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On pseudo Z-symmetric Lorentzian manifolds admitting a type of semi-symmetric metric connection

Hülya Bağdatlı Yılmaz[1]

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Abstract

The paper aims to investigate the general properties of pseudo Z-symmetric Lorentzian manifolds with semi-symmetric metric ρ-connection $\bar{\nabla}$ and examine compatibility conditions. Moreover, such a manifold is applied to general relativity and its physical consequences are given.

Keywords Pseudo Z-symmetric Lorentzian manifold · Lorentzian quasi-Einstein manifold · GRW space-time · Compatibility · Purely electric space-time · Yang pure space.

Mathematics Subject Classification 53B30 · 53B50 · 53C50 · 53C80

1 Introduction

An n-dimensional Lorentzian manifold M, a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, is a special case of the semi-Riemannian manifold. The signature of Lorentzian metric g is $(1, n-1)$. A tangent vector $W_p \in T_pM$ is called *time-like* if $g(W_p, W_p) < 0$, *space-like* if $g(W_p, W_p) > 0$ or $W_p = 0$ and *light-like* if $g(W_p, W_p) = 0, \quad W_p \neq 0$. If W_p is either time-like or light-like, then it is said to be *causal*. A Lorentzian manifold may not generally have a globally time-like vector field. If it admits a globally time-like vector field, such a manifold is said to be *a time-orientable Lorentzian manifold*, known as a *space-time*, which is the stage of the present modeling of the physical world. One of the most essential and basic concepts in physical theories is causality, particularly in all relativistic theories based on a Lorentzian manifold. Thus, a Lorentzian manifold becomes a convenient choice for the study of general relativity.

✉ Hülya Bağdatlı Yılmaz
hbagdatli@marmara.edu.tr

[1] Mathematics Department, Faculty of Sciences, Marmara University, 34722 Istanbul, Turkey

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[17], a new generalized tensor (0, 2) symmetric tensor was introduced by Mantica and Molinari; the new tensor was defined as:

$$Z(X, Y) = S(X, Y) + \phi g(X, Y), \tag{1}$$

where S and ϕ are the Ricci tensor of the manifold and an arbitrary scalar function, respectively and said to be *generalized Z tensor*. The classical Z tensor is defined as

$$Z(X, Y) = S(X, Y) - \frac{r}{n} g(X, Y), \tag{2}$$

where r is the scalar curvature of the manifold. Hereafter, we refer to the generalized Z tensor shortly as the Z tensor. It is notice that the Z tensor is a general notion of the Einstein gravitational tensor in general relativity.

Mantica and Suh [19] introduced a kind of Riemannian manifold whose non-null Z tensor satisfies the condition

$$(\nabla_X Z)(Y, U) = 2\pi(X)Z(Y, U) + \pi(Y)Z(X, U) + \pi(U)Z(Y, X), \tag{3}$$

where π is a non-zero 1-form, a vector field ρ is called the basic vector field of the manifold corresponding to the associated 1-form π and is defined by

$$\pi(X) = g(X, \rho). \tag{4}$$

This manifold that generalizes the notion of both pseudo-Ricci symmetric manifold and pseudo-projective Ricci symmetric manifold is called *a pseudo Z-symmetric manifold* and denoted by $(PZS)_n$. The authors investigated various properties of such a manifold, especially considering the cases with harmonic curvature tensors under some conditions. Moreover, in 2014, they studied the fundamental properties of $(PZS)_4$ space-times [20].

In 1991, the quasi-Einstein manifold, which arises during the study of exact solutions of the Einstein fields equations, and also during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces, is firstly defined by Defever and Deszcz [7], (see also, [3, 8, 9]) as; a semi-Riemannian manifold is called a quasi-Einstein manifold if its Ricci tensor is not identically zero and satisfies the following condition

$$S(X, Y) = ag(X, Y) + b\pi(X)\pi(Y), \tag{5}$$

where a, b are scalars and π is a non-zero 1-form. A Lorentzian quasi-Einstein manifold whose Ricci tensor satisfies (5) and a 1-form π is a unit time-like vector field, is called *a perfect fluid space-time*.

The concept of semi-symmetric linear connection on a differantiable manifold was introduced by Friedman and Schouten [11]. The idea of metric connection with torsion on a Riemannian manifold was introduced by Hayden [15]. Yano systematically studied a semi-symmetric metric connection on a Riemannian manifold [29].

Various physical problems can be expressed using a semi-symmetric metric connection, and it plays an important role in physics, especially general relativity, (see [5, 6, 10, 24],...).

On an n-dimensional semi-Riemannian manifold, the Weyl conformal curvature tensor field C of type (1, 3) is defined by

$$\begin{aligned} C(X,Y)U = R(X,Y)U &- \frac{1}{n-2}\{S(Y,U)X - S(X,U)Y \\ &+ g(Y,U)LX - g(X,U)LY\} \\ &+ \frac{r}{(n-1)(n-2)}\{g(Y,U)X - g(X,U)Y\}, \end{aligned} \tag{6}$$

for all vector fields X, Y, U on the manifold, respectively, where L is the Ricci operator, (see, [26, 31]).

2 Lorentzian manifolds with semi symmetric metric ρ-connection

Let $\bar{\nabla}$ be a linear connection on an n-dimensional Lorentzian manifold M. It is called semi-symmetric if the torsion tensor $\bar{T}$ of the connection $\bar{\nabla}$ satisfies

$$\bar{T}(X,Y) = \pi(Y)X - \pi(X)Y, \tag{7}$$

for $\forall X, Y \in \chi(M)$. Moreover, if

$$\bar{\nabla} g = 0, \tag{8}$$

then it is called *semi symmetric metric connection*, and the relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ of M is given by the following equation, [29],

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)\rho. \tag{9}$$

Let the vector field ρ corresponding to the 1-form π be a unit time-like vector field,

$$\pi(\rho) = g(\rho,\rho) = -1, \tag{10}$$

and parallel with respect to the connection $\bar{\nabla}$,

$$\bar{\nabla}_X \rho = 0. \tag{11}$$

Then such a connection is called *a semi symmetric metric ρ-connection*, [6]. Using (4), (8), (9) and (10), we achieve

$$\left(\bar{\nabla}_X \pi\right) Y = (\nabla_X \pi) Y - \pi(Y)\pi(X) - g(X,Y). \tag{12}$$

Moreover, considering $\left(\bar{\nabla}_X \pi\right) Y = 0$ and the last equation, we obtain

$$(\nabla_X \pi)\, Y = \pi(Y)\pi(X) + g(X, Y). \tag{13}$$

By considering the above equation, we can say that the 1-form π is closed.

From (9), (10) and (11), it follows that

$$\nabla_X \rho = \pi(X)\rho + X, \tag{14}$$

which means that ρ is a torse-forming vector field. The torse-forming vector fields were introduced by Yano [30]. Based on [28], since 1-form π in (13) is closed, ρ is a concircular vector field. Such a vector field has a significant role in the theory of projective and conformal transformations and interesting applications in general relativity, such as trajectories of time-like concircular vector fields in the de Sitter model determine the world lines receding or colliding galaxies satisfying the Weyl hypothesis, [27]. Moreover, a specially concircular vector field Φ is defined by

$$\nabla_X \Phi = \varrho X, \tag{15}$$

where ϱ is some function on the manifold, [28]. It is noticed that since after a suitable normalization, a concircular vector field ρ can be written in the form a specially concircular vector field. According to Chen [4], a Lorentzian n-manifold with $n \geq 3$ is a generalized Robertson–Walker (GRW) space-time if and only if it admits a time-like concircular vector field in the form (15).

By using the argument given above, we can state the following theorem.

Theorem 1 *An n-dimensional Lorentzian manifold with semi-symmetric metric ρ-connection, $n \geq 3$, is a GRW space-time.*

The above theorem is also shown differently in [6].

Let $\bar{R}$ and R denote the curvature tensors of the manifold with respect to the connections $\bar{\nabla}$ and ∇. Then, the relation between the curvature tensors of the manifold M is given by, [6],

$$\begin{aligned} \bar{R}(X, Y)U = {} & R(X, Y)U - \alpha(Y, U) + \alpha(X, U)Y - g(Y, U)AX \\ & + g(X, U)AY, \end{aligned} \tag{16}$$

for $\forall X, Y, U \in \chi(M)$, where α is a tensor of type (0,2), defined as follows;

$$\alpha(X, Y) = g(AX, Y) = \frac{1}{2} g(X, Y), \tag{17}$$

i.e.

$$AX = \frac{1}{2} X. \tag{18}$$

Substituting (17) and (18) in (16), we obtain

$$\bar{R}(X,Y)U = R(X,Y)U - \{g(Y,U)X - g(X,U)Y\}. \tag{19}$$

By using the above equation, we have the following relation of the covariant curvature tensors of type (0,4)

$$\bar{R}(X,Y,U,V) = R(X,Y,U,V) - \{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}, \tag{20}$$

where $\bar{R}(X,Y,U,V) = g(\bar{R}(X,Y)U,V)$ and $R(X,Y,U,V) = g(R(X,Y)U,V)$. Contracting (20) with respect to X and V, we obtain

$$\bar{S}(Y,U) = S(Y,U) - (n-1)g(Y,U), \tag{21}$$

where $\bar{S}$ and S denote the Ricci tensors of the manifold with respect to the connections $\bar{\nabla}$ and ∇, respectively.

Contracting (21) with respect to Y and U gives

$$\bar{r} = r - n(n-1), \tag{22}$$

where $\bar{r}$ and r denote the scalar curvatures of the manifold with respect to the connections $\bar{\nabla}$ and ∇, respectively.

By using (8), (9), (11), (12), from $\bar{R}(X,Y)U = \bar{\nabla}_X\bar{\nabla}_Y U - \bar{\nabla}_Y\bar{\nabla}_X U - \bar{\nabla}_{[X,Y]}U$, we achieve

$$\bar{R}(X,Y)\rho = \bar{R}(\rho,Y)U = 0, \quad \bar{R}(X,Y,U,\rho) = 0, \quad \bar{S}(\rho,Y) = 0. \tag{23}$$

From (19), (20), (21) and (23), we obtain the following equations:

$$S(\rho,Y) = (n-1)\pi(Y), \tag{24}$$
$$R(X,Y)\rho = \pi(Y)X - \pi(X)Y, \tag{25}$$
$$R(\rho,Y)U = g(Y,U)\rho - \pi(U)Y, \tag{26}$$
$$\pi(R(X,Y)U) = g(Y,U)\pi(X) - g(X,U)\pi(Y). \tag{27}$$

3 Pseudo Z-symmetric Lorentzian manifold with semi-symmetric metric ρ-connection and compatibility conditions

In this section we investigate general properties of pseudo Z-symmetric Lorentzian manifolds with semi-symmetric metric ρ-connection $\bar{\nabla}$ and examine compatibility conditions on these manifolds

Let us denote with $(PZSL)_n$ an n-dimensional pseudo Z-symmetric Lorentzian manifold. The $\bar{Z}$ tensor with respect to connection $\bar{\nabla}$ can be expressed as

$$\bar{Z}(X,Y) = \bar{S}(X,Y) + \bar{\phi}g(X,Y), \tag{28}$$

where $\bar{\phi}$ is an arbitrary scalar function.

We will denote the $(PZSL)_n$ manifold admitting the connection $\bar{\nabla}$ shortly as $(PZ\bar{S}L)_n$.

Proposition 2 *For $(PZ\bar{S}L)_n$ manifolds, we have*

1. *$(n-1+\phi)$ is an eigenvalue of the Z tensor corresponding to the eigenvector ρ.*
2. *The scalar function ϕ satisfies the following equation*

$$(\nabla_X \phi) = 4\,(n-1+\phi)\,\pi(X). \tag{29}$$

3. *The expression of the Z tensor on such a manifold is*

$$Z(X,U) = (1-n-\phi)\,\{2\pi(X)\pi(U) + g(X,U)\}\,. \tag{30}$$

4. *The expression of the $\bar{Z}$ tensor on such a manifold is*

$$\bar{Z}(X,U) = 2\,(1-n-\phi)\,\pi(X)\pi(U) + \left\{\bar{\phi} + 2\,(1-n-\phi)\right\} g(X,U). \tag{31}$$

5. *$\bar{\phi}$ is an eigenvalue of the $\bar{Z}$ tensor corresponding to the eigenvector ρ.*

Proof *1.* Substituting (1) and (21) into (28) provides

$$\bar{Z}(X,Y) = Z(X,Y) + \left(\bar{\phi} - \phi - (n-1)\right) g(X,Y). \tag{32}$$

Let us now substitute ρ for Y in (3) and use (10). Then, we have

$$(\nabla_X Z)\,(\rho,U) = 2\pi(X)Z(\rho,U) - Z(X,U) + \pi(U)Z(\rho,X). \tag{33}$$

Now using (1) and (24), we achieve

$$Z(\rho,U) = (n-1+\phi)\pi(U), \tag{34}$$

which implies that $(n-1+\phi)$ is an eigenvalue of the Z tensor corresponding to the eigenvector ρ.

2. Considering (13), if (34) is substituted in (33), one can easily obtain

$$Z(X,U) = -\,(\nabla_X \phi)\,\pi(U) + (n-1+\phi)\,\{2\pi(X)\pi(U) - g(X,U)\}\,. \tag{35}$$

When ρ is written instead of U in the last result, the following holds:

$$(\nabla_X \phi) = 4\,(n-1+\phi)\,\pi(X).$$

3. From (29) and (35), it follows immediately that,

$$Z(X,U) = (1-n-\phi)\,\{2\pi(X)\pi(U) + g(X,U)\}\,.$$

4. Considering (32) and (30), we obtain

$$\bar{Z}(X,U) = 2\,(1-n-\phi)\,\pi(X)\pi(U) + \left\{\bar{\phi} + 2\,(1-n-\phi)\right\} g(X,U).$$

5. By substituting $X = \rho$ in the previous equation, the following relation is obtained

$$\bar{Z}(\rho,U) = \bar{\phi}\pi(U). \tag{36}$$

Hence, we obtain that $\bar{\phi}$ is an eigenvalue of the $\bar{Z}$ tensor corresponding to the eigenvector ρ. □

Remark 1 If ϕ is constant, from (29), we obtain that it must be $\phi = 1-n$ because of $\pi \neq 0$. Then, from (30), it follows that $Z = 0$. Therefore, in this study, we will consider in case where the scalar function ϕ is not constant.

Theorem 3 *For $(P\bar{Z}SL)_n$ manifolds, we have*

1. *Such a manifold is a Lorentzian quasi-Einstein manifold, i.e., a perfect fluid space-time.*
2. *The expressions of the scalar curvatures r and $\bar{r}$ of such a manifold are in the following forms, respectively,*

$$r = (1-n)\,(n-2+2\phi)\,, \tag{37}$$

$$\bar{r} = 2(n-1)\,(1-n-\phi)\,. \tag{38}$$

Proof *1.* Inserting (1) in (30), one straightforwardly obtains

$$S(X,U) = (1-n-2\phi)\,g(X,U) + 2\,(1-n-\phi)\,\pi(X)\pi(U), \tag{39}$$

which implies that such a manifold is a Lorentzian quasi-Einstein manifold.

2. Contracting of (39), we achieve

$$r = (1-n)\,(n-2+2\phi)\,.$$

Similarly, inserting (28) in (31), one easily obtains

$$\bar{S}(X,U) = 2\,(1-n-\phi)\,\{g(X,U) + \pi(X)\pi(U)\}\,. \tag{40}$$

Contracting of (40), we achieve

$$\bar{r} = 2(n-1)\,(1-n-\phi)\,.$$

□

In 1994, Gebarovski (see, [12, 13]) showed that for a GRW space-time which is the warped product $-I \times f^2 M^*$, the fibres are Einstein if and only if $\nabla_m C^m_{jkl} = 0$. In 1999, Sánchez [23] proved that a GRW space-time is a perfect fluid if and only if M^* is an Einstein manifold. Based on these studies, we can say that a GRW space-time is a perfect fluid if and only if $\nabla_m C^m_{jkl} = 0$. Moreover, we know that an n-dimensional Lorentzian manifold with semi-symmetric metric ρ-connection, $n \geq 3$, is a GRW space-time.

By using the argument given above and Theorem 3, we can state the following theorem.

Lemma 4 *A $(PZ\bar{S}L)_n$ manifold has the property div $C = 0$.*

Theorem 5 *For the $(PZ\bar{S}L)_n$ manifold, the Ricci tensor S is R-compatible.*

Proof It is well known that in local coordinates, the divergence of the Weyl conformal tensor satisfies the relation;

$$\nabla_m C^m_{jkl} = \frac{n-3}{n-2}\left[\nabla_m R^m_{jkl} + \frac{1}{2(n-1)}\left\{(\nabla_j r) g_{kl} - (\nabla_k r) g_{jl}\right\}\right]. \tag{41}$$

Since div $C = 0$ according to Lemma 4, one easily obtains

$$\nabla_m R^m_{jkl} = \nabla_j S_{kl} - \nabla_k S_{jl} = \frac{1}{2(n-1)}\left\{(\nabla_k r) g_{jl} - (\nabla_j r) g_{kl}\right\}. \tag{42}$$

By using (29) and (37) after a straightforward calculation, one obtains the following result:

$$\nabla_m R^m_{jkl} = 4(1-n+\phi)\left[\pi_j g_{kl} - \pi_k g_{jl}\right]. \tag{43}$$

By performing the covariant derivative of (43) and using (13) and (29), one easily obtains

$$\begin{aligned}\nabla_i \nabla_m R^m_{jkl} &= 4(1-n+\phi)\pi_i\left[\pi_j g_{kl} - \pi_k g_{jl}\right] \\ &\quad + 4(1-n+\phi)\left[\pi_i \pi_j g_{kl} + g_{ij} g_{kl} - \pi_i \pi_k g_{jl} - g_{ik} g_{jl}\right].\end{aligned}$$

Now a cyclic permutation of the indices i, j, k is performed and the resulting three equations are added to obtain

$$\nabla_i \nabla_m R^m_{jkl} + \nabla_j \nabla_m R^m_{kil} + \nabla_k \nabla_m R^m_{ijl} = 0, \tag{44}$$

which means that according to [16], the Ricci tensor S is R-compatible (Riemann compatible). □

In [16], Mantica and Molinari showed that

$$\nabla_i \nabla_m R^m_{jkl} + \nabla_j \nabla_m R^m_{kil} + \nabla_k \nabla_m R^m_{ijl} = -S_{im} R^m_{jkl} - S_{jm} R^m_{kil} - S_{km} R^m_{ijl}. \tag{45}$$

Since the Ricci tensor S is R-compatible in $(P\bar{Z}SL)_n$ according to Theorem 5, by considering (44) and (45), we can write easily

$$S_{im}R^m_{jkl} + S_{jm}R^m_{kil} + S_{km}R^m_{ijl} = 0. \tag{46}$$

By putting (19) and (21) in (46) and considering the first Bianchi identity, we get

$$\bar{S}_{im}\bar{R}^m_{jkl} + \bar{S}_{jm}\bar{R}^m_{kil} + \bar{S}_{km}\bar{R}^m_{ijl} = 0. \tag{47}$$

The last equation means that the Ricci tensor $\bar{S}$ with respect to the connection $\bar{\nabla}$ is $\bar{R}$-compatible in $(P\bar{Z}SL)_n$.

Conversely, let the Ricci tensor $\bar{S}$ with respect to the connection $\bar{\nabla}$ be $\bar{R}$-compatible in $(P\bar{Z}SL)_n$. Then, substituting (19) and (21) in (47) and considering the first Bianchi identity, we obtain (46). This implies that the Ricci tensor S is R-compatible in $(P\bar{Z}SL)_n$.

Thus, considering Theorem 5 as well as the above discussion, we deduce the following corollary:

Corollary 1 *For the $(P\bar{Z}SL)_n$ manifold, the following are equivalent:*

1. *The Ricci tensor S is R-compatible.*
2. *The Ricci tensor $\bar{S}$ with respect to the connection $\bar{\nabla}$ is $\bar{R}$-compatible.*

Theorem 6 *For $(P\bar{Z}SL)_n$ manifold, $\pi(X)$ is R-compatible.*

Proof In local coordinates, the integrability conditions of (13) are

$$\pi_m R^m_{jkl} = \pi_j g_{kl} - \pi_k g_{jl}. \tag{48}$$

After a straightforward calculation, we achieve

$$\pi_i\pi_m R^m_{jkl} + \pi_j\pi_m R^m_{kil} + \pi_k\pi_m R^m_{ijl} = 0. \tag{49}$$

Based on [16], the above result means that $\pi(X)$ is R-compatible. □

According to [16], a vector field u is Riemannian compatible if and only if it is Weyl compatible and $u_{[a}R^m_{b]}u_m = 0$. Hence $\pi(X)$ is also W-compatible (Weyl compatible). Moreover, based on [6], let M be an n-dimensional Lorentzian manifold endowed with a semi-symmetric metric ρ connection $\bar{\nabla}$, then the Weyl conformal curvature tensors $\bar{C}$ and C with respec to $\bar{\nabla}$ and ∇, respectively, coincide on M. Therefore, we obtain the following corollary:

Corollary 2 *$\pi(X)$ is W-compatible ($\bar{W}$-compatible).*

Theorem 7 *For the $(P\bar{Z}SL)_n$ manifold, $\pi(X)$ is $\bar{R}$-compatible.*

Proof Let us now consider (19) in local coordinates. Thus, we can write

$$\bar{R}^m_{jkl} = R^m_{jkl} - \left\{ g_{kl}\delta^m_j - g_{jl}\delta^m_k \right\}.$$

The last equation is then transvected with π_m so that the following holds:

$$\pi_m \bar{R}^m_{jkl} = \pi_m R^m_{jkl} - \left\{ g_{kl}\pi_j - g_{jl}\pi_k \right\}. \tag{50}$$

Now (48) is substituted in (50) to give

$$\pi_m \bar{R}^m_{jkl} = 0. \tag{51}$$

Therefore, we achieve

$$\pi_i \pi_m \bar{R}^m_{jkl} + \pi_j \pi_m \bar{R}^m_{kil} + \pi_k \pi_m \bar{R}^m_{ijl} = 0. \tag{52}$$

Thus we conclude that $\pi(X)$ is $\bar{R}$-compatible. □

When it is considered the above operations, we obtain the following corollary:

Corollary 3 *For the $(P\bar{Z}SL)_n$ manifold, $\pi(X)$ is R-compatible if and only if $\pi(X)$ is $\bar{R}$-compatible.*

Theorem 8 *For the $(P\bar{Z}SL)_n$ manifold, $\phi \neq 1 - n$, the Z tensor is R-compatible.*

Proof In local coordinates, let us rewrite (30). Then, we achieve

$$Z_{im} = (1 - n - \phi)\left\{2\pi_i \pi_m + g_{im}\right\}.$$

Hence, since ϕ is not constant, $\phi \neq 1 - n$,

$$\pi_i \pi_m = \frac{1}{2(1 - n - \phi)} Z_{im} + \frac{1}{2} g_{im}. \tag{53}$$

Considering that g is R-compatible because of the first Bianchi identity and using (53) in (49), we achieve

$$Z_{im} R^m_{jkl} + Z_{jm} R^m_{kil} + Z_{km} R^m_{ijl} = 0. \tag{54}$$

This means that the Z tensor is R-compatible. □

Theorem 9 *For the $(P\bar{Z}SL)_n$ manifold, $\phi \neq 1 - n$, the $\bar{Z}$ tensor is $\bar{R}$-compatible.*

Proof Let us consider (31) in local coordinates and use the same procedure as the above proof. Thus, we obtain, $\phi \neq 1-n$,

$$\pi_i \pi_m = \frac{1}{2(1-n-\phi)} \bar{Z}_{im} - \left\{ \frac{\bar{\phi}}{2(1-n-\phi)} + 1 \right\} g_{im}. \tag{55}$$

Now (55) is substituted in (52) to give

$$\bar{Z}_{im} \bar{R}^m_{jkl} + \bar{Z}_{jm} \bar{R}^m_{kil} + \bar{Z}_{km} \bar{R}^m_{ijl} = 0, \tag{56}$$

in which we use that g is R-compatible. □

Based on [18], a symmetric tensor is R-compatible if and only if it is W-compatible and it commutes with the Ricci tensor. Thus, we can express the following conclusion by considering that C and $\bar{C}$ coincide and taking into account the last two theorems.

Corollary 4 *The Z tensor and the $\bar{Z}$ tensor are W-compatible ($\bar{W}$-compatible).*

4 Application to general relativity

In general relativity, it is described the matter content of space-time and is specified the physical aspects of space-time by the energy-momentum tensor T. The substance content is supposed to be a liquid with density and pressure, with dynamic and kinematic quantities.

The geometry of space-time is controlled by the Ricci tensor S for the reason that the Ricci tensor S and the energy-momentum tensor T are related to each other by the Einstein equations without the cosmological constant

$$S(X, Y) = \frac{r}{2} g(X, Y) + kT(X, Y), \tag{57}$$

where k is non-zero gravitational constant (see, [21, 22]).

Now, let us consider a $(PZ\overline{S}L)_n$ space-time obeying Einstein's field equation without the cosmological constant. By using Theorem 5, (37), (57) and the first Bianchi identity one gets

$$T_{im} R^m_{jkl} + T_{jm} R^m_{kil} + T_{km} R^m_{ijl} = 0. \tag{58}$$

Thus, we can state the following

Theorem 10 *For the $(PZ\overline{S}L)_n$ space-time, the energy-momentum tensor T is R-compatible.*

Based on [18] and the above theorem, we can express that in the $(PZ\overline{S}L)_n$ manifold, the energy-momentum tensor T is W-compatible.

According to [1] and [25], for a 4-dimensional space-time, given a time-like velocity field u^i with $u^i u_i = -1$, the electric and magnetic parts of Weyl tensors are as

$$E_{kl} = u^j u^m C_{jklm},$$
$$H_{kl} = \frac{1}{2}\epsilon_{jkrs} u^j u^m C^{rs}_{lm},$$

where C^{rs}_{lm} is the component of type (2, 2) of the Weyl tensor, ϵ_{jkrs} is a completely skew-symmetric Levi-Civita tensor.

In [25], it is expressed that on a 4-dimensional space-time a time-like vector field is Weyl-compatible if and only if the magnetic part of the Weyl tensor vanishes. Thus, considering Corollary 2, we can state that the magnetic component of the Weyl tensor is equal to zero. Hence, such a space-time is called purely electric space-time. We can express the above discussion as follows:

Theorem 11 *The* $(P\bar{Z}SL)_4$ *space-time is purely electric space-time.*

In [25], it is shown that a 4-dimensional space-time with a Weyl-compatible time-like vector is type I, D or O. Thus, taking into account of Corollary 2, we conclude that

Theorem 12 *A* $(P\bar{Z}SL)_4$ *space-time is of Petrov types I, D or O.*

In [2], Capozziello et al obtain that a 4-dimensional generalized Robertson–Walker space-time is a perfect fluid space-time if and only if it is a Robertson–Walker space-time. Thus, taking into account that an n-dimensional Lorentzian manifold with semi-symmetric metric ρ-connection, $n \geq 3$, is a GRW space-time and Theorem 3, we deduce that:

Theorem 13 *A* $(P\bar{Z}SL)_4$ *space-time is a Robertson-Walker space-time.*

Guilfoyle and Nolan [14] showed that a 4-dimensional perfect fluid space-time (M, g) with $\sigma + p \neq 0$ is a Yang pure space if and only if (M, g) is a Robertson–Walker space-time. Thus, considering the above theorem, we conclude that the following theorem.

Theorem 14 *A* $(P\bar{Z}SL)_4$ *space-time is a Yang pure space.*

The energy momentum tensor T of type (0, 2), in a perfect fluid space-time, is given by [21]:

$$T(X, Y) = pg(X, Y) + (\sigma + p)\pi(X)\pi(Y), \tag{59}$$

where p and σ are the isotropic pressure and the energy density, respectively. Putting $Y = \rho$ in (39) and (57), we achieve

$$\left\{\frac{r}{2} - \sigma k - 3\right\}\pi(X) = 0. \tag{60}$$

From the above equation, one immediently obtains

$$\sigma = \frac{1}{k}\left\{\frac{r}{2} - 3\right\}. \tag{61}$$

Now, contracting over (59), we achieve

$$T = -\sigma + 3p. \tag{62}$$

Contracting (57) and using (62) gives

$$p = \frac{1}{3k}\{k\sigma - r\}. \tag{63}$$

(61) is substituted in the last result to give

$$p = \frac{1}{3k}\left\{-\frac{r}{2} - 3\right\}. \tag{64}$$

The last equation and (61) lead to

$$3p + \sigma = -\frac{6}{k}. \tag{65}$$

Hence we deduce that:

Theorem 15 *A $(P\bar{Z}SL)_4$ space-time satisfies the state equation $p = -\frac{1}{3}(\sigma + \frac{6}{k})$.*

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Declarations

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Ethical Approval The paper is a theoretical study. There are no applicable neither human or animal studies.

References

1. Bertschinger, E., Hamilton, A.J.S.: Lagrangian evolution of the Weyl tensor. Astroph. J. Geol. **435**, 1–7 (1994)
2. Capozziello, S., Mantica, C.A., Molinari, L.G.: Cosmological perfect uids in f(R)-gravity. Int. J. Geom. Methods Mod. Phys. **16**, 1950008 (2019)
3. Chaki, M.C.: On quasi Einstein manifolds. Publ. Math. Debr. **57**, 297–306 (2000)
4. Chen, Bang-Yen.: A simple characterization of generalized Robertson–Walker spacetimes. Gen. Relat. Gravit. **46**, 1–5 (2014)

5. Chaubey, S.K., De, U.C., Siddiqi, M.D.: Characterization of Lorentzian manifolds with a semi-symmetric linear connection. J. Geom. Phys. **166**, 104269 (2021)
6. Chaubey, S.K., Suh, Y.J., De, U.C.: Characterizasyons of the Lorentzian manifolds admitting a type of semi-symmetric metric connection. Anal. Math. Phys. **10**(4), 61 (2020)
7. Defever, F., Deszcz, R.: On semi-Riemannian manifolds satisfying the condition R. R = Q (S, R). Geom. Topol. Submanifolds **3**, 108–130 (1991)
8. Deszcz, R., Hotlos, G.M., Senturk, Z.: On certain quasi-Einstein semisymmetric hypersurfaces. Ann. Sci. Budapest. Eotovos Sect. Math. **41**, 151–164 (1998)
9. Deszcz, R., Dillen, F., Verstraelen, L., Vrancken, L.: Quasi-Einstein totally real submanifolds of the nearly Kahler 6-sphere. Tohoku Math. J. Second Ser. **51**, 461–478 (1999)
10. Fasihi-Ramandi, G.: Semi-symmetric connection formalism for unification of gravity and electromagnetism. J. Geom. Phys. **144**, 245–250 (2019)
11. Friedmann, A., Schouten, J.A.: Über die Geometrie der halbsymmetrischen Übertragungen. Math. Zeitschr. **21**(1), 211–223 (1924)
12. Gebarowski, A.: On nearly conformally symmetric warped product spacetimes. Soochow J. Math. **20**(1), 61–75 (1994)
13. Gebarowski, A.: Doubly warped products with harmonic Weyl conformal curvature tensor. Colloq. Math. **67**, 73–89 (1994)
14. Guilfoyle, B.S., Nolan, B.C.: Yang's gravitational theory. Gen. Relat. Gravit. **30**(3), 473–495 (1998)
15. Hayden, H.A.: Subspaces of a space with torsion. Proc. Lond. Math. Soc. **34**, 27–50 (1932)
16. Mantica, C.A., Molinari, L.G.: Riemann compatible tensors. Colloq. Math. **128**, 197–210 (2012)
17. Mantica, C.A., Molinari, L.G.: Weakly Z-symmetric manifolds. Acta Math. Hungar. **135**(1–2), 80–96 (2012)
18. Mantica, C.A., Molinari, L.G.: Weyl compatible tensors. Int. J. Geom. Methods Mod. Phys. **11**(08), 1450070 (2014)
19. Mantica, C.A., Suh, Y.J.: Pseudo Z-symmetric Riemannian manifolds with harmonic curvature tensors. Int. J. Geom. Methods Mod. Phys. **9**(1), 1250004 (2012)
20. Mantica, C.A., Suh, Y.J.: Pseudo Z-symmetric space-times. J. Math. Phys. **55**, 042502 (2014)
21. O'neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press, California (1983)
22. Petrov, A.Z.: Einstein Spaces. Pergamon Press, Oxford (1969)
23. Sánchez, M.: On the geometry of generalized Robertson–Walker spacetimes: curvature and Killing fields. Gen. Relat. Gravit. **31**, 1–15 (1999)
24. Schouten, J.A.: Ricci-Calculus. An Introduction to Tensor Analysis and Its Geometrical Applications. Springer, Berlin (1954)
25. Stephani, H., Kramer, D., Mac Callum, M., Hoenselaers, C., Hertl, H.: Exact Solutions of Einstein's Field Equations. Cambridge University Press, London (2003)
26. Singh, H., Khan, Q.: On generalized recurrent Riemannian manifolds. Publ. Math. Debr. **56**(1–2), 87–95 (2000)
27. Takeno, H.: Concircular Scalar Fields in Spherically Symmetric Space–Times. I. Hiroshima University, Hiroshima (1969)
28. Yano, K.: Concircular Geometry. Proc. Imp. Acad.,Tokyo, 16, 195–200, 354–360, 442–448, 505–511 (1940)
29. Yano, K.: On semi-symmetric metric connections. Rev. Roumanie Math. Pures Appl. **15**, 1579–1586 (1970)
30. Yano, K.: On Torse-forming Directions in Riemannian Spaces. Proc. Imp. Acad, Tokyo, 20, 701–705 (1994)
31. Yano, K., Kon, M.: Structure on Manifolds. Series in Pure Math. World Scientific, Singapore (1984)