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COMPUTATION OF THE EXPECTED VALUE OF THE INTERVAL WIDTH IN THE SMALL – SAMPLE CASE

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ÖZET

Istatistiğin esas amacı, örneklemin sahip olduğu bilgilerden yararlanarak, kitle hakkında netice çıkarmak, kitlenin parametrelerini tahmin etmektir. Dikkat etmemiz gereken bir nokta, kitle ortalaması □ için, student's t dağılımına dayanan, küçük örneklem güven aralığı tesadüfi bir genişliğe sahiptir (halbuki geniş-örneklem güven aralığının genişliği tesadüfi değildir). Bu makalenin amacı küçük örneklem göz önüne alındığında, aralık genişliğinin ortalama değeri, diğer bir deyişle beklenen değerini hesaplayabilecek yöntemi ortaya koyabilmektir.

ABSTRACT

The objective of many statistical investigations is to make inferences about population parameters based on sample data. Often these inferences are in the form of estimates, either point estimates[2] on interval estimates.

We must note that the small-sample confidence interval for □, based on student's t (see section IV), possesses a random width [in contrast to the large-sample confidence interval (see section III), where the width is nonrandom]. In this paper our aim is to find the expected value of interval width in the samll-sample case.

I. Necessary Theorems For This Article

Theorem I: Let c be constant.
Then E(c) = c

Theorem II[4]: Let $g(Y_1,Y_2)$ be a function of the random variables Y_1,Y_2 and let c be a constant then
$E[cg(Y_1,Y_2)] = cE[g(Y_1,Y_2)]$

Theorem III: Let Y_1 and Y_2 be random variables with density function $f(y_1,y_2)$ and let $g_1(Y_1,Y_2)$, $g_2(Y_1,Y_2)$,...,$g_k(Y_1,Y_2)$ be functions of Y_1 and Y_2. Then

$E[g_1(Y_1,Y_2) + g_2(Y_1,Y_2) +...+g_k(Y_1,Y_2)]$

$= E[g_1(Y_1,Y_2)] +E[g_2(Y_1,Y_2)]$
$+...+E[g_k(Y_1,Y_2)]$

Theorem IV: Let Y_1 and Y_2 be independent random variables with joint density $f(y_1,y_2)$. Let $g(Y_1)$ and $h(Y_2)$ be functions of Y_1 and Y_2, respectively.
Then
$E[g(Y_1)\, h(Y_2)] = E[g(Y_1)]\, E[h(Y_2)]$
provided the expectations exist.

Theorem V: Let Y_1 and Y_2 be random variables with joint density functions $f(y_1,y_2)$.
Then

$$\begin{aligned} Cov(Y_1,Y_2) &= E[(Y_1 - \mu_1)(Y_2 - \mu_2)] \\ &= E(Y_1,Y_2) - E(Y_1)\, E(Y_2) \end{aligned}$$

Theorem VI: If Y_1 and Y_2 are independent variable then,

$$Cov(Y_1,Y_2) = 0$$

Theorem VII: Let $Y_1,...,Y_n$ and $X_1,...,X_m$ be random variables with $E(Y_i) = \mu_i$ and $E(X_i) = \varepsilon_i$

Define $U_1 = \sum_{i=1}^{n} a_i Y_i \qquad U_2 = \sum_{j=1}^{m} b_j X_j$

for constant $a_1,...,a_n$ and $b_1,...,b_m$. Then

(a) $E(U_1) = \sum_{i=1}^{n} a_i \mu_i$

(b) $V(U_1) = \sum_{i=1}^{n} \sigma^2 V(Y_i) + 2\sum\sum_{i<j} a_i a_j Cov(Y_i,Y_j)$

where the double sum is over all pairs (i,j) with i<j and

(c) $Cov(U_1,U_2) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j Cov(Y_i,X_j)$

Theorem VIII: Let $Y_1,...,Y_n$ be independent normally distributed random with $E(Y_i) = \mu_i$ and
$V(Y_i) = \sigma^2_i$, i = 1,...,n. Define U by

$$U = \sum_{i=1}^{n} a_i Y_i$$

where $a_1,...,a_n$ are constants. Then U is a normally distributed random variable with

$$E(U) = \sum_{i=1}^{n} a_i \mu_i$$

and $\quad V(U) = \sum_{i=1}^{n} a_i^2 \sigma_i^2$

Theorem IX: Let $Y_1,...,Y_n$ be as in Theorem 8, and define Z_i by

$$Z_i = \frac{Y_i - \mu_i}{\sigma_i} \qquad i = 1, \ldots, n.$$

Then $\sum_{i=1}^{n} Z_i^2$ has a χ^2 distribution with n degrees of freedom.

Theorem X: If $Y_1,...,Y_n$ are independent normal variables with common mean μ and common variances σ^2. Then

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \frac{(n-1)S^2}{\sigma^2}$$

has a χ^2 distribution with (n – 1) degrees of freedom. Also, $\bar{Y}$ and S^2 are independent random variables.

Theorem XI: Suppose thet X_n converges in probability to μ_1 and Y_n converges in probability to μ_2. Then
(a) $X_n + Y_n$ converges in probability to $\mu_1+\mu_2$
(b) X_nY_n converges in probability to $\mu_1\mu_2$
(c) X_n/Y_n converges in probability to μ_1/μ_2 that $\mu_2 \neq 0$.
(d) $\sqrt{X_n}$ converges in probability to $\sqrt{\mu_1}$ provided that $P(X_n \geq 0 = 1)$

Theorem XII: The central limit[10] theorem: Let $X_1,...,X_n$ be independent and identically random variables with $E(X_i) = \mu$ and $V(X_i) = \sigma^2 < \infty$. Define Y_n to be

$$Y_n = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \text{ where } \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Then Y_n converges in distribution to standard normal random variable.

Theorem XIII: The sequence of random variables, X1, ..., Xn is said to converge in probability to the constant c if for every positive number c.

$$\lim_{n\to\infty} P[X_n - c > \varepsilon\ ] = 1$$

Theorem XIII : Suppose that X_n converges distribution to a random variable X and Y_n converges in probability to unity. Then X_n/Y_n converges in distribution to X.

Example 1: Suppose that X1, ..., Xn are independent and identically distributed random variable with $E(X_i) = \mu_1$, $E(X_i^2) = \mu_2'$, $E(X_i^3) = \mu_3'$, and $E(X_i^4) = \mu_4'$, all assumed finite. Let S^2 denote the sample variance given by

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$$

Show that S^2 converges in probability to $V(X_i)$ for the solution, first note that we can write

$$S'^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2$$

where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

To show that S'^2 converges in probability to $V(X_i)$ we will apply both theorems 11 and 13. Look at the terms in S'^2. The quantity $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ is the average of n independent and identically distributed variables of the form X_i^2, with $E(X_i^2) = \mu_2'$ and $V(X_i^2) = \mu_4' - (\mu_2')^2$. Since $V(X_i^2)$ is assumed to be finite. Theorem 13 tells us that $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ converges in probability to μ_2'. Now consider the limit of $\overline{X}^2$ as n approaches infinity. Theorem 13 tells us that $\overline{X}$ converges in probability to μ_1 and it follows from theorem 11 part (b), that $\overline{X}^2$ converges in probability to μ^2

This leads to final step. Having shown that $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ and $\overline{X}^2$ converge in probability to μ_2' and μ^2, respectively, it follows from theorem 11 that

$$S'^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2$$

converges in probability to

$$\mu_2' - \mu^2 = V(X_i).$$

This example shows that, for large samples, the sample variance should be closed to the population variance with high probability.

II. Some Common Unbiased Point Estimators

Some estimators[15] for population parameters are selected intitively. For example, it seems natural to use the sample mean, $\overline{Y}$, to estimate the population mean μ, and the sample proportion, $\hat{p} = Y/n$, to

Target Parameter θ	Sample Size(s)	Point Estimator $\hat{\theta}$	Expected Value of $\hat{\theta}$	Variance Of $\hat{\theta}$
μ	n	$\overline{Y}$	μ	$\frac{\sigma^2}{n}$
P	n	$\hat{p} = \frac{Y}{n}$	P	$\frac{pq}{n}$
$\mu_1 - \mu_2$	n_1 and n_2	$\overline{Y}_1 - \overline{Y}_2$	$\mu_1 - \mu_2$	$\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$
$p_1 - p_2$	n_1 and n_2	$\hat{p}_1 - \hat{p}_2$	$p_1 - p_2$	$\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}$

Table 1

estimate a binomial parameter p.[3]
How, then, would you estimate the difference between corresponding parameters for two different populations, say the difference in means, ($\mu_1 - \mu_2$), when the inference is to be based on random samples of n, and n_2 observations selected independently from the two populations?Again, our intuition suggest the point estimators, $(\overline{Y}_1 - \overline{Y}_2)$, the difference in the means, for estimating ($\mu_1 - \mu_2$) and $(\hat{P}_1 - \hat{P}_2)$, the difference in the sample proportions, for estimating ($p_1 - p_2$) means , for estimating $(\mu_1 - \mu_2)$ and $(\hat{p}_1 - \hat{p}_2)$, the difference in the sample proportions , for estimating ($p_1 - p_2$).

Since the four estimators , $\overline{Y}$, $\hat{p}$, $(\overline{Y}_1 - \overline{Y}_2)$, and $(\hat{p}_1 - \hat{p}_2)$ are functions of the random measurements[7] , we could find their expected values and variances using the expectation theorems from 1 to 7. Such an effort would show that all four estimators are unbiased and that they possess the variances shown in Table 1 , when random sampling has been employed[2].

Table 1. Expected Values And Variances Of Some Common Point Estimators

* Note : σ_1^2 and σ_2^2 are the variances of population 1 and 2 respectively.

Using our previous knowledge and theorem 7 , it follows that

$$E(\overline{Y}_1 - \overline{Y}_2) = E(\overline{Y}_1) - E(\overline{Y}_2) = \mu_1 - \mu_2$$

and

$$V(\overline{Y}_1 - \overline{Y}_2) = V(\overline{Y}_1) + V(\overline{Y}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Not all estimatorss are unbiased. Example 2 shows that the sample variance[11]

$$S'^2 = \frac{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}{n},$$

is a biased estimator of σ^2 and , particularly , that this bias can be corrected by dividing the sum of squares of deviations of the measurements about $\overline{Y}$ by $(n - 1)$ rather than n. Because it is most often used in practice to estimate σ^2 , the unbiased estimator[9]

$$S^2 = \frac{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}{n-1}$$

is often calles the sample variance

Example 2 : . Let $Y_1, \ldots, Y_n$ be a random sample with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$, show that

$$S'^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$$

is a biased estimator of σ^2

Solution : It can be shown that

$$\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \sum_{i=1}^{n} Y_i^2 - n\overline{Y}^2$$

Hence

$$E\left[\sum_{i=1}^{n}(Y_i - \overline{Y})^2\right] = E\left[\sum_{i=1}^{n} Y_i^2\right] - nE(\overline{Y}^2)$$

$$= \sum_{i=1}^{n} E(Y_i^2) - nE(\overline{Y}^2)$$

Now note that $E(Y_i^2)$ is the same for $i = 1 , 2 , \ldots , n$ and use the fact that the variance of a sample variable is given by $V(Y) = E(Y^2) - \mu^2$. Then

$$E(Y^2) = V(Y) + \mu^2 \text{ and}$$

$$E\left[\sum_{i=1}^{n}(Y_i - \overline{Y})^2\right] = \sum_{i=1}^{n}(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= n\sigma^2 - \sigma^2$$

$$= (n-1)\,\sigma^2$$

it follows that

$$E(S'^2) = \frac{n-1}{n}\sigma^2$$

and that

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$$

is an unbiased estimator of σ^2

Two final comments can be made concerning the point estimators of Table 1 , the expected value and variances shown in the table are valid regardless of the form of the population probability density functions[6]. Second , all four estimators will possess probability distributions that are approximately normal for "large samples". The central limit theorem justifies this statement for $\overline{Y}$ and $\hat{p}$ while Theorem 8 which attributes normality to all linear functions of normally distributed random variables , justifies the assertion for $(\overline{Y_1} - \overline{Y_2})$ and $(\hat{p}_1 - \hat{p}_2)$. How large is "large"? For most populations , the probability distributions of $\overline{Y}$ will be mound-shaped for relatively small samples , as low as $n = 5$, and will tend rapidly to normality as ths sample size approaches $n = 30$ or larger. However , you will sometimes need to select larger samples from binomial populations because the required sample size depends on p. The binomial probability distribution is perfectly symmetrical about its mean when $p = ½$ and becomes more and more asymmetric as p tends to 0 or 1. As a rough rule you can assume that the distribution of $\hat{p}$ will be mound-shaped and approaching normality for sample sizes such that $p \pm 2\sqrt{pq/n}$ lies in the interval (0 , 1)

Since we know that $\overline{Y}$, $\hat{p}$, $(\overline{Y_1} - \overline{Y_2})$, and $(\hat{p}_1 - \hat{p}_2)$ are unbiased with near-normal (at least mound-shaped) probability distributions for moderate-sized samples , let us now see how we can use this information to answer a practical question. If we uyse an estimator once acquire a single estimate , how good will this estimate be? How mush faith can we place in the validity of our inference?

III A Large-Sample Confidence Interval

The method for finding confidence intervals[1] requires that you find a quantity , called a pivotal quantity[14] that possesses two characteristics : (1) is a function of θ and the sample measurements , and (2) it possesses a probability distribution that is independent of θ[1]. This , procedure called the pivotal method is illustrated by the following example

Example 3 : Let $\hat{\theta}$ be a statistic that is normally distributed with expected value and variance equal to θ and σ_θ respectively. Find a confidence interval for θ that possesses a confidence coefficient equal to $(1 - \alpha)$.

Solution : The quantity $Z = \dfrac{\hat{\theta} - \theta}{\sigma_\theta}$

Has a standard normal distribution. Now select two tail-end values of this distribution $z_{\alpha/2}$ and $-z_{\alpha/2}$ such that

$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$$

See Figure I

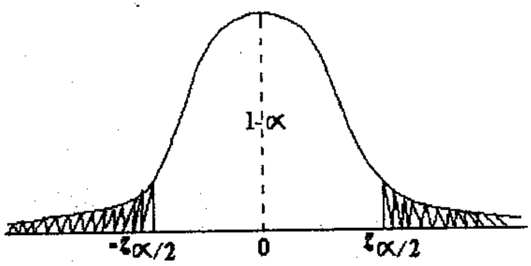

Figure I . Location of $-z_{\alpha/2}$ and $z_{\alpha/2}$

From this point on we employ the following logic. If Y is a randm variable , c is a constant $(c > 0)$, and $P(a < Z < b) = 0{,}7$ Then certainly $P(ac < aY < bc) = 0{,}7$ Similarly , $P(a + c < Y + c < b + c) = 0{,}7$

That is , the probability that $a < Y < b$ is uneffected by a change of a scale or translation of Y. Now let us apply this information in our example. Subtituting for Z in the probability statement

$$P(-z_{\alpha/2} < \frac{\hat{\theta} - \theta}{\sigma_\theta} < z_{\alpha/2}) = 1 - \alpha$$

Multiplying by σ_θ

$$P(-z_{\alpha/2} < \hat{\theta} - \theta < z_{\alpha/2}) = 1 - \alpha$$

and subtracting $\hat{\theta}$ from each term of the inequality

$$P(-\hat{\theta} - z_{\alpha/2} . \sigma_\theta < -\theta < -\hat{\theta} + z_{\alpha/2} . \sigma_\theta) = 1 - \alpha$$

Finally,

$$P(\hat{\theta} - z_{\alpha/2} . \sigma_\theta < \theta < \hat{\theta} + z_{\alpha/2} . \sigma_\theta) = 1 - \alpha$$

Thus, the lower and upper confidence limits for θ are lower confidence limit

$(\text{LCL}) = \hat{\theta} - z_{\alpha/2} . \sigma_\theta$

upper confidence limit

$(\text{UCL}) = \hat{\theta} + z_{\alpha/2} . \sigma_\theta$

Example 3 can be used to find large-sample confidence intervals for μ, P, $(\mu_1 - \mu_2)$, and $(p_1 - p_2)$, the parameters estimated under the conditions described in section II and hence will satisfy the assumptions of example 3

IV Small-Small Confidence Intervals For μ and $\mu_1 - \mu_2$

The following confidence interval is based on the assumption that the experimenter's sample has been randomly selected from a normal population[12]. It is appropriate for samples of any size and works satisfactorily even when the population is nonnormal as long as the departure from normality is not excessive. That is, we rarely the form of the population frequency distribution before we sample. So if a confidence interval is to be of any value, it must "work" reasonably well even when the population is nonnormal. Working "well" means that the confidence coefficient should not be affected by modest departures from normality. Experimental studies indicate that this particular confidence interval will maintain a confidence coefficient close to the experimenter's specified value for most mound-shaped probability distributions

We assume that $Y_1, Y_2, \ldots, Y_n$ represents a random sample selected from a normal population and let $\overline{Y}$ and S^2 represent the sample mean and variance respectively. We would like construct a confidence interval for the population mean when $V(Y_i) = \sigma^2$ is unknown. A pivotal statistic for this situation can be formed using student's t statistic.

Recall from theorem on that $\overline{Y}$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$ are independent and $\frac{(n-1)S^2}{\sigma^2}$ has a X^2 distribution with $(n-1)$ degrees of freedom. Then

$$T = \frac{Z}{\sqrt{\frac{X^2}{n}}} \quad \text{where} \quad Z = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$$

and $X^2 = \frac{(n-1)S^2}{\sigma^2}$

with $V = (n - 1)$ degrees of freedom. Substituting for Z, X^2 and V this quanfity reduces to

$$T = \frac{\sqrt{n}(\overline{Y} - \mu)/\sigma}{\sqrt{\frac{(n-1)(S^2/\sigma^2)}{n-1}}} = \frac{\overline{Y} - \mu}{S/\sqrt{n}}$$

which has a t distribution with $(n - 1)$ degress of freedom. From tables we can find $t_{\alpha/2}$ and $-t_{\alpha/2}$ so that

$$P(-t_{\alpha/2} \le Z \le t_{\alpha/2}) = 1 - \alpha$$

The t distribution has a density function very much like the Standard normal except that tails are thicker, as illustrated in Figure 2

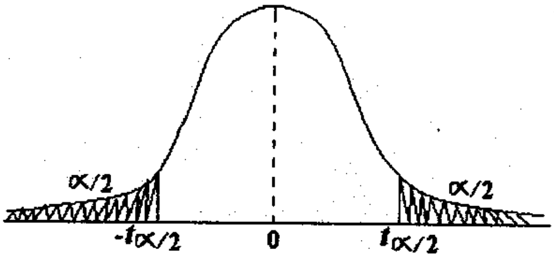

Figure 2. Location of $t_{\alpha/2}$ and $-t_{\alpha/2}$

The confidence interval for μ is developed just as in Example 3. We thus have a confidence interval for μ of the form

$$\overline{y} \pm t_{\alpha/2}\frac{S}{\sqrt{n}}$$

which means that $\overline{y}-t_{\alpha/2}\frac{S}{\sqrt{n}}$ is the lower confidence limit and $\overline{y}+t_{\alpha/2}\frac{S}{\sqrt{n}}$ is the upper confidence limit.

Note that the values of $t_{\alpha/2}$ depend on the degrees of freedom $(n-1)$ as well as the confidence coefficient $(1-\alpha)$

Suppose that we are interested in comparing means from two normal populations, one with mean μ_1 and variance σ_1^2 and the other with mean μ_2 and variance σ_2^2. A confidence interval for $\mu_1-\mu_2$ based on a T random variable can be constructed if we assume that

$\sigma_1^2 = \sigma_2^2 = \sigma$

The large-sample confidence interval for $(\mu_1-\mu_2)$ is developed from the random variable

$$Z=\frac{(\overline{Y}_1-\overline{Y}_2)-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$$

which has approximately a Standard normal distribution, $\overline{Y}_1$ and $\overline{Y}_2$ are the respective sample means obtained from random sampling. Under the assumption $\sigma_1^2 = \sigma_2^2 = \sigma$, the above ratio becomes

$$Z=\frac{(\overline{Y}_1-\overline{Y}_2)-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

Now we need an estimator of the common variance, σ^2, in order to construct a t statistic.

Let $Y_{11}, Y_{12}, \ldots, Y_{2n_1}$ denote the random sample o size n_1 from the first population and $Y_{21}, Y_{22}, \ldots, Y_{2n_2}$ the random sample from the second.

Then

$$\overline{Y}_1=\frac{1}{n_1}\sum_{i=1}^{n_1}Y_{1i} \quad \text{and} \quad \overline{Y}_2=\frac{1}{n_2}\sum_{i=1}^{n_2}Y_{2i}$$

The usual unbiased estimator of the common variance, σ^2 is obtained by pooling the sample data

$$S^2=\frac{\sum_{i=1}^{n_1}(Y_{1i}-\overline{Y}_1)+\sum_{i=1}^{n_2}(Y_{2i}-\overline{Y}_2)^2}{n_1+n_2-2}$$

where S_i^2 is the sample variance from the ith sample, $i = 1, 2$. Note that

$$\frac{(n_1+n_2-2)S^2}{\sigma^2}=\frac{\sum_{i=1}^{n_1}(Y_{1i}-\overline{Y}_1)^2}{\sigma^2}+\frac{\sum_{i=1}^{n_2}(Y_{2i}-\overline{Y}_2)^2}{\sigma^2}$$

is the sum of two independent X^2 random variables with (n_1-1) and (n_2-1) degrees of freedom, respectively. Thus (n_1+n_2-2) degrees of freedom (See Theorems 9 and 10). We now utilize this X^2 variable and the Z defined in the previous paragraph to form a pivotal T statistic. That is

$$T=\frac{Z}{\sqrt{\frac{X^2}{n}}}$$

$$=\frac{(\overline{Y}_1-\overline{Y}_2)-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\ \frac{1}{\sqrt{\frac{(n_1+n_2-2)S^2}{\sigma^2(n_1+n_2-2)}}}$$

$$=\frac{(\overline{Y}_1-\overline{Y}_2)-(\mu_1-\mu_2)}{S\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

has a t distribution with (n_1+n_2-2) degrees of freedom. The confidence interval for $(\mu_1-\mu_2)$ then has the form

$$(\overline{y}_1-\overline{y}_2)\pm t_{\alpha/2}\cdot S\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

where $t_{\alpha/2}$ comes from the t distribution with (n_1+n_2-2) degrees of freedom.

As the sample size, n, gets large, the T random variable converges in distribution to the Standard normal (See example 4). Thus the small-sample confidence intervals of this section for large n (or large n_1 and n_2). The intervals are nearly equivalent when $(n_1+n_2-2) \geq 30$

Example 4 : Suppose that $X_1, \ldots, X_n$ are independent and identically distributed random variables

$E(X_i) = \mu$ and $V(X_i) = \sigma^2$. Define S'^2 as

$$S'^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Show that

$$\sqrt{n}\,\frac{\overline{X} - \mu}{S'}$$

converges in distribution to a Standard normal random variable

Solution : In Example 1 , we showed that S'^2 converges in probability to σ^2. Hence it follows from Theorem 12 that

$$\sqrt{n}\,\frac{\overline{X} - \mu}{\sigma}$$

converges in distribution to a Standard normal random variable. Therefore

$$\sqrt{n}\,\frac{\overline{X} - \mu}{S'} = \sqrt{n}\,\frac{\overline{X} - \mu}{\sigma} \Big/ \frac{S'}{\sigma}$$

converges in distribution to a Standard normal random variable by theorem 14

V Find The Expected Value Of The Interval Width In The Small-Sample Case

This is the aim of this paper , for this purpose the reader should review all previous sections. Note that the small-sample confidence interval for μ based on Student's t (Section IV) , possesses a random width [in contrast to the larger-sample confidence interval (Section 3) , where the width is nonrandom]. Now we can begin to find the expected value of the interval width in the small sample case.

The width of the small sample confidence interval is

$$2t_{\alpha/2}\frac{S}{\sqrt{n}}$$

and has expected value

$$2t_{\alpha/2}\frac{E(S)}{\sqrt{n}}$$

It is necessary then to find $E(S)$, for this purpose

Let $X = \dfrac{(n-1)S^2}{\sigma^2}$ Then

$$f(x) = \frac{x^{\frac{n-1}{2}-1}\cdot e^{-x/2}}{\Gamma\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}} \qquad \text{for } n > 0$$

The density function for

$$Y = S^2 = \frac{\sigma^2 x}{n-1}$$

is obtained by the transformation method

$$g(y) = f\left[\frac{(n-1)y}{\sigma^2}\right]\left[\frac{dy}{dx}\right]$$

$$= \frac{\left(\sigma^{\frac{n-1}{2}}\right)^{\frac{n-1}{2}} y^{\frac{n-1}{2}-1} e^{\frac{(n-1)y}{2\sigma^2}}}{\Gamma\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}}$$

Now $E(S) = E(\sqrt{Y}) = \displaystyle\int_0^{\infty} y^{1/2} g(y)\,dy$

$$= \int_0^{\infty}\frac{y^{1/2}\left(\frac{n-1}{\sigma^2}\right)y^{\frac{n}{2}\frac{1}{2}-1} e^{\frac{-(n-1)y}{2\sigma^2}}}{\Gamma\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}}$$

$$= \frac{\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)(2\sigma^2)^{n/2}}{\Gamma\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}(n-1)^{n/2}}\int_0^{\infty}\frac{y^{\frac{n}{2}-1} e^{\frac{-(n-1)n}{2\sigma^2}dy}}{\Gamma\left(\frac{n}{2}\right)\left(\frac{2\sigma^2}{n-1}\right)^{n/2}}$$

$$= \frac{(n-1)^{-1/2}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)2^{-\frac{1}{2}}(\sigma^2)^{-1/2}} = \frac{\Gamma\left(\frac{n}{2}\right)2^{1/2}\sigma}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{n-1}}$$

Hence

$$E\left[2t_{\alpha/2}\frac{S}{\sqrt{n}}\right] = \frac{2^{3/2}t_{\alpha/2}\sigma\Gamma\left(\frac{n}{2}\right)}{\sqrt{n(n-1)}\,\Gamma\left(\frac{n-1}{2}\right)}$$

Conclusions

the objective of many statistical investigations is to make inferences about population parameters[3] based on a simple data. Often these inferences are in the form of estimates , either point estimates or interval estimates[13]

In this paper we have discussed the fact that we like to have unbiased estimator with small variance. The goodness of an estimator $\hat{\theta}$ can be measured by σ_{θ} because the error of estimation will generally be smaller than $2\sigma_{\theta}$ with high probability

Interval estimates of many parameters , such as μ and p can be derived from the normal central limit theorem**[8]**. But if sample sizes are small , the normality of the population must be assumed and t distribution**[5]** is used in deriving confidence intervals

If sample measurements have been selected from a normal distribution , a confidence interval for σ^2 can be developed through the use of the X^2 distribution

An the expected value of the interval width in the small sample case is calculated successfully for the first time as far as the author of this paper knows

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