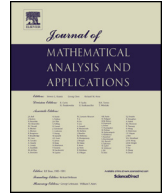




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# Dynamic transitions and bifurcations of 1D reaction-diffusion equations: The non-self-adjoint case



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ABSTRACT

The main goal of this paper is to classify the first transitions of a class of 1D second order reaction diffusion equation with a non-self-adjoint linear part and semilinear nonlinearity on a bounded interval subject to homogeneous boundary conditions. The emphasis is on the interaction of the nonlinear terms, the boundary conditions and the strength of the first order derivative term which makes the linear operator non-self-adjoint. The equations admit a trivial steady state solution which loses stability as a control parameter exceeds a threshold. According to our results, the first transitions of the system are either continuous, jump or mixed type. We compare our results with a recent study where only self adjoint linear operator was considered. In the Dirichlet, Neumann and periodic boundary condition cases, the first transition occurs via a transcritical/pitchfork bifurcation and the dynamics remain unchanged between the self-adjoint and non-self-adjoint cases. However, for the periodic case, with imposed zero mean condition, in the non-self adjoint case, the first transition is always accompanied by a Hopf bifurcation with a stable/unstable bifurcated limit cycle which is different than the self-adjoint case. Finally, we apply our results to determine the first transitions of some well-known reaction diffusion equations such as the Kolmogorov-Fisher, Chaffee-Infante equation and the Burger's equation.

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## 1. Introduction

The dynamical transitions are transitions between the stable states (attractors) of dynamical systems and are ubiquitous in nonlinear sciences [12]. For dissipative systems, usually, when the instability driving mechanism is not strong enough, there is a trivial attractor which is a steady state. As this forcing is increased, the system exhibits new stable states which are periodic solutions, quasi-periodic solutions and even more chaotic attractors [17].

In recent years there have been a growing interest on the first dynamic transitions of nonlinear phenomena [6,9,18,13,15]. Thus it is a growing need to classify the first transitions of partial differential equations

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depending on the nonlinear terms appearing as well as on the boundary conditions. To the best of our knowledge, such a characterization is absent in the literature except for a recent study by the authors [16].

The goal in this paper is simple. We aim to classify a class of second order reaction diffusion equations with semilinear nonlinearities and several main types of homogeneous boundary conditions. Namely, we consider a semilinear reaction diffusion equation in one spatial dimension with a non-self-adjoint second-order linear operator

$$u_t = u_{xx} + 2\gamma u_x + \lambda u + g(u, u_x), \quad (1)$$

where  $(x, t) \in I \times \mathbb{R}^+ \rightarrow u \in \mathbb{R}$  is the unknown function. Here  $\lambda$  and  $\gamma$  are real parameters. In our analysis, we mainly focus on the quadratic and cubic order terms of the nonlinear operator. For this reason, we expand the nonlinear term (upto cubic order) as

$$g(u, u_x) = a_1 u^2 + a_2 u u_x + a_3 u_x^2 + a_4 u^3 + a_5 u^2 u_x + a_6 u u_x^2 + a_7 u_x^3, \quad (2)$$

with constant real coefficients  $a_j$ .

We aim to analyze the effect of following homogeneous boundary conditions:

$$u = 0, \quad \text{at } x = 0, \pi, \quad \text{Dirichlet (D)} \quad (3)$$

$$\frac{\partial u}{\partial x} = 0, \quad \text{at } x = 0, \pi, \quad \text{Neumann (N)} \quad (4)$$

$$u(x + 2\pi) = u(x), \quad \forall x \in \mathbb{R}, \quad \text{Periodic (P)} \quad (5)$$

In the periodic boundary case, we also let

$$m(t) = \int_0^{2\pi} u(x, t) dx,$$

stand for the mean of  $u$ . When (1) has the property that if the initial mean  $m(0) = 0$  implies  $m(t) = 0$ , for all  $t > 0$  then the zero mean condition is usually imposed in the functional setting. As we discuss below, in this case the zero eigenvalue is removed from the spectrum. Hence in addition to (3), (4), (5), we also consider

$$u(x + 2\pi) = u(x), \quad \forall x \in \mathbb{R}, \quad \int_0^{2\pi} u(x, t) dx = 0, \quad \text{Periodic with zero mean (P0)} \quad (6)$$

The parameter  $\lambda$  in (1) plays the role of a control parameter. There is a critical threshold  $\lambda = \lambda_c$  at which the trivial steady state loses its stability. Near this criticality, it is possible to reduce the system to an ODE which approximates the dynamics of the dominant mode. Our main goal is to obtain this reduction and analyze the effects of both nonlinear terms and the boundary conditions on the possible first transition scenarios. For a quick glimpse, we summarize our main results in Table 1.

The current study is a continuation of a recent work by the authors in which only self-adjoint linear operator case ( $\gamma = 0$ ) was considered [16]. As such, this work generalizes and extends the results obtained to the non-self-adjoint case by adding a first order derivative term  $2\gamma \frac{\partial u}{\partial x}$  to the main equation.

Our results in this paper show that the comparison of the self-adjoint and the non-self-adjoint cases are as follows and is also summarized in Table 2.

**Table 1**

The truncated (up to third order) reduced equation, transition type and bifurcation type regarding the first transition of the trivial steady state of (1) for the Dirichlet/Neumann/periodic boundary (DNP) and the periodic boundary with zero mean condition (P0). Here  $u_{k_c}$  and  $z$  denote the approximation of the amplitude of the first critical mode. The exact expressions for the transition numbers  $A$  and  $B$  are given in Section 2.2.

	Reduced equation	Condition	Transition type	Bifurcation type
(DNP)	$\frac{du_{k_c}}{dt} = (\lambda - \lambda_c) u_{k_c} + Au_{k_c}^2 + Bu_{k_c}^3$	$A \neq 0$ $A = 0, B < 0$ $A = 0, B > 0$	Mixed Continuous Jump	Transcritical Supercritical pitchfork Subcritical pitchfork
(P0)	$\frac{dz}{dt} = (\lambda - 1 + 2\gamma i) z + Bz  z ^2$	$\text{Re } B < 0$ $\text{Re } B > 0$	Continuous Jump	Supercritical Hopf Subcritical Hopf

**Table 2**

The comparison of self adjoint ( $\gamma = 0$ ) and non-self-adjoint ( $\gamma \neq 0$ ) cases regarding the transition number and dynamics of the first transition with respect to the boundary condition: Dirichlet (D), Neumann (N), periodic (P), periodic with zero mean (P0).

	(D)	(N)	(P)	(P0)
The transition number changes	Yes	No	No	No
The dynamics of the first transition changes	No	No	No	Yes

- (1) In the homogeneous Dirichlet boundary condition, the addition of the dispersive first derivative term  $2\gamma \frac{\partial u}{\partial x}$  to the equation only changes the transition numbers but not the dynamical behavior of the first transitions. Moreover, we can obtain the transition number in the self-adjoint case as the limit of  $\gamma \rightarrow 0$  in the non-self-adjoint case.
- (2) For the Neumann and periodic boundary condition the addition of the dispersive term, does not change the transition number as well as the dynamical behavior of the first transition.
- (3) However, for periodic boundaries with zero mean condition, the dynamical behavior of the first transition is totally different between the self-adjoint and non-self-adjoint cases. In the non-self-adjoint case, the first transition occurs as a Hopf bifurcation while for the self-adjoint case, the first transition occurs as a bifurcated circle of steady states.

Finally, to show the relevancy of our theoretical results, we also consider several examples, namely the Kolmogorov-Fisher equation, the Chaffee-Infante equation, and the Burger’s equation, which fit into the general framework studied here.

The paper is organized into several sections as follows. The main results are given in Section 2. Section 3 is on the application of the theoretical results. Section 4 discusses the physical/mathematical conclusions and future research directions. We give the center manifold reduction procedure in Section 5.

## 2. Main results

In this section, before stating the main results, we first analyze the linear stability of the trivial steady state. This analysis is presented via the so called principle of exchange of stabilities (PES) condition. For this we need to specify the conditions under which the eigenvalue with the largest real part crosses the imaginary axis. PES condition is in principle a summary of determination of the critical parameter  $\lambda_c$ , the multiplicity and nature of the first critical eigenvalues as well as the transversal crossing of the critical eigenvalue.

Once the PES condition is established, we aim to reduce the equation near the criticality which fully characterizes the local transition near this criticality. This is achieved by center manifold reduction and the proofs are given in Section 5. Then the analysis of this reduced equation gives the type of transition and branches of bifurcating stable/unstable state solutions. We summarize the findings in Theorem 2.1, and Theorem 2.2. Also comparison of non-self-adjoint and self-adjoint cases are presented in Theorem 2.3.

### 2.1. Linear stability analysis

The eigenvalues  $\beta_k$  and the eigenvectors  $e_k(x)$  of the linear operator  $\partial_{xx} + 2\gamma\partial_x + \lambda$  with the boundary conditions (3), (4), (5) can easily be obtained as follows.

$$\beta_k = \lambda - k^2 - \gamma^2, \quad e_k = e^{-\gamma x} \sin kx, \quad k \in \mathbb{Z}_{>0} = \{1, 2, \dots\}, \quad (\text{D})$$

$$\beta_k = \begin{cases} \lambda - k^2 - \gamma^2, & k \geq 1 \\ \lambda, & k = 0 \end{cases}, \quad e_k = \begin{cases} e^{-\gamma x}(\gamma \sin kx + k \cos kx), & k \geq 1 \\ 1, & k = 0 \end{cases}, \quad (\text{N}) \quad (7)$$

$$\beta_k = \lambda - k^2 + i2\gamma k \quad e_k = e^{ikx}, \quad k \in \mathbb{Z}, \quad (\text{P})$$

$$\beta_k = \lambda - k^2 + i2\gamma k \quad e_k = e^{ikx}, \quad k \in \mathbb{Z}_{\neq 0}. \quad (\text{P0})$$

where  $i$  denotes the complex imaginary unit throughout the text.

For (D), (N), and (P) cases, the first critical eigenvalue is simple and real and the following principle of exchange of stabilities (PES) condition

$$\beta_{k_c} \begin{cases} < 0, & \lambda < \lambda_c \\ = 0, & \lambda = \lambda_c \\ > 0, & \lambda > \lambda_c \end{cases} \quad (\text{D}), (\text{N}), (\text{P}) \quad (8)$$

$$\text{Re } \beta_{k_c}|_{\lambda=\lambda_c} < 0, \quad k \neq k_c.$$

is valid.

Here

$$\begin{aligned} k_c = 1, \quad \lambda_c = 1 + \gamma^2, & \quad \text{for (D)} \\ k_c = 0, \quad \lambda_c = 0, & \quad \text{for (N), (P)} \end{aligned} \quad (9)$$

For the (P0) case, the first critical eigenvalues have multiplicity two and the corresponding PES condition is as follows

$$\text{Re } \beta_1 = \text{Re } \beta_{-1} \begin{cases} < 0, & \lambda < 1 \\ = 0, & \lambda = 1 \\ > 0, & \lambda > 1 \end{cases} \quad (\text{P0}) \quad (10)$$

$$\text{Re } \beta_k|_{\lambda=1} < 0, \quad |k| \neq 1,$$

so that

$$k_c = \pm 1, \quad \lambda_c = 1, \quad \text{for (P0)}$$

The first critical mode is given by

$$\begin{aligned} e_{k_c}(x) = e_1(x) = e^{-\gamma x} \sin x, & \quad \text{for (D)} \\ e_{k_c}(x) = e_0(x) = 1, & \quad \text{for (N) and (P)} \end{aligned} \quad (11)$$

In the (P0) case, the first transition has quadratic multiplicity and the critical modes are

$$e_1(x) = e^{ix}, \quad e_{-1}(x) = e^{-ix}, \quad \text{for (P0)} \quad (12)$$

The above PES condition dictates the loss of linear stability of the trivial steady state.

2.2. The transition numbers

We define the transition number in the Dirichlet boundary conditions (D) case as follows.

$$\begin{aligned}
 A_{(D)} &= \frac{2}{\pi} \frac{(1 + e^{-\gamma\pi})}{(\gamma^2 + 9)(\gamma^2 + 1)} [6a_1 - 4\gamma a_2 + 3(\gamma^2 + 1)a_3], \\
 B_{(D)} &= C_{(D)} + \sum_{k=2}^{\infty} B_{(D),k}, \\
 C_{(D)} &= \frac{1}{4\pi} \frac{(1 - e^{-2\gamma\pi})}{\gamma(\gamma^2 + 4)(\gamma^2 + 1)} [6a_4 - 3\gamma a_5 + 2(\gamma^2 + 1)a_6 - \gamma(\gamma^2 + 1)a_7], \\
 B_{(D),k} &= -\frac{8}{\pi^2} \frac{e^{-2\gamma\pi} k^2 \left( (-1)^k - e^{\gamma\pi} \right)^2}{\left( \gamma^2 + (-2 + k)^2 \right)^2 \left( \gamma^2 + (2 + k)^2 \right)^2 (1 - k^2)(\gamma^2 + k^2)} B_{(D),k}^1 B_{(D),k}^2, \quad k \geq 2, \\
 B_{(D),k}^1 &= \left[ \frac{2(-4 - 3\gamma^2 + k^2)a_1 + 4(\gamma + \gamma^3)a_2}{(\gamma^2 + k^2)} + (-4 - 3\gamma^2 + k^2)a_3 \right], \\
 B_{(D),k}^2 &= 2(-4 - 3\gamma^2 + k^2)a_1 + 4\gamma(2 + \gamma^2 - k^2)a_2 + (-4 - 3\gamma^2 + k^2)(2 + \gamma^2 - k^2)a_3.
 \end{aligned} \tag{13}$$

For the Neumann boundary condition (N) and the periodic boundary condition (P), we define the transition number as

$$A_{(N)} = A_{(P)} = a_1, \quad B_{(N)} = B_{(P)} = a_4. \tag{14}$$

Finally, for the periodic boundary with zero mean condition (P0), the transition number is defined as

$$\begin{aligned}
 B &= C_{(P)} + B_{(P),2}, \\
 C_{(P)} &= 3a_4 + ia_5 + a_6 + 3ia_7, \\
 B_{(P),2} &= (2a_1 + ia_2 + 4a_3)(a_1 + ia_2 - a_3)/3.
 \end{aligned} \tag{15}$$

2.3. The main theorems

**Theorem 2.1** (Transition Theorem for the Dirichlet (D), Neumann (N), and periodic (P) boundary conditions). *The trivial steady state solution  $u \equiv 0$  of the equation (1) with boundary conditions (D), (N) and (P) undergoes a first transition at  $\lambda = \lambda_c$  which is described by the following reduced ordinary differential equation*

$$\frac{du_{k_c}}{dt} = (\lambda - \lambda_c) u_{k_c} + Au_{k_c}^2 + Bu_{k_c}^3 + o(3). \tag{16}$$

Here  $u_{k_c}$  describes the amplitude of the first critical mode  $e_{k_c}(x)$ . Also  $A = A_{(D)}$ ,  $A = A_{(N)}$ ,  $A = A_{(P)}$  for (D), (N), and (P) respectively, while  $B = B_{(D)}$ ,  $B = B_{(N)}$ ,  $B = B_{(P)}$  for the conditions (D), (N), and (P).

Moreover the following assertions hold true.

- (1) In the case  $A \neq 0$  the first transition is of **mixed type**. There is a single branch of bifurcated steady state solutions at  $(u, \lambda) = (0, \lambda_c)$  given by

$$u_\lambda(x) = -\frac{(\lambda - \lambda_c)}{A} e_{k_c}(x) + o(|(\lambda - \lambda_c)|).$$

(2) In the case  $A = 0$  and  $B \neq 0$ , the first transition is either of **continuous type** or **jump type** depending on  $B < 0$  or  $B > 0$  respectively. In this case there are two branches of bifurcated steady state solutions at  $(u, \lambda) = (0, \lambda_c)$  given by

$$u_\lambda(x) = \pm \sqrt{-\frac{(\lambda - \lambda_c)}{B}} e_{k_c}(x) + o(\sqrt{|\lambda - \lambda_c|}).$$

In the periodic boundary with imposed zero mean condition, the transition theorem is in contrast with the above results.

**Theorem 2.2** (Transition Theorem for periodic boundary condition with zero mean (P0)). Assume that the dispersive term is non-zero, that is  $\gamma \neq 0$ . Then the trivial steady state solution of the main equation (1) with periodicity and zero mean conditions (P0) undergoes a first transition at  $\lambda = 1$ . The first transition is described by the following ordinary differential equation

$$\frac{dz}{dt} = (\lambda - 1 + 2\gamma i) z + Bz |z|^2 + o(|z|^3), \quad (17)$$

where  $z \in \mathbb{C}$  approximates the amplitude of the dominant mode  $e_1 = e^{ix}$ . In particular, the system exhibits a **Hopf bifurcation** at  $\lambda = 1$  and

(1) In the case  $\text{Re} B < 0$  the first transition is **continuous type** and a **periodic solution** given by

$$u_\lambda(x, t) = 2\sqrt{-\frac{(\lambda - 1)}{\text{Re} B}} \cos(\gamma t + x) + o(|\lambda - 1|), \quad (18)$$

bifurcates on  $\lambda > 1$ .

(2) If  $\text{Re} B > 0$  then the first transition is **jump type** and an **unstable periodic solution** given by (18) bifurcates on  $\lambda < 1$ .

#### 2.4. Comparison of self-adjoint and non-self-adjoint cases

As the parameter  $\gamma$  controls the strength of the dispersive term  $2\gamma \frac{\partial u}{\partial x}$ , it is interesting to compare the results ([16]) for the self-adjoint linear operator case ( $\gamma = 0$ ) with the results obtained in this study for the non-self-adjoint linear operator case ( $\gamma \neq 0$ ). For the  $\gamma = 0$  case, a previous result shows that

$$A_{(D)} |_{\gamma=0} = \frac{8a_1 + 4a_3}{3\pi},$$

$$B_{(D)} |_{\gamma=0} = \frac{3a_4 + a_6}{4} - \frac{a_2^2}{12} + \frac{8}{\pi^2} \sum_{k=3}^{\infty} \frac{\left((-1)^k - 1\right)^2}{k^2 (k^2 - 4)^2} \frac{(2a_1 + k^2 a_3) [2a_1 - (k^2 - 2) a_3]}{(k^2 - 1)}. \quad (19)$$

The conclusions of the next theorem are summarized in Table 2.

**Theorem 2.3** (Comparison of the self-adjoint/non-self-adjoint linear operators).

(1) For the Dirichlet (D) boundary conditions, transition numbers  $A$  and  $B$  are continuous functions of  $\gamma$  at  $\gamma = 0$ . That is,

$$\lim_{\gamma \rightarrow 0} A_{(D)} = A_{(D)} |_{\gamma=0},$$

$$\lim_{\gamma \rightarrow 0} B_{(D)} = B_{(D)} |_{\gamma=0} .$$

Moreover, in this case, the dynamical behavior at the first transition is unchanged for the  $\gamma = 0$  and  $\gamma \neq 0$  cases.

- (2) For the Neumann boundary condition (N) and the periodic boundary condition (P) the transition number  $A$  and  $B$  are independent of  $\gamma$ . The dynamic behavior at the first transitions are unchanged for  $\gamma = 0$  and  $\gamma \neq 0$  cases.
- (3) For the periodic boundary condition with zero mean (P0), the transition number  $B$  is independent of  $\gamma$ . As a result,  $\lim_{\gamma \rightarrow 0} B = B |_{\gamma=0}$ . However the nature of the first transition for the  $\gamma = 0$  and  $\gamma \neq 0$  are totally different. In the  $\gamma = 0$  case, the bifurcated attractor (repeller) is a circle of steady states while in the  $\gamma \neq 0$  case, the bifurcated attractor (repeller) is a time periodic solution.

### 3. Some applications of the main results

Now we demonstrate some applications of our analytical approach. Throughout this section, we will denote the Dirichlet/Neumann/periodic/periodic with mean zero boundary condition cases by (D)/(N)/(P)/(P0).

#### 3.1. The Kolmogorov-Fisher equation

This model is often used in biological models such as population dynamics to explain the traveling wave phenomena in such systems [5]. In the presence of a dispersive term, the equation is given by

$$\frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} + \tilde{\gamma} \frac{\partial u}{\partial x} = \lambda u(1 - u),$$

where  $\lambda, D > 0$  and  $\tilde{\gamma}$  are constants. The equation can be put into the standard form of (1)

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial X^2} + 2\gamma \frac{\partial U}{\partial X} + \lambda U - U^2,$$

by the following change of variables

$$x = \sqrt{D}X, \quad u = \frac{U}{\lambda}, \quad \gamma = \frac{\tilde{\gamma}\lambda\sqrt{D}}{2}.$$

The coefficients of the nonlinear operator are given by  $a_1 = -1$  and  $a_k = 0$  for  $k \neq 1$ .

Our findings show that the first transition of this equation is captured by the following transition numbers.

$$A_{(D)} = -\frac{12}{\pi} \frac{(1 + e^{-\gamma\pi})}{(\gamma^2 + 9)(\gamma^2 + 1)} \neq 0,$$

$$A_{(N)} = A_{(P)} = -1.$$

We also define the critical parameter  $\lambda_c$  to be  $\lambda_c = 1 + \gamma^2$  or  $\lambda_c = 0$  for (D) and (NP) cases respectively.

According to our theoretical results, the equation supplemented with the boundary conditions (D), (N) and (P) undergoes a first transition at  $\lambda = \lambda_c$  which is of mixed type. The bifurcating branch of steady states is given by

$$u_\lambda(x) = \frac{U}{\lambda} = -\frac{(\lambda - \lambda_c)}{A\lambda} e_{k_c}(x) + o(|(\lambda - \lambda_c)|),$$

where  $A = A_{(D)}, A_{(N)}, A_{(P)}$  depending on the case and  $e_{k_c}$  is the first critical mode as given in (11).

### 3.2. The Chaffee-Infante equation

This equation ([2]) has attracted much attention in the dynamical system analysis of partial differential equations [7,17]. Here, we consider a version of the equation modified by a dispersive term as below

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\gamma \frac{\partial u}{\partial x} + \lambda u - bu^3,$$

where  $\gamma$ ,  $\lambda$ ,  $b$  are constant parameters.

The coefficients of the nonlinear operator are given by  $a_4 = -b$  and  $a_k = 0$  for  $k \neq 4$ . By Theorem 2.1, the first transition numbers are defined as

$$\begin{aligned} A_{(D)} &= A_{(N)} = A_{(P)} = 0, \\ B_{(D)} &= -\frac{3b}{2\pi} \frac{(1 - e^{-2\gamma\pi})}{\gamma(\gamma^2 + 4)(\gamma^2 + 1)}, \\ B_{(N)} &= B_{(P)} = -b, \end{aligned}$$

so that

$$\text{sign}(B_{(D)}) = \text{sign}(B_{(N)}) = \text{sign}(B_{(P)}) = -\text{sign}(b).$$

We also define the critical parameter  $\lambda = \lambda_c$  to be  $\lambda_c = 1 + \gamma^2$  for (D) and  $\lambda_c = 0$  for (NP) cases.

By Theorem 2.1 under the (D), (N), (P) boundary conditions, the system undergoes a transition at  $\lambda = \lambda_c$ . Moreover in all the cases, the transition is of continuous type if  $b > 0$  and jump type if  $b < 0$ . The bifurcating branch of steady states is given by

$$u_\lambda(x) = \pm \sqrt{-\frac{(\lambda - \lambda_c)}{B}} e_{k_c}(x) + o(|\lambda - \lambda_c|^{1/2}),$$

where  $B = B_{(D)}, B_{(N)}, B_{(P)}$  depending on the case and  $e_{k_c}$  is as given in (11).

### 3.3. The Burger's equation

Burger's equation is originally proposed to study certain aspects of turbulence of fluid motion [1]. Here, we consider the Burger's equation with a dispersive term as follows.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\gamma \frac{\partial u}{\partial x} + \lambda u + \alpha u \frac{\partial u}{\partial x}.$$

The dynamic transition of the equation has been analyzed in the periodic setting in [11] and the attractor bifurcation of the equation has been considered in [8]. The equation is already in the standard form with  $a_2 = \alpha$ .

(1) For (D) case

$$A_{(D)} = -\frac{8\alpha}{\pi} \frac{(1 + e^{-\gamma\pi})\gamma}{(\gamma^2 + 9)(\gamma^2 + 1)} \neq 0.$$

Hence if  $\alpha\gamma \neq 0$  then the equation undergoes a mixed type first transition. The bifurcated branch of steady state solutions is given by

$$u_\lambda(x) = -\frac{(\lambda - \lambda_c)}{A_{(D)}} e^{-\gamma x} \sin x + o(|(\lambda - \lambda_c)|).$$

- (2) For (N) and (P) cases, since  $a_1 = a_4 = 0$  the method proposed in this work gives no conclusion. Hence in these cases, further analysis is required to obtain the reduced system which gives the dynamics of the first transition. This problem will be pursued in another work.
- (3) For the Burger’s equation, in the periodic setting, it is possible to impose the zero average condition as if the initial average vanishes, then the average is zero for all times. Thus it is possible to consider the (P0) setting. For (P0) case, the transition number is  $B = -\alpha^2/3$  so that  $\text{Re } B < 0$  if  $\alpha \neq 0$ . This implies that the first transition is continuous if  $\alpha \neq 0$  and a stable limit cycle bifurcates on  $\lambda > 1$  given by

$$u(x, t) = 2\sqrt{\frac{3(\lambda - 1)}{\alpha^2}} \cos(\gamma t + x) + o(|\lambda - 1|).$$

#### 4. Conclusions and mathematical/physical remarks

In this section we list some conclusions of the main results. We denote the Dirichlet, Neumann, periodic and the periodic with zero mean cases by (D), (N), (P) and (P0) respectively.

- (1) **The classification of transition types.** For (DNP), we show that the main equation may exhibit all three possible types of transitions, namely continuous, jump and mixed. The (P0) case is in contrast as in this case only continuous and jump type transitions are possible.
- (2) **The components of the transition number.** The form of the transition numbers presented in this study separate the transition number into several terms which measure the contribution of both stable and the unstable modes. We show that the transition number can be written as

$$\begin{cases} A + C + \sum_{k=2}^\infty B_k & \text{for (D),} \\ A + C & \text{for (NP),} \\ C + B_2 & \text{for (P0).} \end{cases}$$

The **physical** meaning of the terms above are as follows.

- (a)  $A$  is the quadratic nonlinear self-interaction of the critical mode,
  - (b)  $C$  is the cubic nonlinear self-interaction of the critical mode,
  - (c)  $B_k$ 's are the quadratic nonlinear interactions of the critical mode with the  $k$ th stable mode.
- (3) **The relation between the nonlinearity and the type of transition.**
    - (a) For (D), if the nonlinear operator has quadratic terms, then generically  $A \neq 0$ , the first transition is mixed and the bifurcation is transcritical. On the other hand, if the nonlinear operator does not possess quadratic terms, but contains cubic terms, then generically the first transition is continuous/jump and the bifurcation is pitchfork.
    - (b) For (NP), only  $u^2$  and  $u^3$  terms of the nonlinear operator play a role in the transition numbers. The nonlinear terms involving the derivative of  $u$  do not effect the transition number in this case.
    - (c) For (P0), the transition number is effected by all the quadratic and cubic nonlinear terms.
  - (4) **The effect of the higher modes on the first transition.**

- (a) In the (D) case, the effect of higher modes ( $B_k$ ) on the first transition is generically irrelevant. The computation of  $B_k$  is needed only when

$$(a_1, a_2, a_3) \neq (0, 0, 0) \quad \text{and} \quad 6a_1 - 4\gamma a_2 + 3(\gamma^2 + 1)a_3 = 0,$$

which is a non-generic condition.

- (b) In the (NP) case, the transition number is solely determined by the self interaction of the critical mode and does not depend on the higher mode interactions at all. That is, the approximation of the center manifold is not needed in this case.
- (c) For the periodic setting (P0), the quadratic nonlinear self-interaction of the critical mode ( $A_{(P)} = 0$ ) is always zero and the transition number is described by the cubic nonlinear self-interaction of the critical mode  $C_{(P)}$  and  $B_2$  the quadratic nonlinear interaction of the critical modes  $e^{\pm ix}$ , with the second modes  $e^{\pm i2x}$ .
- (5) **The stabilizing effect of the dispersive term.** It can be seen that for (D), the critical lambda value for the self-adjoint case is  $\lambda_c = 1$  while  $\lambda_c = 1 + \gamma^2$  for the non-self-adjoint case. This shows that the dispersive term has a stabilizing effect for the Dirichlet boundaries. On the other hand, for (NP) (resp. for (P0)), the critical lambda value is  $\lambda_c = 0$  (resp.  $\lambda_c = 1$ ) for both self-adjoint and non-self-adjoint cases and the dispersive term does not have effect on the critical lambda threshold.
- (6) **The effect of the dispersive term on the character of the critical eigenvalues.**
- (a) The eigenvalues for (DNP) are real for both self-adjoint ( $\gamma = 0$ ) and non-self-adjoint ( $\gamma \neq 0$ ) linearized operator cases. For (DNP) the addition of the dispersive first derivative term  $2\gamma \frac{\partial u}{\partial x}$  to the equation does not change the dynamical behavior of the first transitions. For (D), the transition number changes but for (NP), the transition numbers are unchanged between the cases  $\gamma = 0$  and  $\gamma \neq 0$ . Moreover, for (D), we can obtain the transition number in the self-adjoint case as the limit of  $\gamma \rightarrow 0$  in the non-self-adjoint case. Thus we conclude that the addition of the dispersion term generalizes the previous results for the (DNP) cases.
- (b) However, the situation is different for the (P0) case. In this case, the transition number is identical for both the self-adjoint and non-self-adjoint cases. But the dynamical behavior of the first transition is totally different. In the non-self-adjoint case, the first transition occurs as a Hopf bifurcation while for the self-adjoint case, the first transition occurs as a bifurcated circle of steady states.
- (7) **The effect of the interval length.** A more general problem would be to consider an interval of length  $L$  instead of  $\pi$ . However by defining the variables

$$x' = \frac{\pi}{L}x, \quad t' = \frac{\pi^2}{L^2}t,$$

we can always rescale the problem to the case studied in this paper. Obviously, the transition numbers will also depend on the length of the interval  $L$ . Also the critical lambda value becomes  $\lambda_c = \frac{L^2}{\pi^2}(1 + \gamma^2)$  for (D), and  $\lambda_c = \frac{L^2}{\pi^2}$  for (P0), showing that increasing  $L$  has a stabilizing effect for the trivial solution in the (D) and (P0) case. For (NP) case,  $\lambda_c = 0$  for any  $L > 0$ .

- (8) **Future work.**

- (a) In our study we considered the semilinear equation where  $g(u, u_x)$ . The quasilinear case  $g(u, u_x, u_{xx})$  will be considered in a future work.

(b) In the Neumann and periodic boundary condition cases, the reduced equation can be written as

$$\frac{du_0}{dt} = \lambda u_0 + \left( \frac{1}{2!} \frac{\partial^2 g(u, u_x)}{\partial u^2} \Big|_{u=0} \right) u_0^2 + \left( \frac{1}{3!} \frac{\partial^3 g(u, u_x)}{\partial u^3} \Big|_{u=0} \right) u_0^3 + O(u_0^4),$$

where  $u_0$  approximates the amplitude of the first critical mode. We expect that the general reduced equation in these cases to be in the form

$$\frac{du_0}{dt} = \lambda u_0 + \sum_{n=2}^{\infty} \left( \frac{1}{n!} \frac{\partial^n g(u, u_x)}{\partial u^n} \Big|_{u=0} \right) u_0^n.$$

One problem in these cases is to study the first transition when

$$\frac{\partial^n g(u, u_x)}{\partial u^n} \Big|_{u=0} = 0, \quad \forall n \geq 2.$$

Since under the above condition, all the nonlinear terms of the reduced equation vanish and no information on the bifurcation can be established. This is the case, for example, for the Burger’s nonlinearity  $g(u, u_x) = \alpha u u_x$ .

(c) We plan to expand the results of this study to two/three spatial dimensional problems [18], system of reaction-diffusion equations [9,19], equations containing noise terms [3,4], equations on unbounded intervals and equations containing third and higher order leading linear terms [14,10].

### 5. The reduction procedure

We limit ourselves in the regime where the initial conditions are near the trivial steady state as well as the control parameter is near criticality, i.e.  $|\lambda - \lambda_c|$  is small. Under these conditions, it is sufficient to consider the dynamics of the projection of the solution onto the first critical mode which gives a reduced equation of the system. We will use the central manifold theory to obtain this reduced equation. The analysis of this reduced equation then gives a complete picture on the first transition and the associated bifurcation.

#### 5.1. Dirichlet boundary condition case

We begin by expanding the unknown function  $u$  in terms of the basis functions (7) as

$$u = \sum_{k=1}^{\infty} u_k(t) e^{-\gamma x} \sin kx.$$

To obtain the reduced equation, we write the main equation in the abstract form

$$\frac{du}{dt} = L_\lambda u + G(u), \tag{20}$$

where  $L_\lambda$  and  $G$  are linear and nonlinear operators defined on suitable functional spaces respectively. Now let us consider the adjoint basis eigenfunctions

$$e_k^* = e^{\gamma x} \sin kx, \quad k \geq 1,$$

so that

$$\langle e_j, e_k^* \rangle = \int_0^\pi e^{-\gamma x} \sin jx e^{\gamma x} \sin kx dx = \frac{\pi}{2} \delta_{jk},$$

where  $\delta_{jk}$  is the Kronecker's delta.

We express the nonlinear term as

$$\begin{aligned} G(u) &= a_1 u^2 + a_2 u u_x + a_3 u_x^2 + a_4 u^3 + a_5 u^2 u_x + a_6 u u_x^2 + a_7 u_x^3 + o(|(u, u_x)|^3) \\ &= \sum_{k=1}^{\infty} g_k(t) e^{-\gamma x} \sin kx, \end{aligned}$$

where

$$g_k = \frac{\langle G(u), e_k^* \rangle}{\langle e_k, e_k^* \rangle} = \frac{2}{\pi} \int_0^\pi G(u) e^{\gamma x} \sin kx dx, \quad k \geq 1.$$

Then we project the equation on the first eigenvector by taking the inner product of (20) with  $e_1^*$  to obtain

$$\frac{du_1}{dt} = \beta_1 u_1 + g_1,$$

where  $\beta_1 = \lambda - 1 - \gamma^2$  and

$$g_1 = \frac{2}{\pi} \langle G(u), e_1^* \rangle.$$

We have to obtain an expression for  $g_1$  in terms of  $u_1$ . According to the center manifold theorem, the higher frequency modes  $u_k$  are approximated by  $\Phi_k$ , the components of the center manifold function  $\Phi$  due to the tangency condition. As a result, for  $k \geq 2$ , we have

$$\Phi_k(u_1(t)) = b_k u_1(t)^2 + o(u_1(t)^2), \quad u_1(t) \rightarrow 0.$$

That is, we set

$$\begin{aligned} u &= u_c + \Phi, \\ u_c &= u_1 e^{-\gamma x} \sin x, \\ \Phi &= \sum_{k=2}^{\infty} \Phi_k(u_1) e^{-\gamma x} \sin kx = \sum_{k=2}^{\infty} b_k u_1^2 e^{-\gamma x} \sin kx + o(u_1^2). \end{aligned}$$

The explicit computation of the term  $g_1$  is as follows.

$$\begin{aligned} g_1 &= \frac{2a_1}{\pi} \int_0^\pi [u_1^2 e^{-\gamma x} \sin^3 x + 2u_1 \Phi \sin^2 x] dx \\ &+ \frac{2a_2}{\pi} \int_0^\pi [u_1^2 e^{-\gamma x} \sin^2 x (-\gamma \sin x + \cos x) + u_1 \Phi_x \sin^2 x + u_1 \Phi \sin x (-\gamma \sin x + \cos x)] dx \\ &+ \frac{2a_3}{\pi} \int_0^\pi [u_1^2 e^{-\gamma x} \sin x (-\gamma \sin x + \cos x)^2 + 2u_1 \Phi_x \sin x (-\gamma \sin x + \cos x)] dx \end{aligned}$$

$$\begin{aligned}
 &+ \frac{2a_4}{\pi} \int_0^\pi u_1^3 e^{-2\gamma x} \sin^4 x dx + \frac{2a_5}{\pi} \int_0^\pi u_1^3 e^{-2\gamma x} \sin^3 x (-\gamma \sin x + \cos x) dx \\
 &+ \frac{2a_6}{\pi} \int_0^\pi u_1^3 e^{-2\gamma x} \sin^2 x (-\gamma \sin x + \cos x)^2 dx + \frac{2a_7}{\pi} \int_0^\pi u_1^3 e^{-2\gamma x} \sin x (-\gamma \sin x + \cos x)^3 dx + o(3) \\
 &= A_{(D)} u_1^2 + B_{(D)} u_1^3 + o(3),
 \end{aligned}$$

where

$$\begin{aligned}
 A_{(D)} &= \frac{2a_1}{\pi} \int_0^\pi e^{-\gamma x} \sin^3 x dx + \frac{2a_2}{\pi} \int_0^\pi e^{-\gamma x} \sin^2 x (-\gamma \sin x + \cos x) dx \\
 &+ \frac{2a_3}{\pi} \int_0^\pi e^{-\gamma x} \sin x (-\gamma \sin x + \cos x)^2 dx \\
 &= \frac{2}{\pi} \frac{(1 + e^{-\gamma\pi})}{(\gamma^2 + 9)(\gamma^2 + 1)} [6a_1 - 4\gamma a_2 + 3(\gamma^2 + 1) a_3],
 \end{aligned}$$

and

$$\begin{aligned}
 B_{(D)} &= \frac{4a_1}{\pi} \sum_{k=2}^\infty b_k \int_0^\pi e^{-\gamma x} \sin kx \sin^2 x dx \\
 &+ \frac{2a_2}{\pi} \sum_{k=2}^\infty b_k \int_0^\pi e^{-\gamma x} [(-\gamma \sin kx + k \cos kx) \sin^2 x + \sin kx \sin x (-\gamma \sin x + \cos x)] dx \\
 &+ \frac{4a_3}{\pi} \sum_{k=2}^\infty b_k \int_0^\pi e^{-\gamma x} (-\gamma \sin kx + k \cos kx) \sin x (-\gamma \sin x + \cos x) dx \\
 &+ \frac{2a_4}{\pi} \int_0^\pi e^{-2\gamma x} \sin^4 x dx + \frac{2a_5}{\pi} \int_0^\pi e^{-2\gamma x} \sin^3 x (-\gamma \sin x + \cos x) dx \\
 &+ \frac{2a_6}{\pi} \int_0^\pi e^{-2\gamma x} \sin^2 x (-\gamma \sin x + \cos x)^2 dx + \frac{2a_7}{\pi} \int_0^\pi e^{-2\gamma x} \sin x (-\gamma \sin x + \cos x)^3 dx.
 \end{aligned}$$

By the computation of the above integrals, the coefficient  $B_{(D)}$  has the following form

$$\begin{aligned}
 B_{(D)} &= C_{(D)} + \sum_{k=2}^\infty B_{(D),k} \\
 C_{(D)} &= \frac{1}{4\pi} \frac{(1 - e^{-2\gamma\pi})}{\gamma(\gamma^2 + 4)(\gamma^2 + 1)} [6a_4 - 3\gamma a_5 + 2(\gamma^2 + 1) a_6 - \gamma(\gamma^2 + 1) a_7], \\
 B_{(D),k} &= \frac{4}{\pi} \frac{ke^{-\gamma\pi} \left( (-1)^k - e^{\gamma\pi} \right)}{\left( \gamma^2 + (-2 + k)^2 \right) \left( \gamma^2 + (2 + k)^2 \right)} \\
 &\times \left[ \frac{2(-4 - 3\gamma^2 + k^2)}{\gamma^2 + k^2} a_1 + \frac{4(\gamma + \gamma^3)}{\gamma^2 + k^2} a_2 + (-4 - 3\gamma^2 + k^2) a_3 \right] b_k, \quad k \geq 2.
 \end{aligned} \tag{21}$$

Now to derive the reduced model to the lowest order, we need to find  $b_k$ . Therefore, we differentiate

$$u_k(t) = b_k u_1^2(t) + o(u_1^2),$$

to find

$$\dot{u}_k(t) = \dot{u}_1(2b_k u_1 + o(u_1)),$$

where  $\dot{f} = df/dt$ . This gives

$$\beta_k(\lambda_c) u_k + g_k(t) = \dot{u}_k(t) = (\beta_1(\lambda_c) u_1 + g_1(t))(2b_k u_1 + o(u_1)).$$

Since  $g_1 = O(u_1^3)$  and

$$\beta_k(\lambda_c) = \lambda_c - \gamma^2 - k^2 = 1 - k^2 + O(|\beta_1|),$$

we get

$$\beta_k(\lambda_c) (b_k u_1^2 + o(u_1^2)) + g_k = O(u_1^2 |\beta_1|) + O(u_1^3).$$

The coefficient  $b_k$  can be found by approximating  $g_k$  to the quadratic order in  $u_1$ . Since  $u_k = \Phi_k(u_1) = O(u_1^2)$

$$\begin{aligned} g_k &= \frac{2}{\pi} a_1 u_1^2 \int_0^\pi e^{-\gamma x} \sin^2 x \sin kx dx + \frac{2}{\pi} a_2 u_1^2 \int_0^\pi e^{-\gamma x} (-\gamma \sin x + \cos x) \sin x \sin kx dx \\ &\quad + \frac{2}{\pi} a_3 u_1^2 \int_0^\pi e^{-\gamma x} (-\gamma \sin x + \cos x)^2 \sin kx dx \\ &= u_1^2 \frac{2}{\pi} \frac{ke^{-\gamma\pi} \left( (-1)^k - e^{\gamma\pi} \right)}{\left( \gamma^2 + (-2 + k)^2 \right) (\gamma^2 + k^2) \left( \gamma^2 + (2 + k)^2 \right)} \\ &\quad \times \left[ \begin{array}{l} 2(-4 - 3\gamma^2 + k^2) a_1 - 4\gamma(2 + \gamma^2 - k^2) a_2 \\ + (-4 - 3\gamma^2 + k^2) (2 + \gamma^2 - k^2) a_3 \end{array} \right], \end{aligned}$$

then

$$\begin{aligned} b_k &= \frac{2}{\pi} \frac{ke^{-\gamma\pi} \left( (-1)^k - e^{\gamma\pi} \right)}{(1 - k^2) \left( \gamma^2 + (-2 + k)^2 \right) (\gamma^2 + k^2) \left( \gamma^2 + (2 + k)^2 \right)} \\ &\quad \times \left[ \begin{array}{l} -2(-4 - 3\gamma^2 + k^2) a_1 + 4\gamma(2 + \gamma^2 - k^2) a_2 \\ + (-4 - 3\gamma^2 + k^2) (2 + \gamma^2 - k^2) a_3 \end{array} \right], \quad k \geq 2. \end{aligned}$$

Thus,  $B_{(D),k}$  in (21) is found as in (13).

That finishes the reduction procedure for the Dirichlet boundary condition case.

5.2. Neumann boundary condition case

The proof can be easily done by applying a similar analysis for the Neumann boundary condition (N). In this case, the eigenfunctions are now given by  $e_k(x) = e^{-\gamma x}(\gamma \sin kx + k \cos kx)$ ,  $k \geq 1$ .

In the Neumann case (N), the critical mode is constant  $e_0 = 1$ . We once again approximate write the solution as the center/unstable part  $u_0 e_0 = u_0$  plus the center manifold function  $\Phi$  which approximates the stable part of the solution.

$$u = u_0 + \Phi,$$

$$\Phi = \sum_{k=1}^{\infty} \Phi_k(u_0) e^{-\gamma x}(\gamma \sin kx + k \cos kx) = \sum_{k=1}^{\infty} b_k u_0^2 e^{-\gamma x}(\gamma \sin kx + k \cos kx) + o(u_0^2).$$

For the Neumann boundary conditions, the critical adjoint eigenfunction is

$$e_0^* = e^{2\gamma x}.$$

Taking the inner product of the main equation with  $e_0^*$  gives

$$\frac{du_0}{dt} = \beta_0 u_0 + g_0,$$

where  $\beta_0 = \lambda$  and  $g_0$  is as follows.

$$\begin{aligned} g_0 &= \frac{2\gamma a_1}{e^{2\gamma\pi} - 1} \int_0^\pi (u_0 + \Phi)^2 e^{2\gamma x} dx + \frac{2\gamma a_2}{e^{2\gamma\pi} - 1} \int_0^\pi (u_0 + \Phi) \Phi_x e^{2\gamma x} dx + \frac{2\gamma a_3}{e^{2\gamma\pi} - 1} \int_0^\pi \Phi_x^2 e^{2\gamma x} dx \\ &+ \frac{2\gamma a_4}{e^{2\gamma\pi} - 1} \int_0^\pi (u_0 + \Phi)^3 e^{2\gamma x} dx + \frac{2\gamma a_5}{e^{2\gamma\pi} - 1} \int_0^\pi (u_0 + \Phi)^2 \Phi_x e^{2\gamma x} dx \\ &+ \frac{2\gamma a_6}{e^{2\gamma\pi} - 1} \int_0^\pi (u_0 + \Phi) \Phi_x^2 e^{2\gamma x} dx + \frac{2\gamma a_7}{e^{2\gamma\pi} - 1} \int_0^\pi \Phi_x^3 e^{2\gamma x} dx \\ &= \frac{2\gamma a_1}{e^{2\gamma\pi} - 1} \int_0^\pi (u_0 + \Phi)^2 e^{2\gamma x} dx + \frac{2\gamma a_2}{e^{2\gamma\pi} - 1} \int_0^\pi (u_0 + \Phi) \Phi_x e^{2\gamma x} dx + \frac{2\gamma a_4}{e^{2\gamma\pi} - 1} \int_0^\pi u_0^3 e^{2\gamma x} dx \\ &= A_{(N)} u_0^2 + B_{(N)} u_0^3 + o(3), \end{aligned}$$

where

$$A_{(N)} = \frac{2\gamma a_1}{e^{2\gamma\pi} - 1} \int_0^\pi e^{2\gamma x} dx = a_1,$$

and

$$B_{(N)} = \frac{2\gamma a_2}{e^{2\gamma\pi} - 1} \sum_{k=1}^{\infty} b_k k e^{\gamma\pi} \left( (-1)^k - e^{-\gamma\pi} \right) + a_4.$$

By similar computation, it can be found easily that  $b_k = 0$  for  $k \geq 1$ . Thus,  $B_{(N)}$  in (14) is found as

$$B_{(N)} = a_4.$$

The assertions of Theorem 2.1 for the Neumann boundary condition (N) follow from the above analysis.

### 5.3. Periodic boundary condition with zero mean case

In this case, the system can be reduced as

$$\begin{aligned} u &= u_c + \Phi \\ u_c &= ze^{ix} + \bar{z}e^{-ix}, \\ \Phi(z) &= \sum_{\substack{k=-\infty \\ |k| \neq 1}}^{\infty} \Phi_k(z, \bar{z}) e^{ikx} = \sum_{\substack{k=-\infty \\ |k| \neq 1}}^{\infty} (b_{k,1}z^2 + b_{k,2}z\bar{z} + b_{k,3}\bar{z}^2) e^{ikx} + o(3), \end{aligned}$$

where

$$\Phi_k(z, \bar{z}) = (b_{k,1}z^2 + b_{k,2}z\bar{z} + b_{k,3}\bar{z}^2) + o(3).$$

Now by projecting the main equation on to the space spanned by  $e_1 = e^{ix}$  we find that

$$\frac{dz}{dt} = (\lambda - 1 + i2\gamma)z + g_1.$$

We need to find  $g_1$ . Similarly, we can write the following equation

$$\begin{aligned} g_1 &= \frac{1}{2\pi} \int_0^{2\pi} [a_1u^2 + a_2uu_x + a_3u_x^2 + a_4u^3 + a_5u^2u_x + a_6uu_x^2 + a_7u_x^3 + o(|(u, u_x)|^3)] e^{-ix} dx \\ &+ \frac{ia_2}{2\pi} \int_0^{2\pi} (ze^{ix} + \bar{z}e^{-ix} + \Phi) (ze^{ix} - \bar{z}e^{-ix} + \Phi_x/i) e^{-ix} dx \\ &- \frac{a_3}{2\pi} \int_0^{2\pi} (ze^{ix} - \bar{z}e^{-ix} + \Phi_x/i)^2 e^{-ix} dx + \frac{a_4}{2\pi} \int_0^{2\pi} (ze^{ix} + \bar{z}e^{-ix} + \Phi)^3 e^{-ix} dx \\ &+ \frac{ia_5}{2\pi} \int_0^{2\pi} (ze^{ix} + \bar{z}e^{-ix} + \Phi)^2 (ze^{ix} - \bar{z}e^{-ix} + \Phi_x/i) e^{-ix} dx \\ &- \frac{a_6}{2\pi} \int_0^{2\pi} (ze^{ix} + \bar{z}e^{-ix} + \Phi) (ze^{ix} - \bar{z}e^{-ix} + \Phi_x/i)^2 e^{-ix} dx \\ &- \frac{ia_7}{2\pi} \int_0^{2\pi} (ze^{ix} - \bar{z}e^{-ix} + \Phi_x/i)^3 e^{-ix} dx + o(3) \\ &= (2a_1 + ia_2 + 4a_3) (b_{2,1}z^2 + b_{2,2}z\bar{z} + b_{2,3}\bar{z}^2) \bar{z} + (3a_4 + ia_5 + a_6 + 3ia_7) z^2\bar{z} + o(3). \end{aligned}$$

We need to find  $b_{2,1}, b_{2,2}, b_{2,3}$ . As in the Dirichlet and Neumann type boundary conditions, these coefficients are obtained as follows.

$$b_{2,1}z^2 + b_{2,2}z\bar{z} + b_{2,3}\bar{z}^2 = \frac{1}{3}(a_1 + ia_2 - a_3)z^2.$$

Thus, we find

$$\begin{aligned} g_1 &= \frac{1}{3}(2a_1 + ia_2 + 4a_3)(a_1 + ia_2 - a_3)z^2\bar{z} + (3a_4 + ia_5 + a_6 + 3ia_7)z^2\bar{z} + o(3) \\ &= Bz|z|^2 + o(3), \end{aligned}$$

where  $B$  is obtained as in (15).

#### 5.4. Periodic boundary condition

The proof in this case mimics the proof of the periodic boundary condition with zero mean case and is omitted.

## References

- [1] J.M. Burgers, A mathematical model illustrating the theory of turbulence, in: Richard Von Mises, Theodore Von Kármán (Eds.), *Advances in Applied Mechanics*, vol. 1, Elsevier, January 1948, pp. 171–199.
- [2] N. Chafee, E.F. Infante, A bifurcation problem for a nonlinear partial differential equation of parabolic type, *Appl. Anal.* 4 (1) (January 1974) 17–37.
- [3] Mickaël D. Chekroun, Honghu Liu, Shouhong Wang, *Approximation of Stochastic Invariant Manifolds*. SpringerBriefs in Mathematics, Springer International Publishing, Cham, 2015.
- [4] Mickaël D. Chekroun, Honghu Liu, Shouhong Wang, *Stochastic Parameterizing Manifolds and Non-Markovian Reduced Equations*, SpringerBriefs in Mathematics, Springer International Publishing, Cham, 2015.
- [5] Ronald Aylmer Fisher, The wave of advance of advantageous genes, *Ann. Eugen.* 7 (4) (1937) 355–369.
- [6] Daozhi Han, Marco Hernandez, Quan Wang, Dynamic transitions and bifurcations for a class of axisymmetric geophysical fluid flow, *SIAM J. Appl. Dyn. Syst.* 20 (1) (January 2021) 38–64.
- [7] Daniel Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, vol. 840, Springer, Berlin Heidelberg, 1981.
- [8] Chun-Hsiung Hsia, Xiaoming Wang, On a Burgers' type equation, *Discrete Contin. Dyn. Syst., Ser. B* 6 (5) (2006) 1121–1139.
- [9] Lan Jia, Liang Li, Stability and dynamic transition of vegetation model for flat arid terrains, *Discrete Contin. Dyn. Syst., Ser. B* 27 (6) (2022) 3375–3398.
- [10] Kevin Li, Dynamic transitions of the Swift-Hohenberg equation with third-order dispersion, *Discrete Contin. Dyn. Syst., Ser. B* 26 (12) (2021) 6069–6090.
- [11] Limei Li, Kiah Wah Ong, Dynamic transitions of generalized Burgers equation, *J. Math. Fluid Mech.* 18 (1) (March 2016) 89–102.
- [12] Tian Ma, Shouhong Wang, *Phase Transition Dynamics*, Springer International Publishing, Cham, 2019.
- [13] Yiqiu Mao, Dongming Yan, Chunhsien Lu, Dynamic transitions and stability for the acetabularia whorl formation, *Discrete Contin. Dyn. Syst., Ser. B* 24 (11) (2019) 5989.
- [14] Kiah Wah Ong, Dynamic transitions of generalized Kuramoto-Sivashinsky equation, *Discrete Contin. Dyn. Syst., Ser. B* 21 (4) (March 2016) 1225–1236.
- [15] Zhigang Pan, Lan Jia, Yiqiu Mao, Quan Wang, Transitions and bifurcations in couple stress fluid saturated porous media using a thermal non-equilibrium model, *Appl. Math. Comput.* 415 (February 2022) 126727.
- [16] Taylan Şengül, Burhan Tiryakioglu, Dynamic transitions and bifurcations of 1D reaction–diffusion equations: the self-adjoint case, *Math. Methods Appl. Sci.* 45 (5) (March 2022) 2871–2892.
- [17] Roger Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences., vol. 68, Springer, New York, 1997.
- [18] Quan Wang, Dongming Yan, On the stability and transition of the Cahn-Hilliard/Allen-Cahn system, *Discrete Contin. Dyn. Syst., Ser. B* 25 (7) (2020) 2607.
- [19] Chao Xing, Jiaojiao Pan, Hong Luo, Stability and dynamic transition of a toxin-producing phytoplankton-zooplankton model with additional food, *Commun. Pure Appl. Anal.* 20 (1) (2021) 427–448.