. (2017) provided a bound for the time-delay which ensures single equilibrium consensus of a continuous-time second order protocol in multi-agent systems. The given bound depends on the roots of the characteristic equation.

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While there exist a significant amount of research on the second order traditional consensus problem, the literature on cluster consensus concerning time delays is very limited. Chen et al. (2017) examined the cluster consensus of second-order agents in continuous-time where the authors considered the network as an interconnection of two subgroups and constructed a consensus protocol for each subgroup to agree on two different final values. The authors stated delay-dependent conditions by frequency domain analysis. Erkan et al. studied the cluster consensus problem in discrete-time and introduced novel definitions of primary layer and secondary layer subgraphs which are useful in determining the number of clusters in the network when an averaging based linear consensus protocol is used (Erkan et al., 2018). It is stated that the number of clusters is defined as the sum of the number of primary layer and secondary layer subgraphs of the network graph. Develer and Akar (2018) examined the effect of input delay on the cluster consensus problem in continuous-time and show that the upper bound of time-delay that does not affect the number of clusters.

In this paper, we consider a discrete-time second order consensus protocol in delayed networks. The main contribution of the paper is to obtain conditions on the controller parameters of the second-order cluster consensus problem with transmission delay and investigate the possible adverse effect of delay on the stability.

The remainder of the paper is organized as follows. In Section 2, we review some important concepts of graph theory, notation and basic definitions that are utilized throughout the paper and introduce the delayed model of second-order cluster consensus. In Section 3, we obtain necessary and sufficient conditions on the controller parameters so as to ensure K -cluster consensus for an arbitrary directed graph. In Section 4, numerical examples are provided in order to illustrate the theoretical results. Finally, we conclude the paper in Section 5.

2. MATHEMATICAL PRELIMINARIES

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph where $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the set of vertices and the set of directed edges, respectively. Let $\mathcal{A} = [a_{ij}] \in R^{n \times n}$ denote the non-negative weighted adjacency matrix of the graph which satisfy $a_{ij} > 0 \iff \mathcal{E}_{ji} \in \mathcal{E}$ and $a_{ij} = 0$ if $\mathcal{E}_{ji} \notin \mathcal{E}$ or $i = j$ where $\mathcal{E}_{ij} = (v_i, v_j)$ refers to a direct information flow from vertex i to vertex j . The Laplacian matrix of a directed graph, denoted by $L = [l_{ij}]$, is defined as

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{\substack{k=1 \\ k \neq i}}^n a_{ik}, & i = j. \end{cases}$$

2.1 Graph Partitioning Preliminaries

Definition 1. (Subgraph) A subgraph of a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph, $\mathcal{G}_{sg} = (\mathcal{V}_{sg}, \mathcal{E}_{sg})$, such that $\mathcal{V}_{sg} \subseteq \mathcal{V}$ and $\mathcal{E}_{sg} \subseteq \mathcal{E} \cap (\mathcal{V}_{sg} \times \mathcal{V}_{sg})$.

Definition 2. (Primary layer subgraphs) (Erkan et al. (2018)) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an arbitrary directed graph. There exist l_p ($l_p \geq 1$) subsets in the set \mathcal{V} such that each $\mathcal{V}_{p,i}$, $i = 1, \dots, l_p$, consists of the maximum number of

vertices that form a spanning tree for the subgraph $\mathcal{G}_{p,i}$ and $\forall v_a \in \mathcal{V}_{p,i}$ and $\forall v_b \notin \mathcal{V}_{p,i}$, we have $(v_b, v_a) \notin \mathcal{E}$. The subgraphs $\mathcal{G}_{p,i}$ ($i = 1, \dots, l_p$) are called the primary layer subgraphs of \mathcal{G} where l_p is the number of primary layer subgraphs.

Definition 3. (Secondary layer subgraphs) (Erkan et al. (2018)) Let $\bar{\mathcal{V}}$ be the set which consists of the vertices that

are not in the primary layer subgraphs, i.e., $\bar{\mathcal{V}} = \mathcal{V} \setminus \bigcup_{i=1}^{l_p} \mathcal{V}_{p,i}$.

Then there exist l_s subsets in $\bar{\mathcal{V}}$ such that each subset $\mathcal{V}_{s,i}$, $i = 1, \dots, l_s$, has a spanning tree for its subgraph and there exists exactly one vertex $v_a \in \mathcal{V}_{s,i}$ which satisfies the following

- i) $\forall v_b \in \mathcal{V}_{s,i} \setminus v_a$ and $\forall v_c \in \mathcal{V} \setminus \mathcal{V}_{s,i}$, we have $(v_c, v_b) \notin \mathcal{E}$.
- ii) There exist at least two vertices in at least two different subgraphs v_d and v_e such that $(v_d, v_a) \in \mathcal{E}$ and $(v_e, v_a) \in \mathcal{E}$.
- iii) v_a is reachable by all vertices in $\mathcal{V}_{s,i}$.

The subsets $\mathcal{V}_{s,i}$, $i = 1, \dots, l_s$ are called the secondary layer subgraphs of \mathcal{G} where l_s is the number of secondary layer subgraphs.

Given any directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, it can be partitioned into l_p primary layer and l_s secondary layer subgraphs using the algorithm proposed by Erkan et al. (2018). Let $n_{p,i}$ and $n_{s,j}$, $i = 1, \dots, l_p$, $j = 1, \dots, l_s$, denote the number of vertices in the i -th primary and the j -th secondary layer subgraphs, respectively. Let $n_p = \sum_i n_{p,i}$ and $n_s = \sum_j n_{s,j}$ denote the total number of vertices in primary and secondary layer subgraphs, respectively. Note that $n_p + n_s = n$ is satisfied for any directed graph.

2.2 Mathematical Model

Consider a multi-agent network consisting of n agents. Suppose that the dynamics of the agents are represented by the following equations:

$$\begin{aligned} x_i(k+1) &= x_i(k) + v_i(k) \\ v_i(k+1) &= v_i(k) + u_i(k) \end{aligned} \quad (1)$$

where $x_i(k) \in R^N$ and $v_i(k) \in R^N$, $i = 1, \dots, n$, are the (position and velocity) states of agent i and $u_i(k)$ is the control input at time step k . In delayed networks, consider the input

$$\begin{aligned} u_i(k) &= -v_i(k) + v_i(k - \tau) \\ &+ k_1 \sum_{j=1}^n a_{ij} (x_j(k - \tau) - x_i(k - \tau)) \\ &+ k_2 \sum_{j=1}^n a_{ij} (v_j(k - \tau) - v_i(k - \tau)) \end{aligned} \quad (2)$$

where τ is the uniform bounded time delay in the network, k_1, k_2 are the controller parameters, and a_{ij} are the components of the weighted adjacency matrix of the network. Using the definition of the Laplacian, (1) can be re-written as

$$\det(\lambda I_{2n(\tau+1)} - \Gamma) = \lambda^{2n\tau} \det \left(\begin{bmatrix} (\lambda - 1)I_{n_p} & -I_{n_p} & 0_{n_s} & 0_{n_s} \\ 0_{n_p} & \lambda I_{n_p} & 0_{n_s} & 0_{n_s} \\ 0_{n_p} & 0_{n_p} & (\lambda - 1)I_{n_s} & -I_{n_s} \\ 0_{n_p} & 0_{n_p} & 0_{n_s} & \lambda I_{n_s} \end{bmatrix} - \begin{bmatrix} 0_{2n \times 2n(\tau-1)} & 0_{n_p} & 0_{n_p} & 0_{n_s} & 0_{n_s} \\ -k_1 L_p & I_{n_p} & -k_2 L_p & 0_{n_s} & 0_{n_s} \\ 0_{n_p} & 0_{n_p} & 0_{n_s} & 0_{n_s} & 0_{n_s} \\ -k_1 L_{sp} & -k_2 L_{sp} & -k_1 L_s & I_{n_s} & -k_2 L_s \end{bmatrix} \right) \times \begin{bmatrix} \frac{1}{\lambda} I_{2n} & 0_{2n} & 0_{2n} \\ \vdots & \ddots & 0_{2n} \\ \frac{1}{\lambda^\tau} I_{2n} & \cdots & \frac{1}{\lambda^\tau} I_{2n} \end{bmatrix} \times \begin{bmatrix} I_{2n} \\ 0_{2n(\tau-1) \times 2n} \end{bmatrix}.$$

The third property of Lemma 2 yields

$$\det(\lambda I_{2n(\tau+1)} - \Gamma) = \det \left(\begin{bmatrix} \lambda^\tau (\lambda - 1) I_{n_p} & -\lambda^\tau I_{n_p} \\ k_1 L_p & (\lambda^{\tau+1} - 1) I_{n_p} + k_2 L_p \end{bmatrix} \right) \times \det \left(\begin{bmatrix} \lambda^\tau (\lambda - 1) I_{n_s} & -\lambda^\tau I_{n_s} \\ k_1 L_s & (\lambda^{\tau+1} - 1) I_{n_s} + k_2 L_s \end{bmatrix} \right).$$

Furthermore, from the fourth property of Lemma 2, we have

$$\det(\lambda I_{2n(\tau+1)} - \Gamma) = \lambda^{\tau n} \det((\lambda^{\tau+2} - \lambda^{\tau+1} - \lambda + 1) I_{n_p} + [k_2(\lambda - 1) + k_1] L_p) \times \det((\lambda^{\tau+2} - \lambda^{\tau+1} - \lambda + 1) I_{n_s} + [k_2(\lambda - 1) + k_1] L_s) = \lambda^{\tau n} \prod_{i=1}^{n_p} ((\lambda^{\tau+2} - \lambda^{\tau+1} - \lambda + 1 + [k_2(\lambda - 1) + k_1] \mu_{p,i}) \times \prod_{j=1}^{n_s} ((\lambda^{\tau+2} - \lambda^{\tau+1} - \lambda + 1 + [k_2(\lambda - 1) + k_1] \mu_{s,j}))$$

where $\mu_{p,i}$ and $\mu_{s,j}$ are the eigenvalues of L_p and L_s , respectively. From Lemma 1, the multiplicity of the eigenvalue 0 of L_p is l_p and 0 is not an eigenvalue of L_s . Therefore, the multiplicity of the eigenvalue 1 of Γ is $(\tau + 2)l_p$.

Theorem 4. Suppose that a directed network consists of l_p primary and l_s secondary layer subgraphs. Then the network (1) achieves cluster consensus with $K = l_p + l_s$ equilibria if and only if the roots of the polynomials

$$\lambda^{\tau+2} - \lambda^{\tau+1} - \lambda + 1 + [k_2(\lambda - 1) + k_1] \mu_i \quad (7)$$

are inside the unit circle where μ_i , $i = 1, \dots, n - l_p$, are the nonzero eigenvalues of L .

Proof. (Sufficiency) From Lemma 1, Γ has an eigenvalue 1 with the multiplicity $(\tau + 2)l_p$. Condition of Theorem 4 ensures that the remaining eigenvalues of Γ are inside the unit circle. Following the same procedure as in the proof of Theorem 1 of Xie and Wang (2012), one can show that the system achieves cluster consensus with K equilibria.

(Necessity) Suppose that at least one root of the polynomials given in (7) is not inside the unit circle, which corresponds to one of the following three cases:

Case 1: $d(\Gamma, 1) = (\tau + 2)l_p$ and there exists at least one eigenvalue with magnitude greater than 1, or

Case 2: $d(\Gamma, 1) > (\tau + 2)l_p$ and all other eigenvalues are less than 1 in magnitude, or

Case 3: $d(\Gamma, 1) > (\tau + 2)l_p$ and there exists at least one eigenvalue with magnitude greater than 1.

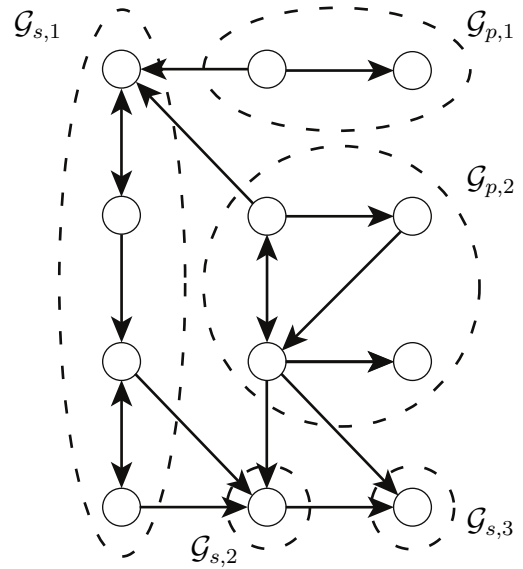


Fig. 1. A directed graph consisting of 2 primary and 3 secondary layer subgraphs.

where $d(A, \lambda)$ denotes the multiplicity of the eigenvalue λ of the matrix A . The system (1) is unstable when the conditions of the Cases 1 or 3 hold. The conditions of Case 2 will result in the number of equilibria to be greater than $K = l_p + l_s$. The proof of the necessity part is completed by contradiction. \square

Remark 1. It is well-studied in the literature that bounded delay does not have an adverse effect on the stability of the first-order single equilibrium consensus problem, i.e., the network achieves consensus so long as the amount of delay is bounded in the network (Cihan and Akar, 2015). Theorem 4 states the conditions for which cluster consensus is achieved in a delayed network. It is further illustrated in the following section that the stability region in the parameter space depends on the uniform delay in the network.

4. NUMERICAL RESULTS

Consider the multi-agent network illustrated in Fig. 1. Using the partitioning algorithm given in Erkan et al. (2018), the network can be partitioned into 2 primary and 3 secondary layer subgraphs. The stability region for the system without delay is given in Fig. 2(a) when the Laplacian matrix is unweighted. The stability regions for the system with $\tau = 1$, $\tau = 5$ and $\tau = 10$ are depicted in Figs. 2(b), (c) and (d), respectively. Note that the stability region shrinks as the amount of delay in the network increases. In order to illustrate this, let the controller parameters be chosen as $k_1 = 0.2$ and $k_2 = 0.5$ and initial conditions of the states $x_i(0)$ and $v_i(0)$ be randomly chosen. The network achieves cluster consensus with $K = l_p + l_s = 5$ equilibria when $\tau = 1$ whereas the network is unstable when $\tau = 5$ as shown in Figs. 3(a) and (b).

5. CONCLUSION

This paper examines the possible detrimental effect of delay on the convergence of a second-order cluster consensus algorithm. While delay does not affect stability in

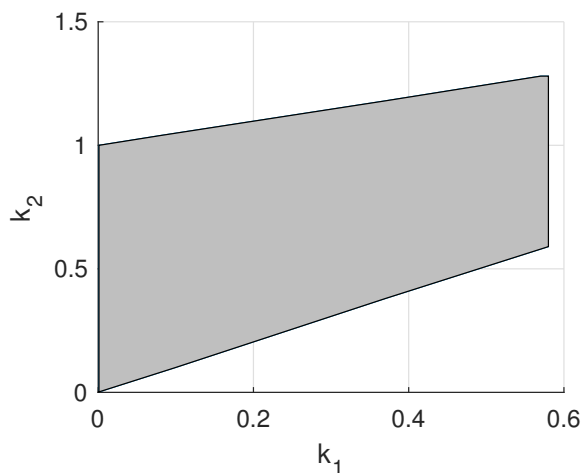
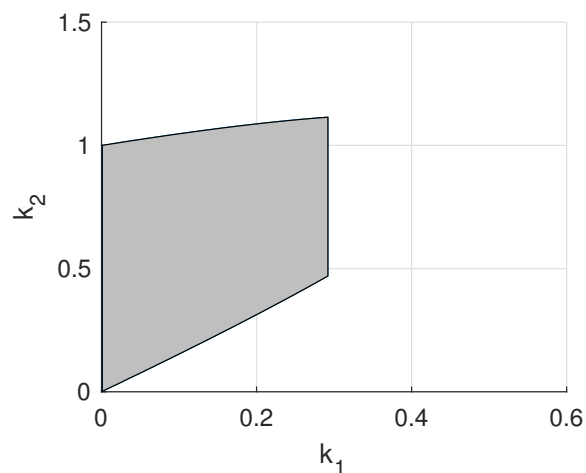
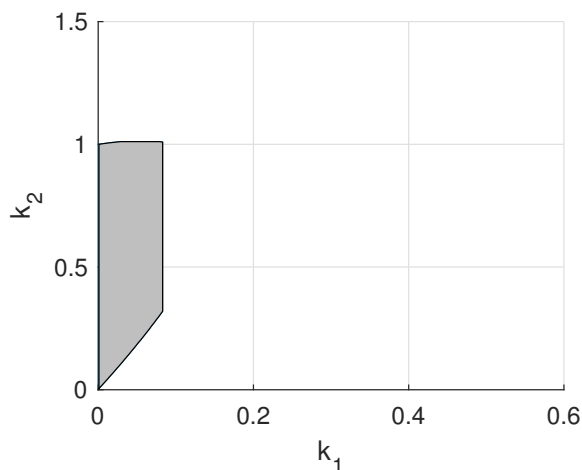
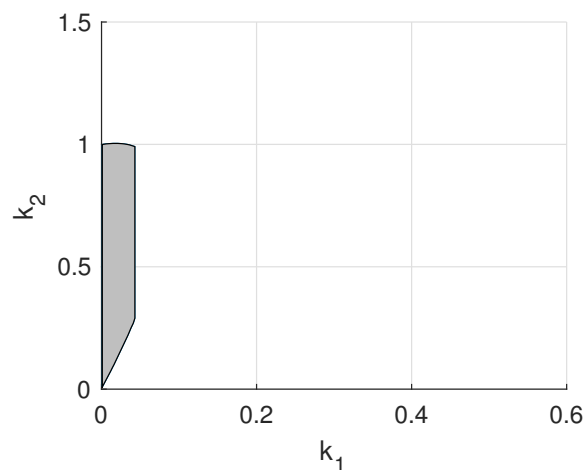
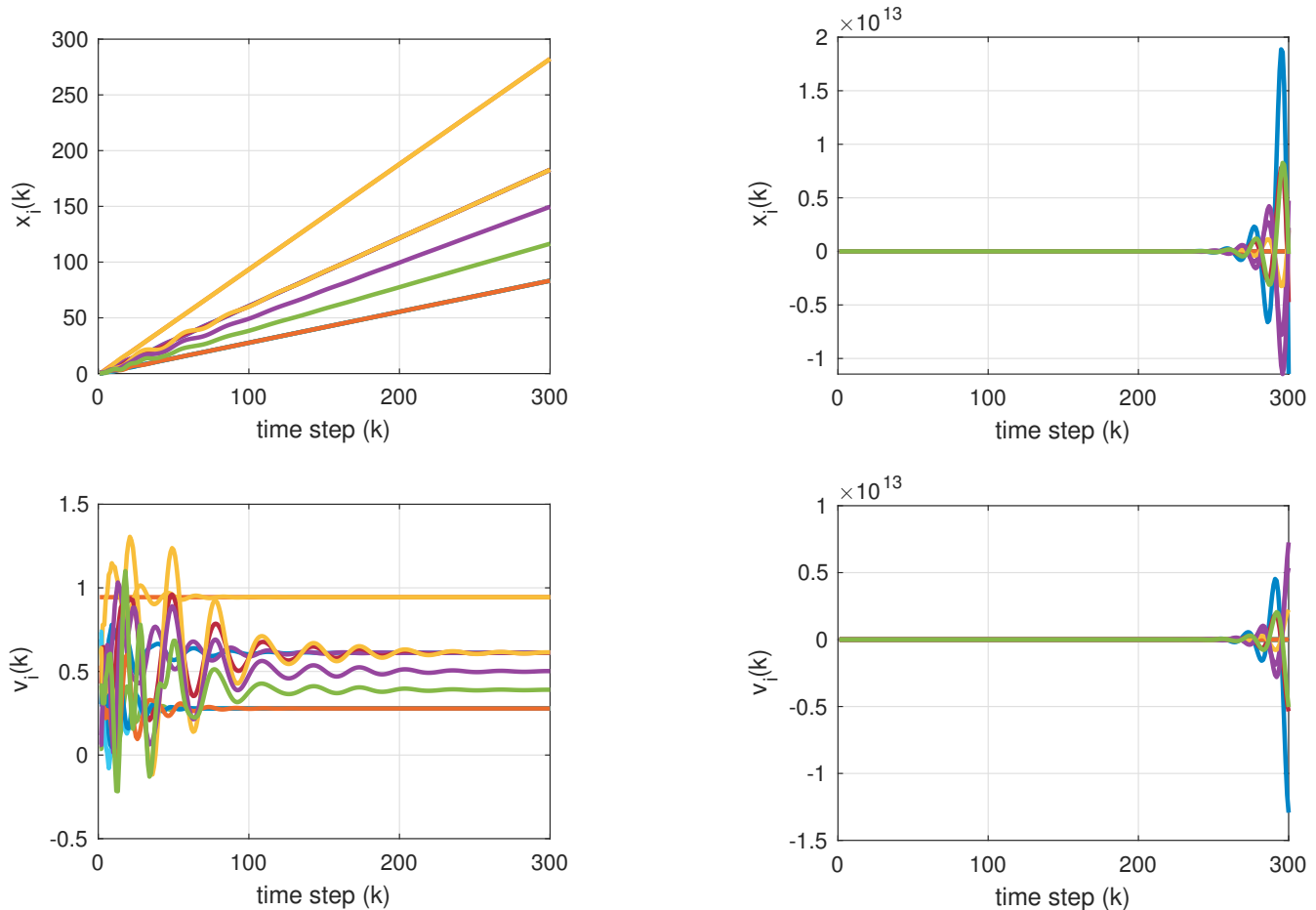
(a) Stability region in the parameter space for $\tau = 0$ (b) Stability region in the parameter space for $\tau = 1$ (c) Stability region in the parameter space for $\tau = 5$ (d) Stability region in the parameter space for $\tau = 10$

Fig. 2. Stability regions in the parameter space for various amounts of delay in the network.

the first-order consensus problem, it is shown that it may lead to instability in the network when agent dynamics are of the second-order. It is verified via simulations that, for a given network, the stability region in the parameter space shrinks as the amount of delay increases. Future research directions include obtaining a systematic method for choosing controller parameters that guarantee stability and investigating the effect of non-uniform delay.

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(a) Evolution of the states for $\tau = 1$ (b) Evolution of the states for $\tau = 5$ Fig. 3. Simulation results for $k_1 = 0.2$ and $k_2 = 0.5$.

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