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Engin Özkan[1,a] and Hakan Akkuş[2,b]

AFFILIATIONS

[1] Department of Mathematics, Faculty of Sciences, Marmara University, İstanbul, Türkiye
[2] Department of Mathematics, Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, Yalnızbağ Campus, 24100 Erzincan, Türkiye

[a] **Author to whom correspondence should be addressed:** engin.ozkan@marmara.edu.tr
[b] hakan.akkus@ogr.ebyu.edu.tr

ABSTRACT

In this study, we define a new generalization of the Fibonacci sequence that gives the copper ratio, and we will call it the copper Fibonacci sequence. In addition, inspired by the copper Fibonacci definition, we also define copper Lucas sequences, and then we give the relationships between the terms of these sequences. We present some properties, such as the Binet formulas, special summation formulas, special generating functions, etc. We find the relationships between the roots of the characteristic equation of these sequences and the general terms of these sequences. What is interesting here is that the relationships obtained from that between the roots of the characteristic equation of these new sequences and the terms of the sequences are satisfied in both roots. In addition, we examine the relationships between these sequences with the classic Fibonacci and Lucas sequences. Moreover, we calculate some identities of these sequences, such as Cassini and Catalan. Then Catalan transformation is applied to these sequences, and their terms are found. Furthermore, we apply Hankel transform to the Catalan transform of these sequences. Besides, we associate the terms of the Hankel transformation of the Catalan copper Fibonacci sequence with the classical Fibonacci numbers and the terms of the Hankel transformation of the Catalan copper Lucas sequence with the terms of the copper Lucas sequence. We present the application of copper Fibonacci and copper Lucas sequences to hyperbolic quaternions. Finally, the terms of the copper Fibonacci and copper Lucas sequences are associated with their hyperbolic quaternion values.

I. INTRODUCTION

Metallic ratios appear in many branches of science, especially chemistry and biology. Some of the most interesting metallic ratios are given in the Table I (see Refs. 1–3 for details).

Number sequences play a vital role in understanding the complexity of any problem consisting of some patterns. The rabbit problem in Leonardo Fibonacci's classic book Liber Abaci can be given as an example. Inspired by the rabbit problem, the Fibonacci sequence was developed, and the relationship between the terms of this sequence is called the golden ratio. In the literature, there are many metallic ratios, such as the golden ratio, copper ratio, and silver ratio. The copper ratio, which we frequently encounter especially in chemistry and biology, is included in many studies and attracts the attention of scientists. In this article, a new sequence called copper Fibonacci is defined with the help of the copper ratio. The ratio of the larger to the smaller of two consecutive terms of this sequence is the copper ratio. In addition, another feature of this sequence is that when special transformations are applied to the terms of the sequence, the results are obtained as single-index terms of the classical Fibonacci sequence.

In mathematics, sequences that consist of some patterns play a vital role in understanding the complexities of any given problem. For example, Fibonacci and Lucas sequences are evaluated in this context.

The Fibonacci and Lucas sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci sequences have been applied in various fields, such as cryptology,[4] phylotaxis,[5] biomathematics,[6] chemistry,[7] engineering,[8] etc. Many generalizations of the Fibonacci sequence have been given. The known examples of such sequences are the Fermat, Fermat–Lucas, Oresme, Pell, and Jacobsthal sequences (see Refs. 9–12 for details).

For $n \in \mathbb{N}$, the Fibonacci numbers F_n and Lucas numbers L_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n \text{ and } L_{n+2} = L_{n+1} + L_n,$$

with the initial conditions $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$.

For F_n and L_n, the Binet formulas are given by the following relations, respectively,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $t^2 - t - 1 = 0$. Here, the α number is the known golden ratio.

Some of the identities between Fibonacci and Lucas sequences are as follows (for more identities, see pages 69–99 in Ref. 9):

$$\text{i. } 5F_n^2 = L_{2n} - 2(-1)^n; \text{ ii. } L_n^2 = L_{2n} + 2(-1)^n;$$
$$\text{iii. } L_n = F_{n+1} + F_{n-1} \text{ iv. } F_{3n} = F_{2n}L_n - F_n(-1)^n.$$

The Cassini formula for the Fibonacci sequence is

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

The interesting feature of the Cassini identity is that it is used in geometric paradoxes (see page 100–108 in Ref. 9 for details).

For $n \in \mathbb{N}$, the bronze Fibonacci numbers BF_n, bronze Lucas numbers BL_n, third-order bronze Fibonacci numbers Bf_n, third-order bronze Lucas numbers Bl_n, and modified third-order bronze Fibonacci numbers MBf_n are defined by the recurrence relations, respectively,

$$BF_{n+2} = 3BF_{n+1} + BF_n, BL_{n+2} = 3BL_{n+1} + BL_n,$$
$$Bf_{n+3} = 3Bf_{n+2} + Bf_{n+1} + Bf_n,$$

$$Bl_{n+3} = 3Bl_{n+2} + Bl_{n+1} + Bl_n,$$

and

$$MBf_{n+3} = 3MBf_{n+2} + MBf_{n+1} + MBf_n,$$

with the initial conditions $BF_0 = 0$, $BF_1 = 1$, $BL_0 = 2$, $BL_0 = 3$, $Bf_0 = 0$, $Bf_1 = 1$, $Bf_2 = 3$, $Bl_0 = 3$, $Bl_1 = 3$, and $Bl_2 = 11$ and $MBf_0 = 1$, $MBf_1 = 2$, and $MBf_2 = 7$.

For BF_n and BL_n, the Binet formulas are given, respectively, by the following relationships:

$$BF_n = \frac{\lambda^n - \psi^n}{\lambda - \psi} \text{ and } BL_n = \lambda^n + \psi^n,$$

where $\lambda = \frac{3+\sqrt{13}}{2}$ and $\psi = \frac{3+\sqrt{13}}{2}$ are the roots of the characteristic equation $r^2 - 3r - 1 = 0$. Here, the λ number is the known bronze ratio.

In Ref. 13, Akbiyik and Jeta defined the third-order bronze Fibonacci Bf_n, bronze Lucas Bl_n, and modified bronze Fibonacci MBf_n numbers. In addition, they found the following important features of these sequences:

$$\text{i. } Bf_{n+3} - Bf_{n+1} = MBf_{n+3} + Bf_{n+2}; \text{ ii. } MBf_{n+1} = Bf_{n+1} - Bf_n;$$
$$\text{iii. } Bl_{n+2} = 2MBf_{n+2} - MBf_{n+1} - MBf_n;$$
$$\text{iv. } MBf_{n+3} = 2Bf_{n+2} + Bf_{n+1} + Bf_n.$$

With the help of the recurrence relation of the Fibonacci sequence, k-sequences were introduced, and these sequences had an important place in number theory.[14] In Ref. 15, Catarino and Ricardo defined the k-bronze Fibonacci numbers. In addition, in this study, their examination of the relationship of this sequence with matrices is used. Moreover, they presented on quaternion Gaussian bronze Fibonacci numbers.[16] In Ref. 17, Jeta defined third-order bronze Fibonacci quaternions and found many properties related to these quaternions. Karaarslan worked on Gaussian bronze Lucas numbers.[18]

Quaternions were first described by Hamilton in 1843. In addition, quaternions are used to control rotational movements, especially in kinematics,[19] 3D games,[20] mechanics,[21] Eulerian angles,[22] and chemistry.[23] In Ref. 24, Horadam defined complex Fibonacci and Fibonacci quaternions, and various features were found.

The algebra of hyperbolic quaternions is an algebra that is not related to the elements of the form over the real numbers,

$$q = xi_1 + yi_2 + zi_3 + ti_4, x, y, z, t \in \mathrm{R}.$$

The properties of the q components are defined in Table I.

In Ref. 25, Macfarlane performed a lot of research on hyperbolic quaternions and their properties. An expression of the general form of hyperbolic quaternions is

$$\hbar = \hbar_1 i_1 + \hbar_2 i_2 + \hbar_3 i_3 + \hbar_4 i_4 = (\hbar_1, \hbar_2, \hbar_3, \hbar_4).$$

Here, $\hbar_1, \hbar_2, \hbar_3,$ and $\hbar_4$ are the terms of the sequence, and $i_1, i_2, i_3,$ and i_4 are hyperbolic quaternions.

As seen above, many generalizations of Fibonacci and Lucas sequences have been given so far.

We separate the article into four parts.

In Sec. II, we give a new generalization inspired by the bronze Fibonacci sequence. We call the sequences the copper Fibonacci sequences and denote them as CF_n, and we define the copper Lucas sequence as CL_n. In addition, we give the generating functions,

TABLE I. Metallic ratios.

..... Ratio	Value	OEİS
Plastic	1.324 717 957 244 7...	A182097
Gold	1.618 033 988 749 895...	A001622
Silver	2.414 213 562 373 095...	A014176
Bronze	3.302 775 637 731 995...	A098316
Copper	4.236 067 977 499 79...	A098317
Nickel	5.192 582 403 567 252...	A098318
Aluminum	6.162 277 660 168 38...	A176398
Iron	7.140 054 944 640 259...	A176439
Tin	8.123 105 625 617 661...	A176458
Lead	9.109 772 228 646 444...	A176522

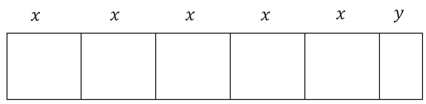

FIG. 1. Copper rectangle.

summation formulas, the Binet formulas, and some properties for these sequences. Then we examine the relationship between copper Fibonacci and copper Lucas sequences. In addition, we obtain the relationships between these sequences with the classic Fibonacci and Lucas sequences. Moreover, we calculate some identities of these sequences.

In Sec. III, we define the Catalan transformation of the copper Fibonacci and copper Lucas sequences, and some properties are given. In addition, we obtain the Catalan generating functions of these sequences. Then Hankel transformations are applied to the Catalan transformations of these sequences. Finally, we associate the terms of the Hankel transformation of the Catalan copper Fibonacci sequence with the classical Fibonacci numbers and the terms of the Hankel transformation of the Catalan copper Lucas sequence with the terms of the copper Lucas sequence.

In Sec. IV, we present on the application of copper Fibonacci and copper Lucas sequences to hyperbolic quaternions. Then we define hyperbolic copper Fibonacci and copper Lucas quaternions. For these quaternions, we give many properties, such as Binet formulas. Finally, the terms of the copper Fibonacci and copper Lucas sequences are associated with their hyperbolic quaternion values.

II. COPPER FIBONACCI AND COPPER LUCAS SEQUENCES

If one removes four largest possible squares from a rectangle with the ratio length/width equal to $2 + \sqrt{5}$, one gets a rectangle with the same ratio length/width. Now let us examine Fig. 1.

According to the rectangle, we obtain

$$\frac{4x + y}{x} = \frac{x}{y} \text{ or } 4 + \frac{y}{x} = \frac{x}{y}.$$

Thus, we find

$$\frac{x}{y} = 2 + \sqrt{5}.$$

Another feature of this rectangle that makes it interesting is this. At the same time, when we remove these squares from Fig. 1 and continue the process indefinitely, we find the same ratio, that is, the copper ratio. We can say that the series we define shortly will be used in many areas of mathematics. Applications in groups, coding theory, and applications in quaternions can be given as examples.

Definition 2.1. For $n \in \mathbb{N}$, the copper Fibonacci sequence CF_n and copper Lucas sequence CL_n are defined, respectively, as

$$CF_{n+2} = 4CF_{n+1} + CF_n, \tag{2.1}$$

with $CF_0 = 0$ and $CF_1 = 1$, and

$$CL_{n+2} = 4CL_{n+1} + CL_n, \tag{2.2}$$

with $CL_0 = 2$ and $CL_1 = 4$.

Then let us give some information about the equations of these sequences.

The characteristic equation of the copper Fibonacci and copper Lucas sequences is

$$r^2 - 4r - 1 = 0. \tag{2.3}$$

The roots of the characteristic equation are as follows:

$$\kappa = 2 + \sqrt{5} \cong 4.236\,067\,977\,499\,79\ldots, \text{ and } \theta = 2 - \sqrt{5}.$$

Here, the κ number is the known copper ratio.

Next, we give the relationships between these roots below:

$$\kappa + \theta = 4, \ \kappa - \theta = 2\sqrt{5}, \ \kappa^2 + \theta^2 = 18, \text{ and } \kappa\theta = -1.$$

The first few values of copper Fibonacci and copper Lucas sequences are

$$0, 1, 4, 17, 72, 305, 1292, 5473, 23\,184, 98\,209, \ldots$$

and

$$2, 4, 18, 76, 322, 1364, 5778, 24\,476, 103\,682, \ldots,$$

respectively.

In addition, the terms of the copper Fibonacci and copper Lucas sequences can be found with the help of the following relations. Let $n \in \mathrm{N}^+$,

$$CF_n = \sum_{q=0}^{\frac{n-1}{2}} \binom{n-1-q}{q} 4^{n-1-2q} \text{ and } CL_n = \sum_{q=0}^{\frac{n}{2}} \frac{n}{n-q} \binom{n-q}{q} 4^{n-2q}.$$

In the following theorem, we express the Binet formulas of the copper Fibonacci sequence CF_n and copper Lucas sequence CL_n.

Theorem 2.1. Let $n \in \mathbb{N}$. We obtain

i. $CF_n = \frac{\kappa^n - \theta^n}{\kappa - \theta}$ and ii.$CL_n = \kappa^n + \theta^n$.

Proof. i. The Binet form of a sequence is as follows:

$$CF_n = S\kappa^n + T\theta^n.$$

The scalars S and T can be obtained by substituting the initial conditions and solving the given system of equations. For $n = 0$, $CF_0 = 0$, and for $n = 1$, $CF_1 = 1$. Hence, $S = \frac{1}{\kappa - \theta}$ and $T = \frac{-1}{\kappa - \theta}$. Thus, we obtain

$$CF_n = \frac{\kappa^n - \theta^n}{\kappa - \theta}.$$

The proof of the other may be found similarly. □

Theorem 2.2. The ratio of the larger to the smaller of two consecutive terms of the copper Fibonacci and copper Lucas sequences converges to the copper ratio. We obtain

$$\lim_{n\to\infty}\frac{CF_{n+1}}{CF_n}=\kappa \text{ and } \lim_{n\to\infty}\frac{CL_{n+1}}{CL_n}=\kappa.$$

Proof. If the Binet formula is used, we have

$$\lim_{n\to\infty}\frac{CF_{n+1}}{CF_n}=\frac{\frac{\kappa^{n+1}-\theta^{n+1}}{\kappa-\theta}}{\frac{\kappa^n-\theta^n}{\kappa-\theta}}=\lim_{n\to\infty}\frac{\kappa^{n+1}\left(1-\left(\frac{\theta}{\kappa}\right)^{n+1}\right)}{\kappa^n\left(1-\left(\frac{\theta}{\kappa}\right)^n\right)}.$$

Thus, we obtain

$$\lim_{n\to\infty}\frac{CF_{n+1}}{CF_n}=\kappa.$$

Similarly, we have

$$\lim_{n\to\infty}\frac{CL_{n+1}}{CL_n}=\kappa.$$

Next, we examine the relationships between the roots of the characteristic equation of these sequences. Let us pay attention to the fact that the relationships obtained from the relationship between the roots of the characteristic equation of the new sequences and the terms of the sequences are satisfied in both roots.

Theorem 2.3. Let $n\in\mathbb{N}$. We obtain

i. $\kappa^{2n}=\frac{CF_{2n}}{2}\kappa\sqrt{5}-\frac{CL_{2n-1}}{4}$,
ii. $\theta^{2n}=\frac{CF_{2n}}{2}(2\sqrt{5}-\kappa)\sqrt{5}-\frac{CL_{2n-1}}{4}$,
iii. $\kappa^{2n+1}=-\frac{CF_{2n}}{2}\sqrt{5}+\kappa\frac{CL_{2n+1}}{4}$,
iv. $\theta^{2n+1}=-\frac{CF_{2n}}{2}\sqrt{5}-\theta\frac{CL_{2n+1}}{4}$,
v. $2\sqrt{5}\ CF_n+CL_n=2\kappa^n$, and
vi. $2\sqrt{5}\ CF_n-CL_n=-2\theta^n$.

Proof. **i.** If the Binet formulas are used, we obtain

$$\begin{aligned}\frac{CF_{2n}}{2}\kappa\sqrt{5}-\frac{CL_{2n-1}}{4}&=2\kappa\sqrt{5}\frac{\kappa^{2n}-\theta^{2n}}{(\kappa-\theta)4}-\frac{\kappa^{2n-1}+\theta^{2n-1}}{4}\\&=\frac{\kappa^{2n+1}-\kappa\theta^{2n}-\kappa^{2n-1}-\theta^{2n-1}}{4}\\&=\frac{\kappa^{2n}\left(\kappa-\frac{1}{\kappa}\right)+\theta^{2n}\left(-\kappa-\frac{1}{\theta}\right)}{4}\\&=\kappa^{2n}.\end{aligned}$$

The proofs of the others may be found similarly. □

Theorem 2.4. Let $s=\kappa$ or $s=\theta$ and $x,y,z,t\in\mathbb{N}$. We obtain

i. $s^x=sCF_x+CF_{x-1}$,
ii. $CF_{x(y-z)}=s^{xz}CF_{xy}+s^{xy}CF_{xz}$
iii. $s^{2x}=s^xCL_x-(-1)^x$,
iv. $s^x=s^yCF_{x-y+1}+s^{y-1}CF_{x-y}$,
v. $s^{xt}=\frac{s^xCF_{xt}}{CF_x}-(-1)^x\frac{CF_{x(t-1)}}{CF_x}$, and
vi. $1+4s+s^{2(2^{x+1}+1)}=s^{2(2^x+1)}CL_{2^{x+1}}$.

Proof. i. For $s=\kappa$, we have

$$\begin{aligned}sCF_x+CF_{x-1}&=\kappa\left(\frac{\kappa^x-\theta^x}{\kappa-\theta}\right)+\left(\frac{\kappa^{x-1}-\theta^{x-1}}{\kappa-\theta}\right)\\&=\frac{\kappa^{x-1}(\kappa^2+1)-\theta^{x-1}(\kappa\theta+1)}{\kappa-\theta}=\kappa^x.\end{aligned}$$

Similarly, when the other root is written, the same equality is obtained.

The proof of the others may be found similarly. □

In the following theorems, we give special relations between the copper Fibonacci CF_n and copper Lucas CL_n sequences.

Theorem 2.5. Let $n\in\mathbb{N}$. The following equation is true:

$$CF_n=\frac{1}{10}CL_{n+1}-\frac{1}{5}CL_n.$$

Proof. The following relation is used for proofs:

$$CF_n=a\,x\,CL_{n+1}+b\,x\,CL_n.$$

For these n values, we obtain

$$CF_0=a\,x\,CL_1+b\,x\,CL_0,$$

$$CF_1=a\,x\,CL_2+b\,x\,CL_1.$$

We find

$$a=\frac{1}{10}\text{ and }b=-\frac{1}{5}.$$

The proof of the other may be found similarly. □

Theorem 2.6. Let $m,n\in\in\mathbb{N}$ and $n>m$. We obtain

i. $CF_{n+m+1}=CF_{n+1}CF_{m+1}+CF_nCF_m$,
ii. $CL_{n+m+1}=CL_{m+1}CF_{n+1}+CL_mCF_n$,
iii. $CF_n^2=\frac{1}{20}CL_{2n}+\frac{1}{10}\frac{CF_n}{CF_{-n}}$,
iv. $CL_n^2=CL_{2n}+2\frac{CL_n}{CL_{-n}}$,
v. $CL_n^2+CL_{n+1}^2=CL_{2n}+CL_{2n+2}$, and
vi. $CF_n^2+CF_{n+1}^2=CF_{2n+1}$.

Proof. If the Binet formula is used, we obtain

$$\begin{aligned}\text{i.}CF_{n+1}CF_{m+1}+CF_nCF_m&=\frac{\kappa^{n+1}-\theta^{n+1}}{\kappa-\theta}\frac{\kappa^{m+1}-\theta^{m+1}}{\kappa-\theta}\\&\quad+\frac{\kappa^n-\theta^n}{\kappa-\theta}\frac{\kappa^m-\theta^m}{\kappa-\theta}\\&=\frac{\kappa^{n+m+1}(\kappa-\theta)-\theta^{n+m+1}(\kappa-\theta)}{(\kappa-\theta)^2}\\&=\frac{\kappa^{n+m+1}-\theta^{n+m+1}}{\kappa-\theta}\\&=CF_{n+m+1},\end{aligned}$$

v. $CF_{-n} = \dfrac{\kappa^{-n} - \theta^{-n}}{\kappa - \theta} = -\dfrac{\kappa^n - \theta^n}{(\kappa - \theta)\kappa^n\theta^n} = -\dfrac{CF_n}{\kappa^n\theta^n}$ so

$$\kappa^n\theta^n = -\frac{CF_{-n}}{CF_n}.$$

Then, we find

$$CF_n^2 = \frac{\kappa^n - \theta^n}{\kappa - \theta}\frac{\kappa^n - \theta^n}{\kappa - \theta} = \frac{\kappa^{2n} + \theta^{2n} - 2\kappa^n\theta^n}{(\kappa - \theta)(\kappa - \theta)} = \frac{1}{20}CL_{2n} - \frac{1}{10}\kappa^n\theta^n.$$

Thus, we obtain

$$CF_n^2 = \frac{1}{20}CL_{2n} + \frac{1}{10}\frac{CF_n}{CF_{-n}}.$$

The proof of the others may be found similarly. □

Theorem 2.7. Let $k, m, n \in \mathbb{N}$ and $k > m, n$. We obtain

i. $4CF_{k+m+n} = CL_kCL_mCF_n + CF_kCL_mCL_n + CL_kCF_mCL_n + 20CF_kCF_mCF_n$ and
ii. $4CL_{k+m+n} = CL_kCL_mCL_n + 20CL_kCF_mCF_n + 20CF_kCL_mCF_n + 20CF_kCF_mCL_n$.

The proofs of Theorem 2.7 are shown using the Binet formulas in a similar way to Theorem 2.6.

In the following theorem, we examine the relationships among copper Fibonacci sequence CF_n, copper Lucas sequence CL_n, Fibonacci sequence F_n, and Lucas sequence L_n.

Theorem 2.8. Let $n \in \mathbb{N}$. We obtain

i. $CF_n = CF_{-n}\left(\frac{F_{3n} - F_{2n}(F_{n+1} + F_{n-1})}{F_n}\right)$,
ii. $CL_n = CL_{-n}\left(\frac{L_n^2 - L_{2n}}{2}\right)$, and
iii. $CL_n^2 - 20CF_n^2 = 4(F_{n+1}F_{n-1} - F_n^2)$.

Proof. If the Binet formula is used, we obtain

i. $CF_{-n} = \dfrac{\kappa^{-n} - \theta^{-n}}{\kappa - \theta} = -\dfrac{\kappa^n - \theta^n}{(\kappa - \theta)\kappa^n\theta^n} = -\dfrac{CF_n}{(-1)^n}$ Thus, we get

$$\begin{aligned} CF_n &= -CF_{-n}(-1)^n \\ &= CF_{-n}\frac{F_{3n} - F_{2n}L_n}{F_n} \\ &= CF_{-n}\left(\frac{F_{3n} - F_{2n}(F_{n+1} + F_{n-1})}{F_n}\right), \end{aligned}$$

ii. $CL_{-n} = \kappa^{-n} + \theta^{-n} = \dfrac{\kappa^n + \theta^n}{\kappa^n\theta^n} = \dfrac{CL_n}{(-1)^n}$ Thus, we have

$$\begin{aligned} CL_n &= CL_{-n}(-1)^n \\ &= CL_{-n}\left(\frac{L_n^2 - L_{2n}}{2}\right), \end{aligned}$$

iii. $$\begin{aligned} CL_n^2 - 20CF_n^2 &= (\kappa^n + \theta^n)^2 - 20\left(\frac{\kappa^n - \theta^n}{\kappa - \theta}\right)^2 \\ &= \kappa^{2n} + \theta^{2n} + 2\kappa^n\theta^n - \kappa^{2n} - \theta^{2n} + 2\kappa^n\theta^n \\ &= 4\kappa^n\theta^n \\ &= 4(-1)^n \\ &= 4(F_{n+1}F_{n-1} - F_n^2). \end{aligned}$$ □

In the following theorems, we obtain special sum formulas of the copper Fibonacci sequence CF_n and copper Lucas sequence CL_n. In addition, we find binomial sum formulas of these sequences.

Theorem 2.9. Let $m, n, p, r \in \mathbb{N}$. We obtain

i. $$\sum_{m=0}^{n} CF_{rm+p} = \begin{cases} \dfrac{-CF_{nr+r+p} + (-1)^r CF_{nr+p} + CF_{r-p} + CF_p}{1 + (-1)^r - CL_r}, & \text{if } p < r, \\ \dfrac{-CF_{nr+r+p} + (-1)^r CF_{nr+p} + CF_{p-r} + CF_p}{1 + (-1)^r - CL_r}, & \text{otherwise,} \end{cases}$$

and

ii. $$\sum_{m=0}^{n} CL_{rm+p} = \begin{cases} \dfrac{-CL_{nr+r+p} + (-1)^r CL_{nr+p} - CL_{r-p} + CL_p}{1 + (-1)^r - CL_r}, & \text{if } p < r, \\ \dfrac{(-1)^r CL_{nr+p} - CL_{p-r} + CL_p - CL_{nr+r+p}}{1 + (-1)^r - CL_r}, & \text{otherwise.} \end{cases}$$

Proof. i. If the Binet formulas are used, we get

$$\begin{aligned} &\sum_{m=0}^{n} CF_{rm+p} \\ &= \sum_{y=0}^{n} \frac{\kappa^{rm+p} - \theta^{rm+p}}{\kappa - \theta} = \frac{\kappa^p}{\kappa - \theta}\sum_{y=0}^{n} (\kappa^r)^m - \frac{\theta^p}{\kappa - \theta}\sum_{y=0}^{n} (\theta^r)^m \\ &= \frac{\kappa^p}{\kappa - \theta}\frac{\kappa^{nr+r} - 1}{\kappa^r - 1} - \frac{\theta^p}{\kappa - \theta}\frac{\theta^{nr+r} - 1}{\theta^r - 1} \\ &= \begin{cases} \dfrac{-CF_{nr+r+p} + (-1)^r CF_{nr+p} + CF_{r-p} + CF_p}{1 + (-1)^r - CL_r}, & \text{if } p < r, \\ \dfrac{-CF_{nr+r+p} + (-1)^r CF_{nr+p} + CF_{p-r} + CF_p}{1 + (-1)^r - CL_r}, & \text{otherwise.} \end{cases} \end{aligned}$$

The proof of the other may be found similarly. □

Corollary 2.1. The sum of the consecutive, odd, and even indexed terms of the copper Fibonacci and copper Lucas sequences are as follows:

i. $\sum_{m=0}^{n} CF_m = \frac{5CF_n + CF_{n-1} - 1}{4}$,
ii. $\sum_{m=0}^{n} CL_m = \frac{5CL_n + CL_{n-1} + 2}{4}$,
iii. $\sum_{m=0}^{n} CF_{2m} = \frac{CF_{2n+2} - CF_{2n} - CF_2}{16}$,
iv. $\sum_{m=0}^{n} CL_{2m} = \frac{CL_{2n+2} - CL_{2n} + CL_2 - 2}{16}$,
v. $\sum_{m=0}^{n} CF_{2m+1} = \frac{CF_{2n+3} - CF_{2n+1} - 2}{16}$, and
vi. $\sum_{m=0}^{n} CL_{2m+1} = \frac{CL_{2n+3} - CL_{2n+1}}{16}$.

Proof. The proofs are found according to the special values of r and p in Theorem 2.9.

Theorem 2.10. Let $n \in \mathbb{N}$. We obtain

i. $\sum_{j=0}^{n}\binom{n}{j}4^{j}CF_{j}=CF_{2n}$,
ii. $\sum_{j=0}^{n}\binom{n}{j}4^{j}CL_{j}=CL_{2n}$,
iii. $\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}CF_{2j}=(-4)^{n}CF_{n}$, and
iv. $\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}CL_{2j}=(-4)^{n}CL_{n}$.

Proof. i. The following equations are obtained with the help of characteristic equations of the copper Fibonacci and copper Lucas sequences:

$$\kappa^{2}=4\kappa+1 \text{ and } \theta^{2}=4\theta+1$$

$$\begin{aligned}\sum_{j=0}^{n}\binom{n}{j}4^{j}CF_{j}&=\sum_{j=0}^{n}\binom{n}{j}4^{j}\left(\frac{\kappa^{j}-\theta^{j}}{\kappa-\theta}\right)\\&=\frac{1}{\kappa-\theta}\sum_{j=0}^{n}\binom{n}{j}4^{j}\kappa^{j}-\frac{1}{\kappa-\theta}\sum_{j=0}^{n}\binom{n}{j}4^{j}\theta^{j}\\&=\frac{1}{\kappa-\theta}\left[(1+4\kappa)^{n}-(1+4\theta)^{n}\right]\\&=\frac{1}{\kappa-\theta}\left[\left(\kappa^{2}\right)^{n}-\left(\theta^{2}\right)^{n}\right]\\&=CF_{2n}.\end{aligned}$$

The proofs of the others are shown similarly. □

In the following theorems, we give special generating functions of the copper Fibonacci CF_n and copper Lucas CL_n sequences.

Theorem 2.11. Let $r,n,p\in\mathbb{N}$ and $p>r$; we obtain

i. $\sum_{n=0}^{\infty}CF_{rn+p}x^{n}=\frac{CF_{p}-x(-1)^{r}CF_{p-r}}{1-xCL_{r}+(-1)^{r}x^{2}}$,
ii. $\sum_{n=0}^{\infty}CL_{rn+p}x^{n}=\frac{CL_{p}-x(-1)^{r}CL_{p-r}}{1-xCL_{r}+(-1)^{r}x^{2}}$,
iii. $\sum_{n=0}^{\infty}\frac{CF_{rn}}{n!}x^{n}=\frac{e^{\kappa^{r}x}-e^{\theta^{r}x}}{\kappa-\theta}$, and
iv. $\sum_{n=0}^{\infty}\frac{CL_{rn}}{n!}x^{n}=e^{\kappa^{r}x}+e^{\theta^{r}x}$.

Proof. If the Binet formula is used, we have

$$\text{i.}\quad\begin{aligned}\sum_{i=0}^{\infty}CF_{rn+p}x^{n}&=\sum_{n=0}^{\infty}\frac{\kappa^{rn+p}-\theta^{rn+p}}{\kappa-\theta}x^{n}\\&=\frac{\kappa^{p}}{\kappa-\theta}\sum_{n=0}^{\infty}(\kappa^{r}x)^{n}-\frac{\theta^{p}}{\kappa-\theta}\sum_{n=0}^{\infty}(\theta^{r}x)^{n}\\&=\frac{\kappa^{p}}{\kappa-\theta}\frac{1}{1-\kappa^{r}x}-\frac{\theta^{p}}{\kappa-\theta}\frac{1}{1-\theta^{r}x}\\&=\frac{CF_{p}-x(-1)^{r}CF_{p-r}}{1-xCL_{r}+(-1)^{r}x^{2}}\end{aligned}$$

and

$$\text{iv.}\quad\begin{aligned}\sum_{n=0}^{\infty}\frac{CL_{rn}}{n!}x^{n}&=\sum_{n=0}^{\infty}\frac{\kappa^{rn}+\theta^{rn}}{n!}x^{n}\\&=\sum_{n=0}^{\infty}\frac{(\kappa^{r}x)^{n}}{n!}+\sum_{n=0}^{\infty}\frac{(\theta^{r}x)^{n}}{n!}\\&=e^{\kappa^{r}x}+e^{\theta^{r}x}.\end{aligned}$$

The proof of the other may be found similarly. □

Corollary 2.2. The generating functions of copper Fibonacci and copper Lucas sequences are as follows:

i. $f(x)=\sum_{n=0}^{\infty}CF_{n}x^{n}=\frac{x}{1-4x-x^{2}}$ and
ii. $l(x)=\sum_{n=0}^{\infty}CL_{n}x^{n}=\frac{2-4x}{1-4x-x^{2}}$.

Proof. The proofs are found according to the special values of r and p in Theorem 2.11. □

In the following theorems, we calculate some identities of the copper Fibonacci CF_n and copper Lucas CL_n sequences.

Theorem 2.12. (Cassini identity). Let $n\in\mathbb{N}$. We obtain

i. $CF_{n+1}CF_{n-1}-CF_{n}^{2}=(-1)^{n}$ and
ii. $CL_{n+1}CL_{n-1}-CL_{n}^{2}=20(-1)^{n-1}$.

Proof. If the Binet formulas are used, we obtain

$$\text{i.}\quad\begin{aligned}CF_{n+1}CF_{n-1}-CF_{n}^{2}&=\frac{\kappa^{n+1}-\theta^{n+1}}{\kappa-\theta}\frac{\kappa^{n-1}-\theta^{n-1}}{\kappa-\theta}-\frac{\kappa^{n}-\theta^{n}}{\kappa-\theta}\frac{\kappa^{n}-\theta^{n}}{\kappa-\theta}\\&=\frac{\kappa^{2n}-\kappa^{n+1}\theta^{n-1}-\theta^{n+1}\kappa^{n-1}+\theta^{2n}}{(\kappa-\theta)^{2}}-\frac{\kappa^{2n}-2\kappa^{n}\theta^{n}+\theta^{2n}}{(\kappa-\theta)^{2}}\\&=\frac{(\kappa\theta)^{n}\frac{-\kappa}{\theta}}{(\kappa-\theta)^{2}}+\frac{(\kappa\theta)^{n}\frac{-\theta}{\kappa}}{(\kappa-\theta)^{2}}+\frac{2(\kappa\theta)^{n}}{(\kappa-\theta)^{2}}\\&=(-1)^{n}\end{aligned}$$

and

$$\text{ii.}\quad\begin{aligned}CL_{n+1}CL_{n-1}-CL_{n}^{2}&=\left(\kappa^{n+1}+\theta^{n+1}\right)\left(\kappa^{n-1}+\theta^{n-1}\right)\\&\quad-(\kappa^{n}+\theta^{n})(\kappa^{n}+\theta^{n})\\&=\kappa^{2n}+\kappa^{n+1}\theta^{n-1}+\theta^{n+1}\kappa^{n-1}+\theta^{2n}-\kappa^{2n}\\&\quad-2\kappa^{n}\theta^{n}-\theta^{2n}\\&=(\kappa\theta)^{n}\left(-\frac{\kappa}{\theta}-\frac{\theta}{\kappa}+2\right)\\&=20(-1)^{n-1}.\end{aligned}$$

□

Theorem 2.13. (Catalan identity). Let $n,i\in\mathbb{N}$ and $i<n$. We obtain

i. $CF_{n+i}CF_{n-i}-CF_{n}^{2}=(-1)^{n-i+1}CF_{i}^{2}$ and
ii. $CL_{n+i}CL_{n-i}-CL_{n}^{2}=20(-1)^{n-i}CF_{i}^{2}$.

Theorem 2.14. (D'ocagne identity). Let $n,i\in\mathbb{N}$ and $i<n$. We obtain

i. $CF_{n+1}CF_{i}-CF_{n}CF_{i+1}=(-1)^{i+1}CF_{n-I}$ and
ii. $CL_{n+1}CL_{i}-CL_{n}CL_{i+1}=20(-1)^{i}CF_{n-i}$.

Theorem 2.15. (Vajda identity). Let $n,i,j\in\mathbb{N}$ and $i,j<n$. We obtain

i. $CF_{n+i}CF_{n+j} - CF_nCF_{n+i+j} = (-1)^n CF_iCF_j$ and
ii. $CL_{n+i}CL_{n+j} - CL_nCL_{n+i+j} = 20(-1)^{n+1}CF_iCF_j$.

Theorem 2.16. (Halton identity). Let $n, i, j \in \mathbb{N}$ and $i, j < n$. We obtain

i. $CF_{n+i}CF_{n-i} - CF_{n+j}CF_{n-j} = \frac{1}{20}\left[(-1)^{n-j}BL_{2j} - (-1)^{n-i}BL_{2i}\right]$ and
ii. $BL_{n+i}BL_{n-i} - BL_{n+j}BL_{n-j} = (-1)^{n-i}BL_{2i} - (-1)^{n-j}BL_{2j}$.

Theorem 2.17. (Padilla identity). Let $n \in \mathbb{N}$. We obtain

i. $CF_{n+2}^3 + CF_{n-1}^3 - 3CF_nCF_{n+1}CF_{n+2} = \frac{1}{20}(CF_{3n+6} - 3CF_{3n+3} + CF_{3n-3} + 3CF_{2n+3} - 3CF_{n+2} + 3CF_{n+1})$ and
ii. $CL_{n+2}^3 + CL_{n-1}^3 - 3CL_nCL_{n+1}CL_{n+2} = CL_{3n+6} + 3CL_{n+2} + CL_{3n-3} + 3CL_{n-1} + 20(-CF_{3n+3} - 3CF_{n-1} + 3CF_{n+1} + 3CF_{2n+3})$.

Theorem 2.18. (Melham identity). Let $n \in \mathbb{N}$. We obtain

i. $CF_{n+1}CF_{n+2}CF_{n+6} - CF_n^3 = \frac{1}{20}(CF_{3n+9} - CF_{3n} - \frac{1}{(-1)^{n+1}}CF_{n+7} - \frac{1}{(-1)^{n+2}}CF_{n+5} + \frac{3}{(-1)^n}CF_n - \frac{1}{(-1)^{n+6}}CF_{n-3})$ and
ii. $CL_{n+1}CL_{n+2}CL_{n+6} - CL_n^3 = CL_{3n+9} - CL_{3n} + \frac{1}{(-1)^{n+1}}CL_{k,n+7} + \frac{1}{(-1)^{n+2}}CL_{k,n+5} - \frac{3}{(-1)^n}CL_n + \frac{1}{(-1)^{n+6}}CL_{n-3}$.

Theorem 2.19. (Gelin-Cesaro identity). Let $n \in \mathbb{N}$. We obtain

i. $CF_{n+2}CF_{n+1}CF_{n-1}CF_{n-2} - CF_n^4 = \frac{1}{20}(\frac{1}{(-1)^{n-3}}CL_{2n+4} + \frac{1}{(-1)^{n-2}}CL_{2n+2} + \frac{1}{(-1)^{2n-3}}CL_6 + \frac{1}{(-1)^n}CL_{2n-2} + CL_2 + \frac{1}{(-1)^{n+1}}CL_{2n-4} + \frac{1}{(-1)^{n-3}}CL_{2n} - 90)$ and
ii. $CL_{n+2}CL_{n+1}CL_{n-1}CL_{n-2} - CL_n^4$

$$= \frac{1}{(-1)^{n-2}}CL_{2n+4} + \frac{1}{(-1)^{n-1}}CL_{2n+2} + \frac{1}{(-1)^{2n-3}}CL_6 + \frac{1}{(-1)^{n+1}}CL_{2n-2} + \frac{1}{(-1)^{2n-1}}18 + \frac{1}{(-1)^{n+2}}CL_{2n-4} - \frac{1}{(-1)^{n-2}}CL_{2n} - 5.$$

The proofs of Theorem 2.13–2.19 are shown using the Binet formulas in a similar way to Theorem 2.12.

III. CATALAN TRANSFORM OF THE COPPER FIBONACCI AND COPPER LUCAS SEQUENCES

In this chapter, we apply Catalan transformation to these sequences, and their terms are found. Then, we find the generating function of the Catalan transformation of these sequences. In addition, the Hankel transform is applied to the Catalan transform of these sequences. Finally, we associate the terms of the Hankel transformation of the Catalan copper Fibonacci sequence with the classical Fibonacci numbers and the terms of the Hankel transformation of the Catalan copper Lucas sequence with the terms of the copper Lucas sequence.

Definition 3.1. (Catalan number). For $n \in \mathrm{N}^+$, the nth Catalan numbers are as follows:

$$C_n = \frac{C(2n, n)}{n+1},$$

and the generating function is obtained as

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

For some natural numbers n, the Catalan numbers are 1, 2, 5, 14, 132, ... (see Ref. 26 for details).

Using the Catalan transform, we define the Catalan transform of the copper Fibonacci and copper Lucas sequences, respectively,

$$CCF_n = \sum_{i=0}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i}CF_i, n \in \mathrm{N}^+ \text{ with } CCF_0 = 0$$

and

$$CCL_n = \sum_{i=0}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i}CL_i, n \in \mathrm{N}^+ \text{ with } CCL_0 = 0.$$

Next, we give the Catalan transformation of the first elements of the copper Fibonacci and copper Lucas sequences. The CCF_n and CCL_n values for some n natural numbers are given below:

$$CCF_0 = 0, CCF_1 = 1, CCF_2 = 5, CCF_3 = 27, CCF_4 = 148,$$
$$CCF_5 = 816, CCF_6 = 4501, CCF_7 = 24\,843,$$

and

$$CCL_0 = 0, CCL_1 = 4, CCL_2 = 22, CCL_3 = 120, CCL_4 = 660, CCL_5 = 3644.$$

We can write CCF_n and CCL_n as the product of $nx1$ type CF_n and CL_n's of the lower triangular Catalan matrix C,

$$\begin{bmatrix} CCF_1 \\ CCF_2 \\ CCF_3 \\ CCF_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} CF_1 \\ CF_2 \\ CF_3 \\ CF_4 \\ \vdots \end{bmatrix} \text{ and } \begin{bmatrix} CCL_1 \\ CCL_2 \\ CCL_3 \\ CCL_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} CL_1 \\ CL_2 \\ CL_3 \\ CL_4 \\ \vdots \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} 1 \\ 5 \\ 27 \\ 48 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 17 \\ 72 \\ \vdots \end{bmatrix} \text{ and } \begin{bmatrix} 4 \\ 22 \\ 120 \\ 660 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 4 \\ 18 \\ 76 \\ 322 \\ \vdots \end{bmatrix}.$$

In the following theorem, we find generating functions of the Catalan transform of the copper Fibonacci and copper Lucas sequences.

Theorem 3.1. We obtain

i. $F(x) = \frac{1-\sqrt{1-4x}}{-3+2x+5\sqrt{1-4x}}$ and
ii. $L(x) = \frac{4\sqrt{1-4x}}{-3+2x+5\sqrt{1-4x}}$.

Proof. i. Let $f(x)$ and $C(x)$ be generating functions of copper Fibonacci and Catalan sequences, respectively. $CF_n(x * C(x))$ shows the generating function of the Catalan copper Fibonacci sequence. Then, we write the following relations for the generating function of the Catalan transform of the copper Fibonacci sequence:

$$F(x) = CCF_n(x) = CCF_n(x * c(x)) = \frac{1-\sqrt{1-4x}}{-3+2x+5\sqrt{1-4x}}.$$

Similarly, the generating function of the Catalan copper Lucas sequence is

$$L(x) = CCL_n(x) = CL_n(x * C(x)) = \frac{4\sqrt{1-4x}}{-3+2x+5\sqrt{1-4x}}.$$ □

Definition 3.2. Let the terms of a sequence be $A = \{v_1, v_2, v_3, \ldots\}$. In Ref. 27, the Hankel transform H_n of the terms of this sequence was defined as follows:

$$H_n = \det \begin{bmatrix} v_1 & v_2 & v_3 & \cdots & v_n \\ v_2 & v_3 & \vdots & v_n & \vdots \\ v_3 & \vdots & v_n & \vdots & \vdots \\ \vdots & v_n & \vdots & \vdots & \vdots \\ v_n & \cdots & \cdots & \cdots & v_{2n-1} \end{bmatrix}.$$

Next, we apply Hankel's work to the Catalan copper Lucas sequence CCL_n and copper Fibonacci sequence CCF_n. We obtain

$$HCCL_1 = H_1 = \det[4] = 4 = CL_1,$$

$$HCCL_2 = H_2 = \det \begin{bmatrix} CCL_1 & CCL_2 \\ CCL_2 & CCL_3 \end{bmatrix} = \det \begin{bmatrix} 4 & 22 \\ 22 & 120 \end{bmatrix} = -4 = -CL_1,$$

$$HCCL_3 = H_3 = \det \begin{bmatrix} CCL_1 & CCL_2 & CCL_3 \\ CCL_2 & CCL_3 & CCL_4 \\ CCL_3 & CCL_4 & CCL_5 \end{bmatrix} = -176 = 10CL_2 - CL_1,$$

and

$$HCCF_1 = \det[CCF_1] = \det[1] = 1,$$

$$HCCF_2 = \det \begin{bmatrix} CCF_1 & CCF_2 \\ CCF_2 & CCF_3 \end{bmatrix} = \det \begin{bmatrix} 1 & 5 \\ 5 & 27 \end{bmatrix} = 2,$$

$$HCCF_3 = \det \begin{bmatrix} CCF_1 & CCF_2 & CCF_3 \\ CCF_2 & CCF_3 & CCF_4 \\ CCF_3 & CCF_4 & CCF_5 \end{bmatrix} = \det \begin{bmatrix} 1 & 5 & 27 \\ 5 & 27 & 148 \\ 27 & 148 & 816 \end{bmatrix} = 5,$$

$$HCCF_4 = \det \begin{bmatrix} CCF_1 & CCF_2 & CCF_3 & CCF_4 \\ CCF_2 & CCF_3 & CCF_4 & CCF_5 \\ CCF_3 & CCF_4 & CCF_5 & CCF_6 \\ CCF_4 & CCF_5 & CCF_6 & CCF_7 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 5 & 27 & 148 \\ 5 & 27 & 148 & 816 \\ 27 & 148 & 816 & 4501 \\ 148 & 816 & 4501 & 24\,843 \end{bmatrix} = 13.$$

In the following theorem, we obtain a very interesting property.

Theorem 3.2. Applying the Hankel transform to the Catalan transform of the copper Fibonacci sequence, the following equation can be obtained:

$$HCCF_n = F_{2n-1}.$$

Here, $n \in \mathbb{N}$ and F_n is the classic Fibonacci sequence.

Proof. $HCCF_n \neq 0$. Let us write it as $HCCF_n = \det A_n \det B_n$. Here, the properties of matrices A_n and B_n are as follows.

A_n is the matrix with the principal diagonal $\{1, 1, 1, \ldots\}$, an *nxn* type lower triangular matrix, and a first column containing $\{CCF_1, CCF_2, CCF_3, \ldots\}$. B_n is the matrix with the principal diagonal$\{1, 2, \frac{5}{2}, \frac{13}{5}, \frac{34}{13}, \frac{89}{34}, \ldots\}$, an *nxn* type upper triangular matrix, and a first row containing $\{CCF_1, CCF_2, CCF_3, \ldots\}$. Hence,

$$HCCF_n = \det \begin{bmatrix} CCF_1 & 0 & 0 & 0 & 0 \\ CCF_2 & 1 & 0 & 0 & 0 \\ CCF_3 & \ldots & 1 & 0 & 0 \\ CCF_4 & \ldots & \cdots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & 1 \end{bmatrix}$$

$$\times \det \begin{bmatrix} CCF_1 & CCF_2 & CCF_3 & CCF_4 & \ldots \\ 0 & 2 & \ddots & \vdots & \ldots \\ 0 & 0 & \frac{5}{2} & \ldots & \ldots \\ 0 & 0 & 0 & \frac{13}{5} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}.$$

TABLE II. Hyperbolic quaternion units.

	i_1	i_2	i_3	i_4
i_1	i_1	i_2	i_3	i_4
i_2	i_2	i_1	i_4	$-i_3$
i_3	i_3	$-i_4$	i_1	i_2
i_4	i_4	i_3	$-i_2$	i_1

Thus, we obtain

$$\begin{aligned} HCCF_n &= \det A_n \det B_n = \det B_n = \{b_{11}b_{22}b_{33}b_{44}\ldots b_{nn}\} \\ &= \{1, 2, 5, 13, 34, 89, \ldots\} \\ &= F_{2n-1}. \end{aligned}$$

□

IV. APPLICATIONS

In this chapter, we present on the application of copper Fibonacci and copper Lucas sequences to hyperbolic quaternions. Then we define hyperbolic copper Fibonacci and copper Lucas quaternions. For these quaternions, we give many properties, such as Binet formulas. Finally, the terms of the copper Fibonacci and copper Lucas sequences are associated with their hyperbolic quaternion values.

Definition 4.1. For $n \in \mathbb{N}$, the hyperbolic copper Fibonacci $\check{H}CF_n$ and hyperbolic copper Lucas $\check{H}CL_n$ quaternions are defined, respectively, by

$$\begin{aligned} \check{H}CF_n &= CF_n i_1 + CF_{n+1} i_2 + CF_{n+2} i_3 + CF_{n+3} i_4 \\ &= (CF_n, CF_{n+1}, CF_{n+2}, CF_{n+3}) \end{aligned}$$

and

$$\begin{aligned} \check{H}CL_n &= CL_n i_1 + CL_{n+1} i_2 + CL_{n+2} i_3 + CL_{n+3} i_4 \\ &= (CL_n, CL_{n+1}, CL_{n+2}, CL_{n+3}), \end{aligned}$$

where CF_n is the nth copper Fibonacci number, CL_n is the nth copper Lucas number, and i_1, i_2, i_3, and i_4 are the hyperbolic quaternion units listed in Table II.

Let now give the first four terms of the hyperbolic copper Fibonacci quaternions $\check{H}CF_n$ and hyperbolic copper Lucas quaternions $\check{H}CL_n$, respectively,

- $\check{H}CF_0 = i_2 + 4i_3 + 17i_4, \check{H}CL_0 = 2i_1 + 4i_2 + 18i_3 + 76i_4$,
- $\check{H}CF_1 = i_1 + 4i_2 + 17i_3 + 72i_4, \check{H}CL_1 = 4i_1 + 18i_2 + 76i_3 + 322i_4$,
- $\check{H}CF_2 = 4i_1 + 17i_2 + 72i_3 + 305i_4, \check{H}CL_2 = 18i_1 + 76i_2 + 322i_3 + 1364i_4$,
- $\check{H}CF_3 = 17i_1 + 72i_2 + 305i_3 + 1292i_4, \check{H}CL_3 = 76i_1 + 322i_2 + 1364i_3 + 5778i_4$.

Definition 4.2. For $n \in \mathbb{N}$, the conjugate of the hyperbolic copper Fibonacci quaternions $\check{H}CF_n$ and hyperbolic copper Lucas quaternions $\check{H}CL_n$ are defined, respectively, by

$$\begin{aligned} \check{H}CF_n^* &= CF_n i_1 - CF_{n+1} i_2 - CF_{n+2} i_3 - CF_{n+3} i_4 \\ &= (CF_n, -CF_{n+1}, -CF_{n+2}, -CF_{n+3}) \end{aligned}$$

and

$$\begin{aligned} \check{H}CL_n^* &= CL_n i_1 - CL_{n+1} i_2 - CL_{n+2} i_3 - CL_{n+3} i_4 \\ &= (CL_n, -CL_{n+1}, -CL_{n+2}, -CL_{n+3}). \end{aligned}$$

Definition 4.3. For $n \in \mathbb{N}$, the norms of the hyperbolic copper Fibonacci quaternions $\check{H}CF_n$ and hyperbolic copper Lucas quaternions $\check{H}CL_n$ quaternions are defined, respectively, by

$$\|\check{H}CF_n\| = \sqrt{CF_n^2 + CF_{n+1}^2 + CF_{n+2}^2 + CF_{n+3}^2}$$

and

$$\|\check{H}CL_n\| = \sqrt{CL_n^2 + CL_{n+1}^2 + CL_{n+2}^2 + CL_{n+3}^2}.$$

In the following theorems, we examine the relationships between the $\check{H}CF_n$, $\check{H}CF_n^*$, $\check{H}CL_n$, and $\check{H}CL_n^*$ quaternions.

Theorem 4.1. Let $n \in \mathbb{N}$. The following equations are true:

i. $\check{H}CF_{n+2} = 4\check{H}CF_{n+1} + \check{H}CF_n$,
ii. $\check{H}CL_{n+2} = 4\check{H}CL_{n+1} + \check{H}CL_n$,
iii. $\check{H}CF_{n+2}^* = 4\check{H}CF_{n+1}^* + \check{H}CF_n^*$, and
iv. $\check{H}CL_{n+2}^* = 4\check{H}CL_{n+1}^* + \check{H}CL_n^*$.

Proof. i. If the definition is used, we have

$$\begin{aligned} &4\check{H}CF_{n+1} + \check{H}CF_n \\ &= 4(CF_{n+1}i_1 + CF_{n+2}i_2 + CF_{n+3}i_3 + CF_{n+4}i_4) \\ &\quad + (CF_n i_1 + CF_{n+1}i_2 + CF_{n+2}i_3 + CF_{n+3}i_4) \\ &= (4CF_{n+1} + CF_n)i_1 + (4CF_{n+2} + CF_{n+1})i_2 + (4CF_{n+3} + CF_{n+2})i_3 \\ &\quad + (4CF_{n+4} + CF_{n+3})i_4. \end{aligned}$$

Since $BC_{n+2} = 4CF_{n+1} + CF_n$, we obtain

$$\check{H}CF_{n+2} = 4\check{H}CF_{n+1} + \check{H}CF_n.$$

The proofs of the others may be found similarly. □

Theorem 4.2. We obtain

i. $\check{H}CF_n + \check{H}CF_n^* = 2CF_n i_1$,
ii. $\check{H}CF_n^2 = 2CF_n\check{H}CF_n + (\|\check{H}CF_n{}^2\| - 2CF_n^2)i_1$,
iii. $\check{H}CL_n + \check{H}CL_n^* = 2CL_n i_1$, and
iv. $\check{H}CL_n^2 = 2CL_n\check{H}CL_n + (\|\check{H}CL_n{}^2\| - 2CL_n^2)i_1$.

Proof. ii. If the definition is used, we have

$$\begin{aligned} \check{H}CF_n^2 &= (CF_n i_1 + CF_{n+1}i_2 + CF_{n+2}i_3 + CF_{n+3}i_3) \\ &\quad \times (CF_n i_1 + CF_{n+1}i_2 + CF_{n+2}i_3 + CF_{n+3}i_4) \\ &= (CF_n^2 + CF_{n+1}^2 + CF_{n+2}^2 + CF_{n+3}^2)i_1 + (2CF_nCF_{n+1})i_2 \\ &\quad + (2CF_nCF_{n+2})i_3 + (2CF_nCF_{n+3})i_4 \\ &= 2CF_n(CF_n i_1 + CF_{n+1}i_2 + CF_{n+2}i_3 + CF_{n+3}i_4) \\ &\quad + (-CF_n^2 + CF_{n+1}^2 + CF_{n+2}^2 + CF_{n+3}^2)i_1. \end{aligned}$$

Thus, we obtain

$$\check{H}CF_n^2 = 2CF_n\check{H}CF_n + \left(\left\|\check{H}CF_n{}^2\right\| - 2CF_n^2\right)i_1.$$

The proofs of the others may be found similarly. □

Theorem 4.3. We obtain

i. $i_1\check{H}CF_n - i_2\check{H}CF_{n+1} - i_3\check{H}CF_{n+2} - i_4\check{H}CF_{n+3} = (CF_{n+2} + 16CF_{n+1})i_1$ and
ii. $i_1\check{H}CL_n - i_2\check{H}CL_{n+1} - i_3\check{H}CL_{n+2} - i_4\check{H}CL_{n+3} = (CL_{n+2} + 16CL_{n+1})i_1$.

Proof. i. If the definition is used, we have

$$\begin{aligned} &i_1\check{H}CF_n - i_2\check{H}CF_{n+1} - i_3\check{H}CF_{n+2} - i_4\check{H}CF_{n+3} \\ &= (CF_n - CF_{n+2} - CF_{n+4} + CF_{n+6})i_1 \\ &\quad + (CF_{n+1} - CF_{n+1} - CF_{n+5} + CF_{n+5})i_2 \\ &\quad + (CF_{n+2} + CF_{n+4} - CF_{n+2} - CF_{n+4})i_3 \\ &\quad + (CF_{n+3} - CF_{n+3} + CF_{n+3} - CF_{n+3})i_4 \\ &= (CF_n - CF_{n+2} - CF_{n+4} + CF_{n+6})i_1 \\ &= (CF_{n+2} + 16CF_{n+1})i_1. \end{aligned}$$

The proof of the other may be found similarly. □

In the following theorem, the Binet formulas of the $\check{H}CF_n$, $\check{H}CF_n^*$, $\check{H}CL_n$, and $\check{H}CL_n^*$ quaternions are expressed.

Theorem 4.4. Let $n \in \mathbb{N}$. We obtain

i. $\check{H}CF_n = \frac{\overline{\kappa}\kappa^n - \overline{\theta}\theta^n}{\kappa - \theta}$,
ii. $\check{H}CL_n = \overline{\kappa}\kappa^n + \overline{\theta}\theta^n$,
iii. $\check{H}CF_n^* = \frac{\overline{\kappa}^*\kappa^n - \overline{\theta}^*\theta^n}{\kappa - \theta}$, and
iv. $\check{H}CL_n^* = \overline{\kappa}^*\kappa^n + \overline{\theta}^*\theta^n$,

where

$$\overline{\kappa} = i_1 + \kappa i_2 + \kappa^2 i_3 + \kappa^3 i_4 = (1, \kappa, \kappa^2, \kappa^3), \overline{\kappa}^* = (1, -\kappa, -\kappa^2, -\kappa^3),$$

$$\overline{\theta} = i_1 + \theta i_2 + \theta^2 i_3 + \theta^3 i_4 = (1, \theta, \theta^2, \theta^3) \text{ and } \overline{\theta}^* = (1, -\theta, -\theta^2, -\theta^3).$$

Proof. i. The Binet form of the hyperbolic copper Fibonacci quaternions is

$$\check{H}CF_n = x\kappa^n + y\theta^n.$$

With the initial conditions, the following equations are obtained:

$$\check{H}CF_0 = i_2 + 4i_3 + 17i_4 = x + y$$

and

$$\check{H}CF_1 = i_1 + 4i_2 + 17i_3 + 72i_4 = x\kappa + y\theta.$$

Thus, we have

$$x = \frac{\check{H}CF_1 - \theta\check{H}CF_0}{\kappa - \theta} = \frac{i_1 + \kappa i_2 + \kappa^2 i_3 + \kappa^3 i_4}{\kappa - \theta} = \frac{\overline{\kappa}}{\kappa - \theta},$$

$$y = \frac{\check{H}CF_1 - \kappa\check{H}CF_0}{-\kappa + \theta} = \frac{i_1 + \theta i_2 + \theta^2 i_3 + \theta^3 i_4}{-(\kappa - \theta)} = \frac{-\overline{\theta}}{\kappa - \theta}.$$

Hence, we obtain

$$\check{H}CF_n = \frac{\overline{\kappa}\kappa^n - \overline{\theta}\theta^n}{\kappa - \theta}.$$

The proofs of the others may be found similarly. □

In the following theorems, we associate the terms of the copper Fibonacci and copper Lucas sequences with their hyperbolic quaternion values.

Theorem 4.5. Let $a, b \in \mathbb{N}$ and $a < b$. We obtain

i. $\check{H}CF_{a+2b} = \frac{CF_{2b}}{4}\check{H}CL_{a+1} - \frac{CL_{2b-1}}{4}\check{H}CF_a$,
ii. $\check{H}CL_{a+2b} = 2\sqrt{5}\frac{CF_{2b}}{4}\check{H}CF_{a+1} - \frac{CL_{2b-1}}{4}\check{H}CL_a$,
iii. $\check{H}CF_{a+2b} = \frac{CF_{2b}}{4}\check{H}CF_{a+2} - \frac{CL_{2b-2}}{4}\check{H}CF_a$, and
iv. $\check{H}CL_{a+2b} = \frac{CF_{2b}}{4}\check{H}CL_{a+2} - \frac{CF_{2b-2}}{4}\check{H}CL_a$.

Proof. iv. With the Binet formula, we have

$$\begin{aligned} \frac{CF_{2b}}{4}\check{H}CL_{a+2} - \frac{CF_{2b-2}}{4}\check{H}CL_a &= \frac{1}{4}\frac{\kappa^{2b} - \theta^{2b}}{\kappa - \theta}\left(\overline{\kappa}\kappa^{a+2} + \overline{\theta}\theta^{a+2}\right) \\ &\quad - \frac{1}{4}\frac{\kappa^{2b-2} - \theta^{2b-2}}{\kappa - \theta}\left(\overline{\kappa}\kappa^a + \overline{\theta}\theta^a\right) \\ &= \frac{\overline{\kappa}\kappa^{a+2b}\left(\kappa^2 - \frac{1}{\kappa^2}\right) - \overline{\theta}\theta^{2a+b}\left(\theta^2 - \frac{1}{\theta^2}\right)}{4(\kappa - \theta)}. \end{aligned}$$

Since $\theta^2 = \frac{1}{\kappa^2}$, $\kappa^2 = \frac{1}{\theta^2}$, and $\kappa + \theta = 4$, we obtain

$$\check{H}CL_{a+2b} = \frac{CF_{2b}}{4}\check{H}CL_{a+2} - \frac{CF_{2b-2}}{4}\check{H}CL_a.$$

The proofs of the others may be found similarly. □

Theorem 4.6. Let $a, b, c, d \in \mathbb{N}$ and $c < a, b$. We obtain

i. $\check{H}CF_{a+b} = CF_a\check{H}CF_{b+1} + CF_{a-1}\check{H}CF_b$,
ii. $\check{H}CF_{2a+b} = CL_a\check{H}CF_{a+b} - (-1)^a\check{H}CF_b$,
iii. $\check{H}CF_{ac+d} = \frac{CF_{ac}}{CF_a}\check{H}CF_{a+d} - (-1)^a\frac{CF_{ac-a}}{CF_a}\check{H}CF_d$,
iv. $(-1)^{ac}CF_{ab-ac}\check{H}CF_d = \check{H}CF_{ac+d}CF_{ab} - \check{H}CF_{ab+d}CF_{ac}$, and
v. $\check{H}CF_{a+b}\check{H}CL_{a+c} - \check{H}CF_{a+c}\check{H}CL_{a+b} = CF_{b-c}(-1)^{a+c}(0, 8, 26, 86)$.

Proof. i. If Binet formulas are used, we get

$$\begin{aligned} &CF_a\check{H}CF_{b+1} + CF_{a-1}\check{H}CF_b \\ &= \left(\frac{\kappa^a - \theta^a}{\kappa - \theta}\right)\left(\frac{\overline{\kappa}\kappa^{b+1} - \overline{\theta}\theta^{b+1}}{\kappa - \theta}\right) + \left(\frac{\kappa^{a-1} - \theta^{a-1}}{\kappa - \theta}\right)\left(\frac{\overline{\kappa}\kappa^b - \overline{\theta}\theta^b}{\kappa - \theta}\right) \\ &= \frac{1}{(\kappa - \theta)(\kappa - \theta)}\left[\overline{\kappa}\kappa^{a+b}\left(\kappa + \frac{1}{\kappa}\right) + \overline{\theta}\theta^{a+b}\left(\theta + \frac{1}{\theta}\right)\right]. \end{aligned}$$

Since $\frac{1}{\kappa} = -\theta$ and $\frac{1}{\theta} = -\kappa$, we obtain

$$\check{H}CF_{a+b} = CF_a\check{H}CF_{b+1} + CF_{a-1}\check{H}CF_b.$$

The proofs of the others may be found similarly. □

V. CONCLUSIONS

In this study, we defined a new generalization of Fibonacci sequences that give the copper ratio, and we called it the copper Fibonacci sequence. In addition, inspired by the copper Fibonacci definition, we also defined copper Lucas sequences, and then we gave the relationships between the terms of these sequences. We presented some properties, such as the Binet formulas, special summation formulas, special generating functions, etc. We found the relationships between the roots of the characteristic equation of these sequences with these sequences. What is interesting here is that the relationships obtained from the relationship between the roots of the characteristic equation of these new sequences and the terms of the sequences satisfied for both roots. In addition, we examined the relationships between these sequences with the classic Fibonacci and Lucas sequences. Moreover, we calculated some identities of these sequences, such as Cassini and Catalan. Then Catalan transformation was applied to these sequences, and their terms were found. Furthermore, we applied the Hankel transform to the Catalan transform of these sequences. We associated the terms of the Hankel transformation of the Catalan copper Fibonacci sequence with the classical Fibonacci numbers and the terms of the Hankel transformation of the Catalan copper Lucas sequence with the terms of the copper Lucas sequence. Finally, we presented on the application of copper Fibonacci and copper Lucas sequences to hyperbolic quaternions. We defined hyperbolic copper Fibonacci and copper Lucas quaternions. For these quaternions, we given many properties, such as Binet formulas. Finally, the terms of the copper Fibonacci and copper Lucas sequences are associated with their hyperbolic quaternion values. If this study is examined, such features can be found in other sequences, such as copper Pell and copper Pell–Lucas sequences.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Engin Özkan: Conceptualization (equal); Formal analysis (equal); Supervision (equal); Writing – review & editing (equal). **Hakan Akkuş**: Conceptualization (equal); Data curation (equal); Formal analysis (equal); Funding acquisition (equal); Resources (equal); Writing – original draft (lead).

DATA AVAILABILITY

The authors confirm that the data supporting the findings of this study are available within the paper.

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