

INTERIOR STRUCTURAL BIFURCATION OF 2D SYMMETRIC INCOMPRESSIBLE FLOWS

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ABSTRACT. The structural bifurcation of a 2D divergence free vector field $\mathbf{u}(\cdot, t)$ when $\mathbf{u}(\cdot, t_0)$ has an interior isolated singular point \mathbf{x}_0 of zero index has been studied by Ma and Wang [23]. Although in the class of divergence free fields which undergo a local bifurcation around a singular point, the ones with index zero singular points are generic, this class excludes some important families of symmetric flows. In particular, when $\mathbf{u}(\cdot, t_0)$ is anti-symmetric with respect to \mathbf{x}_0 , or symmetric with respect to the axis located on \mathbf{x}_0 and normal to the unique eigendirection of the Jacobian $D\mathbf{u}(\cdot, t_0)$, the vector field must have index 1 or -1 at the singular point. Thus we study the structural bifurcation when $\mathbf{u}(\cdot, t_0)$ has an interior isolated singular point \mathbf{x}_0 with index -1. In particular, we show that if such a vector field with its acceleration at t_0 both satisfy the aforementioned symmetries then generically the flow will undergo a local bifurcation. Under these generic conditions, we rigorously prove the existence of flow patterns such as pairs of co-rotating vortices and double saddle connections. We also present numerical evidence of the Stokes flow in a rectangular cavity showing that the bifurcation scenarios we present are indeed realizable.

1. Introduction. The main objective of this study is to classify the local bifurcations of incompressible 2D flows subject to certain symmetry constraints. A remarkable result of the geometric theory of incompressible 2D flows developed by Ma and Wang [24] gives necessary and sufficient conditions for the structural stability of 2D incompressible flows [22] and is an extension of the classical structural stability theorem of Peixoto [26]. Namely, a 2D divergence free vector field is structurally stable if and only if it satisfies these three properties: 1) \mathbf{u} is regular, 2) interior saddle points of \mathbf{u} are self-connected, 3) each saddle point on the boundary is connected to a saddle point on the same component of the boundary. However this is a global result and is not applicable when only local information about the vector field is available.

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The same research program also produced results on the local bifurcations around isolated singular points of 2D incompressible flows both near the boundaries (Ghil, Ma and Wang in [11]) and away from boundaries (Ma and Wang [23]). For this, consider the Taylor expansion of a 2D divergence-free vector field $\mathbf{u}(\cdot, t)$ at $t = t_0$,

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}^0(\mathbf{x}) + \mathbf{u}^1(\mathbf{x})(t - t_0) + o(|t - t_0|),$$

where

$$\mathbf{u}^0(\mathbf{x}) = \mathbf{u}(\mathbf{x}, t_0), \quad \mathbf{u}^1(\mathbf{x}) = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t_0).$$

Assume that \mathbf{u}^0 has an interior ($\mathbf{x}_0 \in \overset{\circ}{M}$) singular point ($\mathbf{u}^0(\mathbf{x}_0) = 0$) which is simple ($D\mathbf{u}^0(\mathbf{x}_0) \neq 0$) degenerate ($\det D\mathbf{u}^0(\mathbf{x}_0) = 0$), and isolated. Let the unit vector \mathbf{e}_1 point in the unique eigendirection of $D\mathbf{u}^0(\mathbf{x}_0)$ and \mathbf{e}_2 be normal to it. Ma and Wang [23] proved that if \mathbf{u}^0 has index zero at \mathbf{x}_0 (i.e. \mathbf{x}_0 is a degenerate cusp of \mathbf{u}^0) and if the acceleration $\mathbf{u}^1(\mathbf{x}_0)$ does not vanish in the \mathbf{e}_2 direction, then a structural bifurcation occurs at time t_0 .

This result is conclusive when there is no symmetry present since vector fields having singular points with zero index are generic in the class of all 2D divergence free vectors that undergo a local bifurcation. However there are certain classes of symmetric flows which do not allow such singular points. Most notably, when the flow is anti-symmetric with respect to the singular point (see equation (19)), or when the flow is axisymmetric with respect to \mathbf{e}_2 direction (see equation (18)).

Thus the main purpose of this paper is to extend the results of Ma-Wang [23] to flows under such symmetry constraints. We show that when such symmetries are present, the vector field $\mathbf{u}(\cdot, t_0)$ has generically index -1 or 1 at \mathbf{x}_0 . Suppose that \mathbf{u}^0 has \mathbf{x}_0 as a simple isolated interior degenerate singular point with index -1 (a saddle) or 1 (a center). In this case we show that a generic perturbation does not give rise to a local bifurcation. In particular, if the acceleration $\mathbf{u}^1(\mathbf{x}_0)$ does not vanish in the \mathbf{e}_2 direction, no local bifurcations will occur. Thus we find sufficient conditions on the acceleration field for the existence of a local bifurcation. These sufficient conditions reduce to the condition

$$\frac{\partial \mathbf{u}^1 \cdot \mathbf{e}_2}{\partial \mathbf{e}_1} \Big|_{\mathbf{x}_0} \neq 0,$$

when the acceleration itself satisfies the anti-symmetry condition (19).

The bifurcation scenarios we obtain in Theorem 3.2 are as follows. In the case when \mathbf{x}_0 is a saddle of \mathbf{u}^0 , the bifurcation occurs via the separation of the saddle into two saddles and a center, see Figure 2. In the case when \mathbf{x}_0 is a center of \mathbf{u}^0 , the bifurcation occurs via the separation of the center into two centers and a saddle (a “figure-eight”-shaped pattern), see Figure 3. We discuss the genericity of these two bifurcation scenarios for symmetric flows in Theorem 3.3.

Structural stability and bifurcation for 2-D incompressible flows with symmetry was also studied by Hsia et al. [18]. The symmetry considered in their paper differs drastically from the current work. In their work, the flow must be globally symmetric in the whole domain while we allow flows locally symmetric around a singular point. Moreover, that paper is on the global bifurcations while our results are local and as a result we obtain finer details on the bifurcations.

The consideration of the streamline topology near an interior point has been very fruitful in the study of local bifurcations of 2D divergence-free vector fields. There are many studies which examine streamline patterns and their bifurcations in two-dimensional incompressible flows near or away from the boundaries; see among

many others [1, 2, 27] for a detailed discussion. Brøns and Hartnack [5] was the first to investigate streamline patterns and their bifurcations near simple-degenerate critical point from a topological view. In this approach, the Taylor expansion of the stream function is considered to obtain local information about a velocity field. By considering the coefficients in the Taylor series as bifurcation parameters, a series of non-linear coordinate transformations are applied to simplify the stream function to obtain the normal form, i.e. the simplest possible higher-order terms near the interior point. This approach was used to analyze a variety of specific steady flows, for example, a flow near a fixed wall [16], slip flows [29] a flow close to an axisymmetric flow [3], vortex breakdown [7, 4] and the flow in a half-filled annulus between rotating co-axial cylinders [10]. Also, Brøns et al. [6], Dam et al. [9] and Heil et al. [17] have studied flow bifurcations using time as a parameter. The investigation of streamline topology and their bifurcation were extended to non-simple degenerate critical points close to stationary walls as well as away from the boundaries by Deliceoğlu and Gürcan [12, 14, 13].

We would also like to mention some of the recent papers that uses the geometric theory of 2D divergence free vector fields developed by Ma and Wang [24]. The studies [30, 21, 8, 25] investigate the boundary layer separation in 2D Navier Stokes equations while in [31] the interior separation is studied for the Boussinesq model and in [20] the theory is applied to the formation of cyclones.

The paper is outlined as follows. We give the setup of the problem in Section 2 and list our main results in Section 3. The proofs of the main theorems is given in Section 4. In Section 5, we demonstrate numerical evidence showing the validity of our main theorem in an application. Finally Section 6 discusses the conclusions of this work.

2. Setup of the problem. Let $M \subset \mathbb{R}^2$ be a closed and bounded domain with C^r ($r \geq 1$) boundary ∂M . Let TM denote the tangent bundle of M and $C^r(TM)$ denote the r times continuously differentiable vector fields on M . Let

$$D^r(TM) = \{\mathbf{v} \in C^r(TM) \mid \mathbf{v} \cdot \mathbf{n} |_{\partial M} = 0, \operatorname{div} \mathbf{v} = 0\},$$

where \mathbf{n} is the outer normal on ∂M .

Now consider $\mathbf{u} \in C^1([0, T], D^r(TM))$, i.e. a vector field in $D^r(TM)$ parametrized by t . We say $\mathbf{u}(x, t)$ has a local structural bifurcation in a neighborhood $U \subset M$ of \mathbf{x}_0 at t_0 if for any sufficiently close t^- and t^+ to t_0 with $t^- < t_0 < t^+$, $\mathbf{u}(\cdot, t^-)$ and $\mathbf{u}(\cdot, t^+)$ are not topologically equivalent, i.e. no homeomorphism can be found which maps orbits of $\mathbf{u}(\cdot, t^-)$ to orbits of $\mathbf{u}(\cdot, t^+)$ preserving the orientation.

Consider the Taylor expansion of $\mathbf{u} \in C^1([0, T], D^r(TM))$ (where r is as large as necessary) at $t = t_0$,

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}^0(\mathbf{x}) + \mathbf{u}^1(\mathbf{x})(t - t_0) + o(|t - t_0|), \tag{1}$$

where

$$\mathbf{u}^0(\mathbf{x}) = \mathbf{u}(\mathbf{x}, t_0), \quad \mathbf{u}^1(\mathbf{x}) = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t_0). \tag{2}$$

We will think of the acceleration field \mathbf{u}^1 as a small perturbation of the steady state flow \mathbf{u}^0 near t_0 . In this paper we will consider the local bifurcations of flows having an isolated interior singular point $\mathbf{x}_0 \in \overset{\circ}{M}$ at $t = t_0$, i.e.

$$\mathbf{u}^0(\mathbf{x}_0) = 0. \tag{3}$$

It is known (see Ma and Wang [11]) that \mathbf{x}_0 is connected to $2n$ ($n \in \mathbb{N} \cup \{0\}$) orbits of \mathbf{u}^0 , n of the orbits are stable (meaning that the ω -limit set of those orbits are $\{\mathbf{x}_0\}$) and the rest are unstable (meaning that the α -limit set of those orbits are $\{\mathbf{x}_0\}$). Moreover the stable and unstable orbits connected to \mathbf{x}_0 alternate when a closed curve is traced around \mathbf{x}_0 . Furthermore,

$$\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = 1 - n,$$

where $\text{ind}(\mathbf{u}^0, \mathbf{x}_0)$ is the index of \mathbf{u}^0 at \mathbf{x}_0 defined via the Brouwer degree.

A point \mathbf{x}_0 is called a degenerate singular point of \mathbf{u}^0 if the Jacobian determinant $\det D\mathbf{u}^0(\mathbf{x}_0)$ vanishes and non-degenerate otherwise. Since local structural bifurcations do not occur around non-degenerate interior singular points, one focuses on degenerate ones to study such bifurcations. Hence we will assume

$$\det D\mathbf{u}^0(\mathbf{x}_0) = 0. \quad (4)$$

Generically it suffices to consider the case of simple degeneracy

$$D\mathbf{u}^0(\mathbf{x}_0) \neq 0, \quad (5)$$

that is the Jacobian matrix does not vanish completely at \mathbf{x}_0 .

If $\mathbf{x}_0 \in \dot{M}$ is an isolated singular point with Jacobian $D\mathbf{u}^0(\mathbf{x}_0) \neq 0$, then a characterization given by Ma and Wang (Lemma 3.1, [23]) states that one of the following must hold:

1. either $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = 1$ and \mathbf{x}_0 is a degenerate center,
2. or, $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = -1$ and \mathbf{x}_0 is a degenerate saddle such that 4 orbits connected to \mathbf{x}_0 are tangent to each other at \mathbf{x}_0 ,
3. or $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = 0$ and \mathbf{x}_0 is a cusp such there are two orbits connected to \mathbf{x}_0 and the angle between these orbits is zero.

Thus isolated simple degenerate singular points of \mathbf{u}^0 must have index -1 , 0 or 1 .

Due to simple degeneracy of \mathbf{u}^0 at \mathbf{x}_0 , see (4) and (5), there exists unit vectors $\mathbf{e}_1, \mathbf{e}_2$ satisfying

$$D\mathbf{u}^0(\mathbf{x}_0)\mathbf{e}_1 = 0, \quad D\mathbf{u}^0(\mathbf{x}_0)\mathbf{e}_2 = \alpha\mathbf{e}_1, \quad (6)$$

for some constant $\alpha \neq 0$.

Now we consider an orthogonal coordinate system (x, y) with origin at \mathbf{x}_0

$$\mathbf{x}_0 = 0, \quad (7)$$

and the x and y axes pointing in the direction of \mathbf{e}_1 and \mathbf{e}_2 respectively. In view of (6), we have

$$D\mathbf{u}^0(0) = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, \quad \alpha \neq 0. \quad (8)$$

Now assume further that

$$\frac{\partial^m(\mathbf{u}^0 \cdot \mathbf{e}_1)}{\partial x^m}(\mathbf{x}_0) = \begin{cases} 0, & 1 \leq m < k, \\ \neq 0, & m = k, \end{cases} \quad (9)$$

and

$$\frac{\partial^m(\mathbf{u}^0 \cdot \mathbf{e}_2)}{\partial x^m}(\mathbf{x}_0) = \begin{cases} 0, & 1 \leq m < n, \\ \neq 0, & m = n. \end{cases} \quad (10)$$

Under the above conditions (3)–(10) (Ma and Wang [23] Equation 3.10), the Taylor series expansion of \mathbf{u}^0 at \mathbf{x}_0 becomes

$$\mathbf{u}^0(x, y) = \begin{cases} \alpha y + \lambda x^k + f(x, y) \\ \beta x^n - \lambda k y x^{k-1} + g(x, y) \end{cases}, \tag{11}$$

where $f(x, y) = O(|x|^{k+1}) + yO(|x|) + O(y^2)$ and $g(x, y) = O(|x|^{n+1}) + yO(|x|^k) + O(y^2)$

$$\alpha, \beta, \lambda \in \mathbb{R}, \quad \alpha \neq 0, \quad \beta \neq 0, \quad \lambda \neq 0, \tag{12}$$

$$n, k \in \mathbb{Z}, \quad n \geq 2, \quad k \geq 2. \tag{13}$$

As stated before, the index of \mathbf{u}^0 at \mathbf{x}_0 must be one of -1, 0, 1. The following connection between the index of \mathbf{u}^0 at \mathbf{x}_0 and the coefficients of its Taylor series expansion at $\mathbf{x} = \mathbf{x}_0$ is made in [23].

Lemma 2.1 (Ma and Wang [23] Lemma 3.3). *Consider the following complementary assumptions.*

- (S1) $2k > n + 1$, n is even.
- (S2) $2k > n + 1$, n is odd, $\alpha\beta > 0$.
- (S3) $2k > n + 1$, n is odd, $\alpha\beta < 0$.
- (S4) $2k = n + 1$, $\lambda^2 k + \alpha\beta > 0$.
- (S5) $2k = n + 1$, $\lambda^2 k + \alpha\beta = 0$.
- (S6) $2k = n + 1$, $\lambda^2 k + \alpha\beta < 0$.
- (S7) $2k < n + 1$.

The index of \mathbf{u}^0 given by (11)–(13) at $\mathbf{x}_0 = 0$ is

$$\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = \begin{cases} 0, & \text{if (S1) holds,} \\ -1, & \text{if (S2), (S4) or (S7) holds,} \\ 1, & \text{if (S3) or (S6) holds.} \end{cases} \tag{14}$$

Some remarks are in order:

Remark 1. Since $\alpha\beta \neq 0$ by (12), the assumptions given in Theorem 2.1 are indeed complementary.

Remark 2. Note that under the assumption (S2) or (S7), the index of \mathbf{u}^0 at \mathbf{x}_0 is -1 and \mathbf{u}^0 looks as shown in Figure 1. Under the assumption (S7), we have

$$\mathbf{u}_t(x, y) = \begin{cases} \alpha y + \lambda x^k + t f(x, y), \\ -\lambda k y x^{k-1} + t [\beta x^n + g(x, y)], \end{cases}$$

where $0 \leq t \leq 1$. Since $2k - 1 < n$, there exist a neighborhood $U \subset M$ of $\mathbf{x}_0 (= 0)$, such that $\mathbf{u}_t(x, y)$ has a unique singular point $\mathbf{x} = (0, 0)$ in U for all $t \in [0, 1]$. By the homotopy invariance of the index, we have

$$\text{ind}(\mathbf{u}_0, x_0) = \text{ind}(\mathbf{u}_1, x_0) = \text{ind}(\mathbf{u}, x_0). \tag{15}$$

In a neighborhood of x_0 , orbits of $\mathbf{u}_0 = (\alpha y + \lambda x^k, -\lambda k y x^{k-1})$ are given by the following equations:

$$y \left(\frac{\alpha}{2} y + \lambda x^k \right) = C, \quad 0 \leq |C| < \delta. \tag{16}$$

Apparently, we can see from (16) that if n is odd, k is even and $\alpha\lambda < 0$ (respectively, $\alpha\lambda > 0$), then the flow pattern looks as shown in Figure Figure 1(b) [respectively, as shown in Figure Figure 1(c)]. If both n and k are odd, the streamline pattern is as shown as in Figure Figure 1(d). We note that the degenerate flow patterns in Figure 1 (b-c) were observed by Bakker

[2] and Hartnack [16] near a fixed wall. In this study, they appear away from the boundaries.

Remark 3. Under the assumption (S5), the index of \mathbf{u}^0 can not be determined from the Taylor expansion (11)–(13) alone and the higher order terms are required. In fact, in this case the truncated vector field (i.e. \mathbf{u}^0 without the higher order terms) has a set of zeros on the curve $y = -\frac{\lambda}{\alpha}x^k$ and thus \mathbf{x}_0 is not an isolated zero.

Remark 4. We would like to point out the index formula (14) is not given correctly in Ma and Wang [23] where it is stated that the index is always -1 if $\lambda^2k + \alpha\beta \neq 0$, see (S4) and (S6) cases. We fix this issue by carrying out the proof of Theorem 2.1 under the assumptions (S4) and (S6) in Section 4.

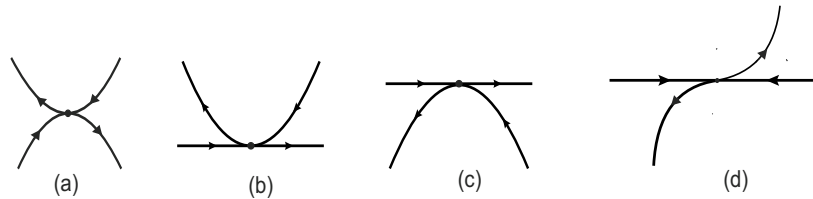


FIGURE 1. The topological structure of \mathbf{u}_0 near the origin, under the assumptions: (a) (S2), (b) (S7), k is even, $\alpha\lambda < 0$, (c) (S7), k is even, $\alpha\lambda > 0$, (d) (S7), k is odd.

Now, consider \mathbf{u}^1 in (2) has the Taylor expansion at $\mathbf{x} = \mathbf{x}_0$ given by

$$\mathbf{u}^1(x, y) = \begin{cases} \lambda_1 + O(|\mathbf{x}|), \\ \lambda_0 + \lambda_2x + \lambda_3y + O(|\mathbf{x}|^2). \end{cases} \tag{17}$$

A genericity argument given in Ma and Wang [23] states that one only needs to study local bifurcations under the assumption (S1), i.e. when \mathbf{u}^0 has \mathbf{x}_0 as a simple degenerate cusp. In that case, they prove the following theorem.

Theorem 2.2 (Ma and Wang [23] Theorem 4.5). *Assume (S1) holds so that $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = 0$ by Theorem 2.1 and $\lambda_0 = \mathbf{u}^1(\mathbf{x}_0) \cdot \mathbf{e}_2 \neq 0$. Then under the assumptions (1)–(10), the vector field \mathbf{u} has a bifurcation in its local structure at (\mathbf{x}_0, t_0) . Moreover, for all t sufficiently close to t_0 , $\mathbf{u}(\mathbf{x}, t)$ has no singular points for any $t < t_0$ (resp. $t > t_0$) and exactly two non-degenerate singular points, one saddle and one center for all $t > t_0$ (resp. $t < t_0$).*

However, symmetries considered in this paper destroy the possibility of local bifurcations around cusps. Hence, one has to study local bifurcations around saddles and centers. The situation can be compared to the occurrence of pitchfork bifurcations when the symmetry of the problem does not allow transcritical bifurcations.

To illustrate our point, we consider two classes of symmetric flows \mathbf{u}^0 which do not fit into the picture of Theorem 2.2. That is (S1) assumption can not be satisfied.

A vector field $\mathbf{u} = (u, v)$ has reflectional symmetry about the y -axis if,

$$u(x, y) = u(-x, y), \quad v(x, y) = -v(-x, y), \tag{18}$$

and has anti-symmetry with respect to the origin if,

$$\mathbf{u}(-\mathbf{x}) = -\mathbf{u}(\mathbf{x}). \tag{19}$$

When the vector field $\mathbf{u}^0 = (u^0, v^0)$ given by (11) has reflectional symmetry about the y -axis, it is evident that n is odd and k is even. On the other hand, when it has anti-symmetry with respect to the origin, both n and k are odd.

Thus the assumption (S1) for \mathbf{u}^0 which although is generic can not satisfy the above mentioned symmetries.

3. Main results. Our main results are the following two theorems which are complementary to [Theorem 2.2](#).

Our first result shows that a generic perturbation of the steady state \mathbf{u}^0 will not cause a local bifurcation.

Theorem 3.1. *Suppose one of the conditions (S2), (S3), (S4), (S6), (S7) holds so that $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = \pm 1$ by [Theorem 2.1](#). If $\lambda_0 = \mathbf{u}^1(\mathbf{x}_0) \cdot \mathbf{e}_2 \neq 0$ then under the assumptions (1)–(10), the vector field \mathbf{u} has no local bifurcation at (\mathbf{x}_0, t_0) .*

Under the assumptions of [Theorem 3.1](#), the topological structure of the flow does not change as t crosses t_0 . Merely, there is a unique singular point with index $\text{ind}(\mathbf{u}^0, \mathbf{x}_0)$ of the perturbed flow near \mathbf{x}_0 which becomes degenerate at $t = t_0$ and is non-degenerate when $t \neq t_0$ is sufficiently close to t_0 .

Thus one needs to consider another class of perturbations \mathbf{u}^1 which will give rise to a local bifurcation. We address this issue in the next theorem.

Theorem 3.2. *Suppose one of the conditions (S2), (S3), (S4), (S6), (S7) holds so that $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = \pm 1$ by [Theorem 2.1](#).*

In addition to $\lambda_0 = 0$, assume also the following conditions on \mathbf{u}^1 given by (17).

$$\begin{aligned} 2\lambda\lambda_1 + \alpha\lambda_2 &\neq 0 && \text{if } k = 2, \\ \lambda_2 = \frac{\partial \mathbf{u}^1 \cdot \mathbf{e}_2}{\partial x} \Big|_{\mathbf{x}_0} &\neq 0 && \text{if } k > 2. \end{aligned} \tag{20}$$

Then under the assumptions (1)–(10), \mathbf{u} has a bifurcation in its local structure at (\mathbf{x}_0, t_0) and the following statements hold true.

- If $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = -1$, then for all t sufficiently close to t_0 , $\mathbf{u}(\mathbf{x}, t)$ has exactly one non-degenerate saddle point for all $t < t_0$ (resp. $t > t_0$) and exactly three non-degenerate singular points, two saddles and a center for all $t > t_0$ (resp. $t < t_0$). The topological structure of the bifurcation is as shown in [Figure 2](#).*

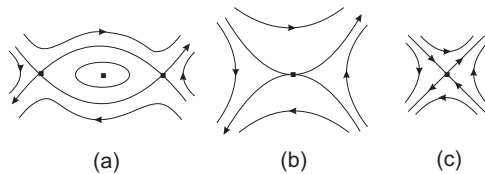


FIGURE 2. Structural bifurcation diagram of the unfolding of codimension-one singularities for the $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = -1$ case in [Theorem 3.2](#): (a) $t = t_0 + \epsilon$, (b) $t = t_0$, (c) $t = t_0 - \epsilon$.

- If $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = 1$, then for all t sufficiently close to t_0 , $\mathbf{u}(\mathbf{x}, t)$ has exactly one non-degenerate center point for all $t < t_0$ (resp. $t > t_0$) and exactly three non-degenerate singular points, two centers and a saddle for all $t > t_0$ (resp. $t < t_0$). The topological structure of the bifurcation is as shown in [Figure 3](#).*

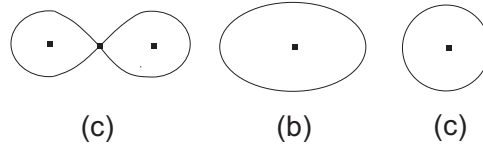


FIGURE 3. Structural bifurcation diagram of the unfolding of codimension-one singularities for the $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = 1$ case in **Theorem 3.2**: (a) $t = t_0 + \epsilon$, (b) $t = t_0$, (c) $t = t_0 - \epsilon$.

Remark 1. If the acceleration field \mathbf{u}^1 is subject to the symmetry constraint (19), then $\lambda_1 = 0$ is automatically satisfied and the assumption (20) reduces to $\lambda_2 \neq 0$ which is equivalent to the condition

$$\frac{\partial \mathbf{u}^1 \cdot \mathbf{e}_2}{\partial x} \Big|_{\mathbf{x}_0} \neq 0.$$

Remark 2. As stated in the introduction, 2D divergence-free vector fields turn unstable if a connection between two saddle points, see Ma and Wang [24]. This is in contrast with the bifurcation scenario given in Figure 2 and deserves an explanation. In [24], such an instability is proven by a saddle-breaking technique which shows that any small (non-symmetric) perturbation near a saddle point may lead to the break-down of the saddle connection. However, as discussed by Hsia, Liu and Wang in [18], in the case of symmetric flows such connections are stable.

Genericity of structural bifurcation in symmetric flows. Now we discuss the genericity of the structural bifurcation given by **Theorem 3.2** for symmetric flows.

First consider the space of anti-symmetric vector fields

$$\mathcal{S}_a = \left\{ \begin{array}{l} \mathbf{u} \in C^1([0, T], D^r(TM)), \mathbf{u}(-\mathbf{x}) = -\mathbf{u}(\mathbf{x}), \\ \mathbf{u}^0(\mathbf{x}_0) = 0, \det D\mathbf{u}^0(\mathbf{x}_0) = 0, \mathbf{u}^0 = \mathbf{u}(\cdot, t_0) \end{array} \right\}.$$

which contains all anti-symmetric smooth 2D divergence-free vector fields that undergo a local bifurcation at (\mathbf{x}_0, t_0) .

In a similar fashion, consider the space of vector fields symmetric with respect to the y -axis that undergo a local bifurcation

$$\mathcal{S}_r = \left\{ \begin{array}{l} \mathbf{u} = (u, v) \in C^1([0, T], D^r(TM)), (u, v)(x, y) = (u, -v)(-x, y) \\ \mathbf{u}^0(\mathbf{x}_0) = 0, \det D\mathbf{u}^0(\mathbf{x}_0) = 0, \mathbf{u}^0 = \mathbf{u}(\cdot, t_0) \end{array} \right\}.$$

Theorem 3.3. *There is an open and dense subset $\widetilde{\mathcal{S}}_a$ of \mathcal{S}_a and an open and dense subset $\widetilde{\mathcal{S}}_r$ of \mathcal{S}_r such that one of the local bifurcation scenarios stated in **Theorem 3.2** must hold for any $\mathbf{u} \in \widetilde{\mathcal{S}}_a$ and for any $\mathbf{u} \in \widetilde{\mathcal{S}}_r$.*

Remark 3. The result given in **Theorem 3.3** is valid even if one considers that instead of the whole flow \mathbf{u} only \mathbf{u}^0 and \mathbf{u}^1 are symmetric.

Remark 4. When $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = -1$, the local bifurcation diagrams for symmetric flows become as given in **Figure 4** and **Figure 5**.

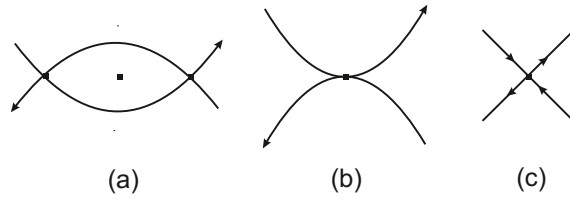


FIGURE 4. Schematic illustration of the unfolding of codimension-one singularities for flows with reflectional symmetry (18) in the $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = -1$ case. (a) $t = t_0 - \epsilon$, (b) $t = t_0$, (c) $t = t_0 + \epsilon$.

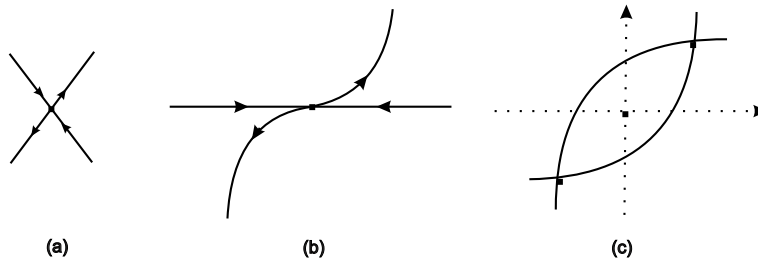


FIGURE 5. Schematic illustration of the unfolding of codimension-one singularities for flows with anti-symmetry (19) in the $\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = -1$ case. (a) $t = t_0 - \epsilon$, (b) $t = t_0$, (c) $t = t_0 + \epsilon$.

4. Proofs.

Proof of Theorem 2.1 under the assumptions (S4) and (S6). Assume $2k = n + 1$. Let $\epsilon > 0$ be sufficiently small and consider the following perturbation \mathbf{u}_ϵ^0 of \mathbf{u}^0 given in (11).

$$\mathbf{u}_\epsilon^0(\mathbf{x}) = \mathbf{u}^0 + (0, -\epsilon)^T.$$

In a small neighborhood of $(\mathbf{x}) = 0$, the singular points of \mathbf{u}_ϵ^0 satisfy the equations

$$y = -\frac{\lambda x^k}{\alpha} + O(|x|^{k+1}). \tag{21}$$

$$\beta x^n + \frac{1}{\alpha} k \lambda^2 x^{2k-1} = \epsilon + O(|x|^{2k}). \tag{22}$$

When $2k - 1 = n$ and $\alpha\beta + k\lambda^2 \neq 0, k \geq 2$, the only solution of (21) and (22) is

$$\mathbf{x}_\epsilon \sim \left((C\epsilon)^{1/(2k-1)}, -\frac{\lambda}{\alpha} (C\epsilon)^{k/(2k-1)} \right),$$

where $C = \alpha(\alpha\beta + k\lambda^2)^{-1}$.

Now it is easy to check that

$$\det D\mathbf{u}_\epsilon^0(\mathbf{x}_\epsilon) \sim -(C\epsilon)^{(2k-2)/(2k-1)} (2k - 1)(\lambda^2 k + \alpha\beta).$$

and

$$\text{sign det } D\mathbf{u}_\epsilon^0(\mathbf{x}_\epsilon) = -\text{sign}(\lambda^2 k + \alpha\beta),$$

so that

$$\text{ind}(\mathbf{u}_\epsilon^0, \mathbf{x}_\epsilon) = -1.$$

Now,

$$\text{ind}(\mathbf{u}^0, \mathbf{x}_0) = \text{ind}(\mathbf{u}_\epsilon^0, \mathbf{x}_\epsilon) = -1$$

follows from the invariance of index under small perturbations.

Proof of Theorem 3.1. Assume that $\lambda_0 \neq 0$ and consider the singular points of the vector field $\mathbf{u}^0 - \epsilon \mathbf{u}^1 = 0$ for sufficiently small $|\epsilon|$.

$$\begin{aligned} \alpha y + \lambda x^k &= \epsilon \lambda_1 + O(|x|^{k+1}) + yO(|x|) + O(y^2), \\ \beta x^n - \lambda k y x^{k-1} &= \epsilon \lambda_0 + O(|x|^{n+1}) + yO(|x|^k) + O(y^2) \end{aligned} \quad (23)$$

By the implicit function theorem, the first equation in (23) can be solved uniquely for y in a small neighborhood of $(\mathbf{x}, \epsilon) = (0, 0)$.

$$y(x, \epsilon) = -\frac{1}{\alpha} (\lambda x^k - \epsilon \lambda_1) + o(|x|^k + |\epsilon|). \quad (24)$$

Using (24) and $\epsilon x^{k-1} = o(|\epsilon|)$ since $k \geq 2$, the second equation of (23) reduces to

$$\beta x^n + \frac{\lambda^2 k}{\alpha} x^{2k-1} = \epsilon \lambda_0 + o(|x|^n + |x|^{2k-1} + |\epsilon|). \quad (25)$$

The equation (25) has the solution

$$\epsilon = C x^m + o(|x|^m), \quad C \neq 0, \quad m = \min\{2k-1, n\} \quad (26)$$

which is the unique solution in a small neighborhood of $x = 0$ and which can be inverted

$$x(\epsilon) = \frac{1}{C^{1/m}} \epsilon^{1/m} + o(|\epsilon|^{1/m}), \quad m = \min\{2k-1, n\}$$

in a small neighborhood of $\epsilon = 0$ as both n and $2k-1$, hence m are odd. In (26), $C \neq 0$ follows from the assumptions (12) and the fact that we have excluded the case

$$\lambda^2 k + \alpha \beta \neq 0 \quad \text{when } 2k-1 = n$$

in the assumptions. Let $y(\epsilon)$ be the solution determined by (24) corresponding to $x(\epsilon)$ and denote the solution

$$\mathbf{x}(\epsilon) = (x(\epsilon), y(\epsilon)).$$

Let us denote the Jacobian determinant of the perturbed field by

$$J = \det D(\mathbf{u}^0 - \epsilon \mathbf{u}^1).$$

By (24) and (26), a straightforward computation yields

$$J(\mathbf{x}(\epsilon)) = -\lambda^2(k^2 + k(k-1))x(\epsilon)^{2k-2} - n\alpha\beta x(\epsilon)^{n-1} + o(|\epsilon|^{\frac{m-1}{m}}).$$

From the above relation, we see that

$$\text{ind}(\mathbf{u}^0 - \epsilon \mathbf{u}^1, \mathbf{x}(\epsilon)) = J(\mathbf{x}(\epsilon)) = \begin{cases} -\text{sign}(\alpha\beta), & 2k > n+1 \\ -\text{sign}(\lambda^2 k + \alpha\beta), & 2k = n+1 \\ -1, & 2k < n+1 \end{cases}$$

which equals to $\text{ind}(\mathbf{u}^0, \mathbf{x}_0)$ by Theorem 2.1 for all ϵ sufficiently small.

Thus there is a unique solution of the perturbed field with the same index as $\text{ind}(\mathbf{u}^0, \mathbf{x}_0)$. Hence there is no bifurcation. This finishes the proof.

Proof of Theorem 3.2. Now suppose $\lambda_0 = 0$ and $\lambda_2 \neq 0$. In this case, the equations for the singular points of the field $\mathbf{u}^0 - \epsilon \mathbf{u}^1 = 0$ are as follows.

$$\begin{aligned} \alpha y + \lambda x^k &= \epsilon \lambda_1 + O(|x|^{k+1}) + yO(|x|) + O(y^2) + \epsilon O(|\mathbf{x}|), \\ \beta x^n - \lambda k y x^{k-1} &= \epsilon \lambda_2 x + \epsilon \lambda_3 y + O(|x|^{n+1}) + yO(|x|^k) + \\ &O(y^2) + \epsilon O(|\mathbf{x}|^2). \end{aligned} \tag{27}$$

Note that since the first equation of (27) is the same as the first equation of (23), we still have the unique solution for y in terms of x and ϵ given by (24) in a small neighborhood of $(\mathbf{x}, \epsilon) = (0, 0)$. Plugging (24) into the second equation of (27), and using

$$\begin{aligned} yO(|x|^k) &= O(|x|^{2k}) + \epsilon O(|x|^k), \\ O(y^2) &= O(x^{2k}) + \epsilon O(|x|^k) + O(\epsilon^2), \\ \epsilon y &= \epsilon O(|x|^k) + O(\epsilon^2), \end{aligned} \tag{28}$$

yields

$$\begin{aligned} x (\alpha \beta x^{n-1} + k \lambda^2 x^{2k-2} - \epsilon (\alpha \lambda_2 + k \lambda \lambda_1 x^{k-2})) &= \\ O(|x|^{n+1}) + yO(|x|^{2k}) + \epsilon O(|x|^2) + O(\epsilon^2). \end{aligned} \tag{29}$$

As $\alpha \lambda_2 \neq 0$, the equations (27) have always the solution given by

$$\mathbf{x}_0(\epsilon) = (x_0(\epsilon), y_0(\epsilon)) = (O(|\epsilon|), \epsilon \lambda_1 / \alpha + o(|\epsilon|)),$$

which can be obtained from the balance between the ϵx term and the $O(\epsilon^2)$ term in (29). For this, we look for a solution of (29) of the form $x_0 = z(\epsilon)\epsilon$, $z(\epsilon) = O(1)$ as $\epsilon \rightarrow 0$, and use the Implicit Function Theorem to deduce the existence of such a solution. The exact computation of the lowest order approximation of $x_0(\epsilon)$ requires the higher order terms in (29), which is not required for what follows. The vector field $\mathbf{u}^0 - \epsilon \mathbf{u}^1$ has the Jacobian

$$J(\mathbf{x}_0(\epsilon)) = \alpha \epsilon \lambda_2 + o(|\epsilon|) + \begin{cases} 0, & k > 2, \\ 2\epsilon \lambda \lambda_1, & k = 2. \end{cases}$$

at the singular point $\mathbf{x}_0(\epsilon)$.

Since both $n - 1$ and $2k - 2$ are both odd, there are two other solutions $\mathbf{x}_\pm(\epsilon)$ of (27) determined by the solutions $x_\pm(\epsilon)$ of (29) near the origin which bifurcate on one side of ϵ . To the lowest order approximation x_\pm can be obtained from the truncated equation of (29)

$$\alpha \beta x^{n-1} + k \lambda^2 x^{2k-2} - \epsilon (\alpha \lambda_2 + k \lambda \lambda_1 x^{k-2}) = 0. \tag{30}$$

Let $y_\pm(\epsilon)$ correspond to $x_\pm(\epsilon)$ via (24) and define

$$\mathbf{x}_\pm(\epsilon) = (x_\pm(\epsilon), y_\pm(\epsilon)).$$

The nature of these solutions depend on the parameters which we investigate in detail below. In particular, we have to consider the cases $k = 2$ and $k > 2$ separately.

The perturbed field $\mathbf{u}^0 - \epsilon \mathbf{u}^1$ has one singular point on $\epsilon < 0$ (resp. $\epsilon > 0$) which is non-degenerate and the index of $\mathbf{u}^0 - \epsilon \mathbf{u}^1$ at that singular point is equal to $\text{ind}(\mathbf{u}^0, \mathbf{x}_0)$. On $\epsilon > 0$ (resp. $\epsilon < 0$) there are three non-degenerate singular points such that sum of the indexes of $\mathbf{u}^0 - \epsilon \mathbf{u}^1$ at these singular points equals $\text{ind}(\mathbf{u}^0, \mathbf{x}_0)$. This result is a direct consequence of the homotopy invariance of index sums in a small domain. Hence there remains to describe the details of the bifurcation in each

case by explicitly computing the bifurcated solutions and their Jacobians which we present below. These details also verify the above claims in each case.

Case $k = 2$. By the condition (20) we have $2\lambda\lambda_1 + \alpha\lambda_2 \neq 0$. Assume without loss of generality that $2\lambda\lambda_1 + \alpha\lambda_2 > 0$. The case $2\lambda\lambda_1 + \alpha\lambda_2 < 0$ is dealt similarly.

Notice that since $n \geq 2$, (S2) and (S3) assumptions need not be checked when $k = 2$.

Subcase $k = 2$ and $n = 3$. This subcase corresponds to either (S4) or the (S6) assumptions. In either case, we have $\alpha\beta + 2\lambda^2 \neq 0$. From (30),

$$\begin{aligned} x_{\pm}(\epsilon) &= \pm \left(\frac{(2\lambda\lambda_1 + \alpha\lambda_2)}{\alpha\beta + 2\lambda^2} \epsilon \right)^{1/2} + o(|\epsilon|^{1/2}), \\ J(\mathbf{x}_{\pm}(\epsilon)) &= -2(2\lambda\lambda_1 + \alpha\lambda_2)\epsilon + o(|\epsilon|). \end{aligned} \tag{31}$$

By (31), the bifurcated solutions are as shown in Table 1.

TABLE 1. The bifurcated solutions for $k = 2, n = 3, 2\lambda\lambda_1 + \alpha\lambda_2 > 0$.

	$\epsilon < 0$	$\epsilon > 0$
$\alpha\beta + 2\lambda^2 > 0$	$\mathbf{x}_0(\epsilon)$ is a saddle	$\mathbf{x}_{\pm}(\epsilon)$ are saddles, $\mathbf{x}_0(\epsilon)$ is a center
$\alpha\beta + 2\lambda^2 < 0$	$\mathbf{x}_0(\epsilon)$ is a center	$\mathbf{x}_{\pm}(\epsilon)$ are centers, $\mathbf{x}_0(\epsilon)$ is a saddle

Subcase $k = 2$ and $n > 3$. This subcase corresponds to (S7) assumption. Since by (12), we have $\lambda \neq 0$ and

$$\begin{aligned} x_{\pm}(\epsilon) &= \pm \left(\frac{(2\lambda\lambda_1 + \alpha\lambda_2)}{2\lambda^2} \epsilon \right)^{1/2} + o(|\epsilon|^{1/2}), \\ J(\mathbf{x}_{\pm}(\epsilon)) &= -2(2\lambda\lambda_1 + \alpha\lambda_2)\epsilon + o(|\epsilon|). \end{aligned} \tag{32}$$

By (32), the bifurcated solutions are as shown in Table 2.

TABLE 2. The bifurcated solutions for $k = 2, n > 3, 2\lambda\lambda_1 + \alpha\lambda_2 > 0$.

$\epsilon < 0$	$\epsilon > 0$
$\mathbf{x}_0(\epsilon)$ is a saddle	$\mathbf{x}_{\pm}(\epsilon)$ are saddles, $\mathbf{x}_0(\epsilon)$ is a center

Case $k > 2$. In this case, by (20) we have $\lambda_2 \neq 0$. Since $\alpha \neq 0$, without loss of generality $\alpha\lambda_2 > 0$. The case $\alpha\lambda_2 < 0$ is dealt similarly.

Subcase $k > 2$ and $2k < n + 1$. This subcase corresponds to (S7) assumption. Since $k \neq 0$ and $\lambda \neq 0$ by (12) and (13), we have

$$\begin{aligned} x_{\pm}(\epsilon) &= \pm \left(\frac{\epsilon\alpha\lambda_2}{k\lambda^2} \right)^{\frac{1}{2k-2}} + o(|\epsilon|^{\frac{1}{2k-2}}), \\ J(\mathbf{x}_{\pm}(\epsilon)) &= -(2k - 2)\epsilon\alpha\lambda_2 + o(|\epsilon|). \end{aligned} \tag{33}$$

By (33), the bifurcated solutions of are as shown in Table 3.

TABLE 3. The bifurcated solutions for $k > 2, 2k < n + 1, \alpha\lambda_2 > 0$.

$\epsilon < 0$	$\epsilon > 0$
$\mathbf{x}_0(\epsilon)$ is a saddle	$\mathbf{x}_{\pm}(\epsilon)$ are saddles, $\mathbf{x}_0(\epsilon)$ is a center

Subcase $k > 2$ and $2k > n + 1$. This subcase corresponds to (S2) and (S3) assumptions. Since $\beta \neq 0$ by (12), we have

$$\begin{aligned}
 x_{\pm}(\epsilon) &= \pm \left(\frac{\epsilon \lambda_2}{\beta} \right)^{\frac{1}{n-1}} + o(|\epsilon|^{\frac{1}{n-1}}), \\
 J(\mathbf{x}_{\pm}(\epsilon)) &= -(n-1)\alpha\epsilon\lambda_2 + o(|\epsilon|).
 \end{aligned}
 \tag{34}$$

By (34), the bifurcated solutions are as shown in Table 4.

TABLE 4. The bifurcated solutions for $k > 2, 2k > n + 1, \alpha\lambda_2 > 0$.

	$\epsilon < 0$	$\epsilon > 0$
$\alpha\beta > 0$	$\mathbf{x}_0(\epsilon)$ is a saddle	$\mathbf{x}_{\pm}(\epsilon)$ are saddles, $\mathbf{x}_0(\epsilon)$ is a center
$\alpha\beta < 0$	$\mathbf{x}_0(\epsilon)$ is a center	$\mathbf{x}_{\pm}(\epsilon)$ are centers, $\mathbf{x}_0(\epsilon)$ is a saddle

Subcase $k > 2$ and $2k = n + 1$. This subcase corresponds to (S4) and (S6) assumptions. Under both assumptions we have $\alpha\beta + k\lambda^2 \neq 0$ and we have

$$\begin{aligned}
 x_{\pm}(\epsilon) &= \pm \left(\frac{\epsilon\alpha\lambda_2}{\alpha\beta + k\lambda^2} \right)^{\frac{1}{2k-2}} + o(|\epsilon|^{\frac{1}{2k-2}}), \\
 J(\mathbf{x}_{\pm}(\epsilon)) &= -(2k-2)\alpha\epsilon\lambda_2 + o(|\epsilon|).
 \end{aligned}
 \tag{35}$$

By (35), the bifurcated solutions are as shown in Table 5.

TABLE 5. The bifurcated solutions for $k > 2, 2k = n + 1, \alpha\lambda_2 > 0$.

	$\epsilon < 0$	$\epsilon > 0$
$\alpha\beta + k\lambda^2 > 0$	$\mathbf{x}_0(\epsilon)$ is a saddle	$\mathbf{x}_{\pm}(\epsilon)$ are saddles, $\mathbf{x}_0(\epsilon)$ is a center
$\alpha\beta + k\lambda^2 < 0$	$\mathbf{x}_0(\epsilon)$ is a center	$\mathbf{x}_{\pm}(\epsilon)$ are centers, $\mathbf{x}_0(\epsilon)$ is a saddle

Proof of Theorem 3.3. Let

$$\widetilde{\mathcal{S}}_a = \left\{ \begin{array}{l} \mathbf{u} \in \mathcal{S}_a, D\mathbf{u}^0(\mathbf{x}_0) \neq 0, (10) \text{ holds with } n = 3, \\ (9) \text{ holds with } k = 3, \lambda^2 k + \alpha\beta \neq 0, \lambda_2 \neq 0 \end{array} \right\}.$$

It is easy to see that the space $\widetilde{\mathcal{S}}_a$ is open and dense in \mathcal{S}_a . Moreover for $\mathbf{u} \in \widetilde{\mathcal{S}}_a$ the assumptions (S1) and (S5) of Theorem 2.1 can not hold. Hence one of the assumptions (S2), (S3), (S4), (S6) or (S7) must hold. Finally for $\mathbf{u} \in \widetilde{\mathcal{S}}_a$, the condition (20) holds, see Remark 1. Now Theorem 3.2 gives the desired result.

Similarly we define

$$\widetilde{\mathcal{S}}_r = \left\{ \begin{array}{l} \mathbf{u} \in \mathcal{S}_r, D\mathbf{u}^0(\mathbf{x}_0) \neq 0, (10) \text{ holds with } n = 3, \\ (9) \text{ holds with } k = 2, \lambda^2 k + \alpha\beta \neq 0, (20) \text{ holds.} \end{array} \right\}$$

which is clearly an open and dense subset of \mathcal{S}_r . Once again the conditions of Theorem 3.2 are satisfied. This finishes the proof.

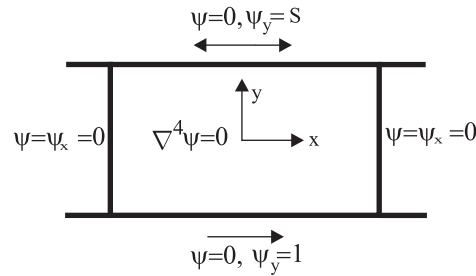


FIGURE 6. The illustration of the dimensionless boundary value problem.

5. Numerical evidence. In this section we present numerical evidence to show that the bifurcations given by [Theorem 3.2](#) is actually observed in certain flow scenarios.

The flow scenario we consider is the steady flow inside a double-lid-driven rectangular cavity with two solid walls and two moving lids. This flow is governed by the Stokes equation which becomes the biharmonic equation for the streamfunction ψ ,

$$\nabla^4 \psi = 0, \quad (36)$$

with boundary conditions as described in [Figure 6](#). The parameters controlling the flow are: the cavity aspect ratio A and the speed ratio of the lids S .

Following Gürçan [\[15\]](#) and Joseph and Sturges [\[19\]](#) the streamfunction for any value of S can be written as:

$$\psi = \sum_{n=-\infty}^{\infty} \{A_n e^{s_n(y-A)} + B_n e^{-s_n(y+A)}\} \frac{\phi_1^n(x, s_n)}{s_n^2}, \quad (37)$$

where the complex coefficients A_n and B_n are determined from the boundary conditions using a truncation technique employing Smith's bi-orthogonality relation [\[28\]](#). s_n are complex eigenvalues and the functions ϕ_1^n are even Papkovitch-Fadle eigenfunctions, satisfying the no-slip boundary conditions on the side walls, see Gürçan [\[15\]](#) for more detail.

Gürçan [\[15\]](#) examined various sequences of flow transitions in this model when the lids move in opposite directions. When S is fixed at the speed ratio -1 , the two bifurcations that are shown in [Theorem 3.2](#) can indeed be realized as codimension one bifurcations. For $A = 0.5$, a single eddy occupies the cavity as shown in [Figure 7\(a\)](#). As aspect ratio is increased, there is a pitchfork bifurcation at $A = 0.931$ as the center becomes a saddle with two sub-eddies (see [Figure 7\(b\)](#)). [Figure 7\(c\)](#) and [Figure 7\(d\)](#) demonstrate an example of a topological saddle point bifurcation. In particular, [Figure 7\(c\)](#) shows a separatrix with one saddle point and two sub-eddies with centers ($S = -1$, $A = 0.3$). As A is decreased to around 0.16, a bifurcation appears above the cavity resulting in two separatrices with a “treble eddy” emerging as demonstrated in [Figure 7\(d\)](#). We note that these flow separations are exactly as described by our main theorems.

6. Conclusion. In this work, we extend the results obtained in Ma and Wang [\[23\]](#) on the local bifurcations of flows to certain symmetric flows which are not considered in the previous studies since they are non-generic with respect to arbitrary divergence-free perturbations. In particular we study the local bifurcations at time

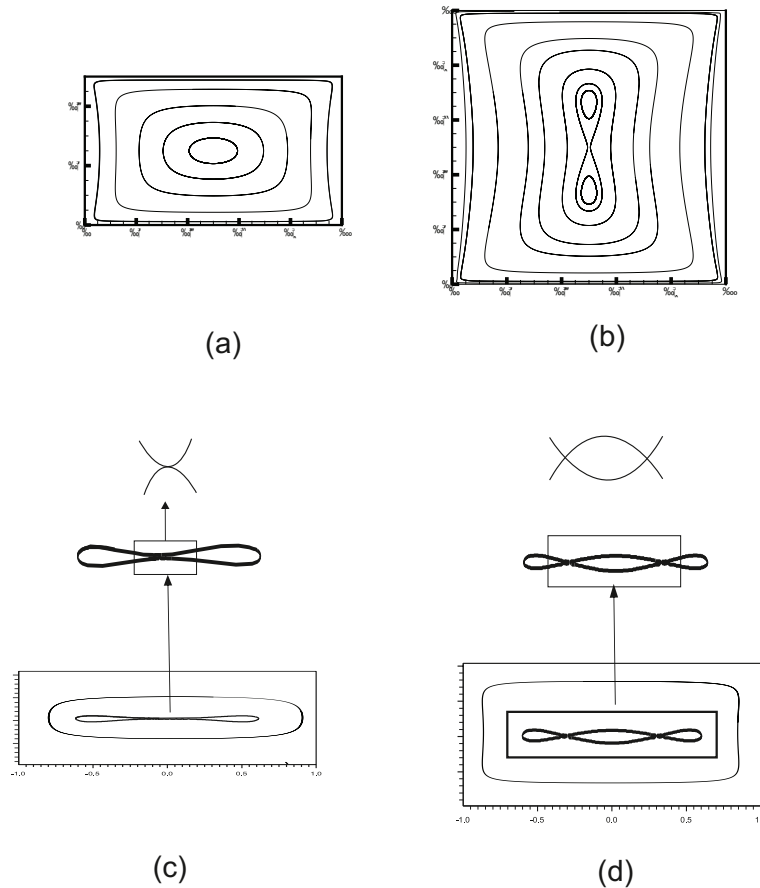


FIGURE 7. The streamlines patterns in rectangular cavities of various aspect ratios A and various lid speed ratios S . (a) $(A, S) = (0.5, -1)$, (b) $(A, S) = (1, -1)$, (c) $(A, S) = (0.3, -1)$, (d) $(A, S) = (0.16, -1)$.

t_0 of the flow \mathbf{u} when $\mathbf{u}(\cdot, t_0)$ has a simple interior degenerate singular point of index -1 (a saddle) or 1 (a center). We show that if the acceleration field at t_0 does not satisfy a certain genericity condition then there is no bifurcation. Thus we give sufficient conditions on the acceleration field for which the flow will undergo a local bifurcation. We also show that the two structural bifurcation scenarios we obtain are indeed generic for flows with certain symmetries. Finally, we present numerical evidence of the Stokes flow in a rectangular cavity showing that the bifurcation scenarios we analytically prove are indeed realizable.

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