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On graded strongly quasi primary ideals

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Abstract. In this article, we introduce graded strongly quasi primary ideals which is an intermediate class of graded primary ideals and graded quasi primary ideals. Let G be a group with identity e, R be a G-graded commutative ring with nonzero unity 1 and P be a proper graded ideal of R. Then P is said to be a graded strongly quasi primary ideal if $xy \in P$ for $x, y \in h(R)$ implies either $x^2 \in P$ or $y^n \in P$ ($x^n \in P$ or $y^2 \in P$) for some $n \in \mathbb{N}$. We give many properties of graded strongly quasi primary ideals and investigate the relations between graded strongly quasi primary ideals and other classical graded ideals such as graded primary, graded 2-prime and graded quasi primary ideals.

Key Words: Graded strongly quasi primary ideals; graded primary ideals; graded 2-prime ideals; graded quasi primary ideals.
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1 Introduction

Throughout this article, G will be a group with identity e and R a commutative ring with nonzero unity 1. R is said to be G-*graded* if $R = \bigoplus_{g\in G} R_g$ with $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$ where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Also, we set $h(R) = \bigcup_{g\in G} R_g$. The support of (R, G) is defined as $supp(R, G) = \left\{g \in G : R_g \neq \{0\}\right\}$. Moreover, it has been proved in [7] that R_e is a subring of R and $1 \in R_e$. Let P be an ideal of a graded ring R. Then P is said to be a graded ideal if $P = \bigoplus_{g\in G}(P \cap R_g)$, i.e., for $x \in P$, $x = \sum_{g\in G} x_g$ where $x_g \in P$ for all $g \in G$. It is known that an ideal of a graded ring need not be graded. Let R be a G-graded ring and P be a graded ideal of R. Then R/P is G-graded by $(R/P)_g = (R_g + P)/P$ for all $g \in G$. If R and S are G-graded rings, then $R \times S$ is a G-graded ring by $(R \times S)_g = R_g \times S_g$ for all $g \in G$.

Lemma 1.1. *([4], Lemma 2.1) Let R be a G-graded ring.*

1. *If P and Q are graded ideals of R, then $P + Q$, PQ and $P \bigcap Q$ are graded ideals of R.*

2. *If $x \in h(R)$, then Rx is a graded ideal of R.*

Let P be a proper graded ideal of R. Then the graded radical of P is denoted by $Grad(P)$ and it is defined as follows:
$Grad(P) = \left\{ x = \sum_{g \in G} x_g \in R : \text{ for all } g \in G, \text{ there exists } n_g \in \mathbb{N} \text{ such that } x_g^{n_g} \in P \right\}$. Note that $Grad(P)$ is always a graded ideal of R (see [9]).

Graded prime ideals have been introduced and studied in [9]. A proper graded ideal P of a graded ring R is said to be a graded prime if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in P$. In 2004, graded primary ideals have been introduced and studied in [8]. A proper graded ideal P of R is said to be a graded primary if for $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in Grad(P)$. In 2020, graded quasi primary ideals have been introduced in [2]. A proper graded ideal P of R is said to be a graded quasi primary ideal if $xy \in P$ for $x, y \in h(R)$ implies $x \in Grad(P)$ or $y \in Grad(P)$. Note that the class of graded quasi primary ideals properly contains the class of graded primary ideals.

The concept of graded prime ideals and its generalizations have an outstanding location in graded commutative algebra. They are valuable tools to determine the properties of graded commutative rings. Various generalizations of graded prime ideals have been studied. Indeed, a nonzero proper graded ideal P of R is called a graded 2-prime ideal if whenever $x, y \in h(R)$ and $xy \in P$, then $x^2 \in P$ or $y^2 \in P$.

Our goal in this article is following [6] to introduce an intermediate class of graded ideals between graded primary ideals and graded quasi primary ideals. A proper graded ideal P of R is said to be a graded strongly quasi primary ideal if $xy \in P$ for $x, y \in h(R)$ implies either $x^2 \in P$ or $y^n \in P$ ($x^n \in P$ or $y^2 \in P$) for some $n \in \mathbb{N}$. We give many properties of graded strongly quasi primary ideals and investigate the relations between graded strongly quasi primary ideals and other classical graded ideals such as graded primary, graded 2-prime and graded quasi primary ideals.

Among several results, we show that the location of graded strongly quasi primary ideals is described as follows:

Graded primary ideals $\Rightarrow$ **graded strongly quasi primary ideals** $\Rightarrow$ graded quasi primary ideals.

Graded 2-prime ideals $\Rightarrow$ **graded strongly quasi primary ideals** $\Rightarrow$ graded quasi primary ideals.

And the converse of each implication is not true in general (Example 2.4, Example 2.5 and Example 2.4). On the other hand, we prove that if $(Grad(P))^2 \subseteq P$, then graded quasi primary, graded 2-prime and graded strongly quasi primary ideals are equivalent (Corollary 2.3). In Proposition 2.9, we show that graded strongly quasi primary ideals have a similar property as that of graded primary ideals in [[8], Proposition 1.15 (ii)]. In Proposition 2.10, we study graded strongly quasi primary ideals under graded homomorphisms. Also, we study graded strongly quasi primary ideals of $S^{-1}R$ where S is a multiplicatively closed subset of $h(R)$ (Proposition 2.17). Furthermore, we prove that if R is a graded domain, then every proper graded ideal of R is a graded strongly quasi primary ideal of R if and only if for any $x, y \in h(R)$, either y divides x or x divides y^n for some $n \in \mathbb{N}$ (Proposition 2.22). Finally, we study graded strongly quasi primary ideals over idealization (Proposition 2.24 and Proposition 2.25).

2 Graded strongly quasi primary ideals

In this section, we introduce and study the concept of graded strongly quasi primary ideals.

Definition 2.1. Let R be a G-graded ring and P be a proper graded ideal of R. Then P is said to be a graded strongly quasi primary ideal if $xy \in P$ for $x, y \in h(R)$ implies either $x^2 \in P$ or $y^n \in P$ ($x^n \in P$ or $y^2 \in P$) for some $n \in \mathbb{N}$.

Proposition 2.2. *Let R be a G-graded ring and P be a proper graded ideal of R.*

1. *If P is a graded primary ideal of R, then P is a graded strongly quasi primary ideal of R.*

2. *If P is a graded 2-prime ideal of R, then P is a graded strongly quasi primary ideal of R.*

3. *If P is a graded strongly quasi primary ideal of R, then P is a graded quasi primary ideal of R.*

Proof. (1) and (2) are obvious. For (3): let $x, y \in h(R)$ such that $xy \in P$. Since P is graded strongly quasi primary ideal of R, $x^2 \in P$ or $y^n \in P$ for some $n \in \mathbb{N}$. Then we get $x \in Grad(P)$ or $y \in Grad(P)$, as desired. □

Corollary 2.3. *Let R be a G-graded ring and P be a proper graded ideal of R. If $(Grad(P))^2 \subseteq P$, then the following statements are equivalent:*

1. *P is a graded quasi primary ideal of R.*

2. *P is a graded 2-prime ideal of R.*

3. *P is a graded strongly quasi primary ideal of R.*

Proof. (1) $\Rightarrow$ (2): Let $x, y \in h(R)$ such that $xy \in P$. Since P is graded quasi primary, we have $x \in Grad(P)$ or $y \in Grad(P)$. As $(Grad(P))^2 \subseteq P$, we obtain that $x^2 \in (Grad(P))^2 \subseteq P$ or $y^2 \in (Grad(P))^2 \subseteq P$. Hence, P is a graded 2-prime ideal of R.

(2) $\Rightarrow$ (3): The result holds by Proposition 2.2 (2).

(3) $\Rightarrow$ (1): The result holds by Proposition 2.2 (3). □

The next example shows that the converse of Proposition 2.2 (1) is not true in general.

Example 2.4 ([6]). Let $R = \left\{a_0 + a_1X + a_2X^2 + ... + a_nX^n : a_1 \in 3\mathbb{Z} \text{ and } a_0, a_2, ..., a_n \in \mathbb{Z}\right\}$ and $G = \mathbb{Z}$. Then R is G-graded by $R_0 = \mathbb{Z}$, $R_1 = 3\mathbb{Z}X$, $R_j = \mathbb{Z}X^j$ for $j \geq 2$ and $R_j = \{0\}$ otherwise. Let $P = (9X^2, X^3, X^4, X^5, X^6)$. Then P is a graded ideal of R with $Grad(P) = (3X, X^2, X^3)$. Note that P is a graded quasi primary ideal of R and $(Grad(P))^2 = (9X^2, 3X^3, X^4, X^5, X^6) \subseteq P$. So, by Corollary 2.3, P is a graded strongly quasi primary ideal of R. On the other hand, P is not graded primary ideal of R since $X^2, 9 \in h(R)$ with $(X^2)9 = 9X^2 \in P$, but $X^2 \notin P$ and $9^n \notin P$ for all $n \in \mathbb{N}$.

Let $Q = (27X, 27X^2, X^3, X^4, X^5, X^6)$. Then Q is a graded ideal of R with $Grad(Q) = (3X, X^2, X^3)$. Note that Q is a graded quasi primary ideal of R which is not graded strongly quasi primary ideal of R since $3X, 9 \in h(R)$ with $(3X)9 = 27X \in Q$, but $(3X)^2 = 9X^2 \notin Q$ and $9^n \notin Q$ for all $n \in \mathbb{N}$.

The next example shows that the converses of Proposition 2.2 (2) and (3) are not true in general.

Example 2.5. Consider $R = K[X, Y]$, where K is a field, and $G = \mathbb{Z}$. Then R is G-graded by $R_n = \bigoplus_{i+j=n, i,j \geq 0} KX^iY^j$ for all $n \in \mathbb{Z}$. Note that $deg(X) = deg(Y) = 1$. Let $P = (X^3, XY, Y^3)$. Then P is a graded ideal of R with $Grad(P) = (X, Y)$ is a graded maximal ideal of R, and so P is a graded primary ideal of R by ([8], Proposition 1.11). Hence, by Proposition 2.2 (1), P is a graded strongly quasi primary ideal of R. On the other hand, P is not graded 2-prime ideal of R since $X, Y \in h(R)$ with $XY \in P$, but $X^2 \notin P$ and $Y^2 \notin P$.

Let P be a proper graded ideal of R. Then the graded ideal generated by n-th powers of homogeneous elements of P is denoted by $P_n = \{a^n : a \in h(R) \bigcap P\}$. It is easy to note that $P_n \subseteq P^n \subseteq P$ and also the equality holds if $n = 1$. Indeed, if $a \in P$, then $a = \sum_{g\in G} a_g$ where $a_g \in R_g \bigcap P$ for all $g \in G$ as P is graded, and then $a_g \in h(R) \bigcap P$ for all $g \in G$, which implies that $a_g \in P_1$ for all $g \in G$, and then $a \in P_1$. So, $P \subseteq P_1 \subseteq P$, which means that $P_1 = P$.

Remark 2.6. In [1], the definition of P_n was as follows: $P_n = \{a^n : a \in P\}$, and it has been proved that if $n!$ is a unit in R, then $P_n = P^n$ [[1]), Theorem 5]. In fact, the next example shows that this result will not be true for our definition of P_n.

Example 2.7. Consider $R = \mathbb{Q}[X, Y]$ and $G = \mathbb{Z}$. Then R is graded by $R_n = \bigoplus_{i+j=n,i,j\geq 0} \mathbb{Q}X^iY^j$ for all $n \in \mathbb{Z}$. Then $P = \langle X, Y^2\rangle$ is a graded ideal of R. However, $P_2 = \langle X^2, Y^4\rangle \subsetneq \langle X^2, XY^2, Y^4\rangle = P^2$.

Proposition 2.8. *Let R be a G-graded ring and P be a proper graded ideal of R. Then the following statements are equivalent.*

1. *P is a graded strongly quasi primary ideal of R.*
2. *For every $a \in h(R)$, either $(a) \subseteq (P : a)$ or $(P : a) \subseteq Grad(P)$.*
3. *For any graded ideals I and J of R with $IJ \subseteq P$, either $I_2 \subseteq P$ or $J \subseteq Grad(P)$.*
4. *For every $a \in h(R)$, either $a^n \in P$ for some $n \in \mathbb{N}$ or $(P : a)_2 \subseteq P$.*

Proof. (1) $\Rightarrow$ (2): Let $a \in h(R)$. If $a^2 \in P$, then $(a) \subseteq (P : a)$. Suppose that $a^2 \notin P$. Let $b \in (P : a)$. Since $(P : a)$ is a graded ideal of R, $b_g \in (P : a)$ for all $g \in G$, and then $ab_g \in P$ for all $g \in G$. Since P is a graded strongly quasi primary ideal and $a^2 \notin P$, we have $b_g \in Grad(P)$ for all $g \in G$, which implies that $b \in Grad(P)$. So, $(P : a) \subseteq Grad(P)$.

(2) $\Rightarrow$ (3): Suppose that $J \nsubseteq Grad(P)$. Then there exists $b \in J - Grad(P)$, and then there exists $g \in G$ such that $b_g \in J - Grad(P)$. Let $x \in I$. Then $x_h \in I$ for all $h \in G$ as I is graded, and then $x_hb_g \in P$ for all $h \in G$. Since $b_g \in (P : x_h) - Grad(P)$ for all $h \in G$, we have $(P : x_h) \nsubseteq Grad(P)$ for all $h \in G$. Then by (2), we have $(x_h) \subseteq (P : x_h)$ for all $h \in G$, and so $x_h^2 \in P$ for all $h \in G$. Hence, $I_2 \subseteq P$.

(3) $\Rightarrow$ (4): Let $a \in h(R)$. If $a \in Grad(P)$, then $a^n \in P$ for some $n \in \mathbb{N}$. Suppose that $a \notin Grad(P)$. Let $I = (P : a)$ and $J = (a)$. Then I and J are graded ideals of R with $IJ = (P : a)(a) \subseteq P$. Since $J \nsubseteq Grad(P)$, by (3), we have $I_2 = (P : a)_2 \subseteq P$.

(4) $\Rightarrow$ (1): Let $a, b \in h(R)$ with $ab \in P$ and $b \notin Grad(P)$. Then $a \in (P : b)$, and so by (4), $a^2 \in (P : b)_2 \subseteq P$. Hence, P is a graded strongly quasi primary ideal of R. □

In the next result, we show that graded strongly quasi primary ideals have a similar property as that of graded primary ideals in [[8], Proposition 1.15 (ii)].

Proposition 2.9. *Let P be a graded strongly quasi primary ideal of R and $a \in h(R)$ such that $(a) = (a^2)$. If $a \notin P$, then $(P : a)$ is a graded strongly quasi primary ideal of R.*

Proof. First note that $(P : a)$ is a graded ideal of R. Since $a \notin (P : a)$, by Proposition 2.8, $Grad((P : a)) = Grad(P)$. Let $x, y \in h(R)$ such that $xy \in (P : a)$ and $y^n \notin (P : a)$ for all $n \in \mathbb{N}$. Then $(ax)y \in P$ and $y^n \notin P$ for all $n \in \mathbb{N}$. Since P is a graded strongly quasi primary ideal of R, $(ax)^2 = x^2a^2 \in P$, and so $x^2 \in (P : a^2) = (P : a)$. Hence, $(P : a)$ is a graded strongly quasi primary ideal of R. □

Let R and S be two G-graded rings. A ring homomorphism $f : R \to S$ is said to be a graded homomorphism if $f(R_g) \subseteq S_g$ for all $g \in G$ (see [7]).

Proposition 2.10. *Let $f : R \to S$ be a graded ring homomorphism with $f(1_R) = 1_S$.*

1. *If f is a graded epimorphism and P is a graded strongly quasi primary ideal of R containing $Ker(f)$, then $f(P)$ is a graded strongly quasi primary ideal of S.*

2. *If I is a graded strongly quasi primary ideal of S, then $f^{-1}(I) = R$ or $f^{-1}(I)$ is a graded strongly quasi primary ideal of R.*

Proof. 1. Clearly, $f(P)$ is a graded ideal of S. Let $xy \in f(P)$, where $x, y \in h(S)$. Then $x = f(a), y = f(b)$ for some $a, b \in h(R)$, and so $f(ab) = f(a)f(b) \in f(P)$. As $Ker(f) \subseteq P$, we have $ab \in P$. Since P is a graded strongly quasi primary ideal, we have either $a^2 \in P$ or $b^n \in P$ for some $n \in \mathbb{N}$, and then $x^2 \in f(P)$ or $y^n \in f(P)$. Hence, $f(P)$ is a graded strongly quasi primary ideal of S.

2. Clearly, $f^{-1}(I)$ is a graded ideal of R. Let $ab \in f^{-1}(I)$ for some $a, b \in h(R)$. Then $f(ab) = f(a)f(b) \in I$, which implies that $(f(a))^2 = f(a^2) \in I$ or $(f(b))^n = f(b^n) \in I$ for some $n \in \mathbb{N}$, and so $a^2 \in f^{-1}(I)$ or $b^n \in f^{-1}(I)$. Hence, $f^{-1}(I)$ is a graded strongly quasi primary ideal of R.

□

Corollary 2.11. *Let R be a G-graded ring and I be a proper graded ideal of R.*

1. *Let P be a graded ideal of R containing I. Then, P is a graded strongly quasi primary ideal of R if and only if P/I is a graded strongly quasi primary ideal of R/I.*

2. *If P is a graded strongly quasi primary ideal of R and S is a graded subring of R with $S \nsubseteq P$, then $S \bigcap P$ is a graded strongly quasi primary ideal of S.*

Proof. 1. Consider the graded natural homomorphism $f : R \to R/I$ defined by $f(x) = x + I$ for each $x \in R$. Then the result holds by Proposition 2.10.

2. Consider the graded homomorphism $f : S \to R$ defined by $f(x) = x$ for each $x \in S$. Then the result hods by Proposition 2.10 (2).

□

Let R be a G-graded ring. Then $R[X]$ is G-graded by $(R[X])_g = R_g[X]$ for all $g \in G$. Note that $X = 1.X \in R_eX \subseteq R_e[X] = (R[X])_e$, and so $X \in h(R[X])$, and then (X) is a graded ideal of $R[X]$. Moreover, if P is a graded ideal of R, then (P, X) is a graded ideal of $R[X]$.

Corollary 2.12. *Let R be a G-graded ring and P be a proper graded ideal of R. Then the following statements are equivalent.*

1. *P is a graded strongly quasi primary ideal of R.*

2. *(P, X) is a graded strongly quasi primary ideal of $R[X]$.*

Proof. The result holds by Corollary 2.11 (1) and from the isomorphism $(P, X)/(X) \cong P$ in $R[X]/(X) \cong R$. □

Let R be a ring and P a proper ideal of R. For any $f(X) = a_0 + a_1X + ... + a_nX^n \in R[X]$, the content $c(f)$ of f is defined as $c(f) = (a_0, a_1, ..., a_n)$. Also, note that $P[X] = \{f \in R[X] : c(f) \subseteq P\}$ is an ideal of $R[X]$. Moreover, for any $f(X) = \sum_{i=0}^{\infty} a_iX^i \in R[[X]]$, the content $c(f)$ of f is defined by $c(f) = (a_i : i \in N)$. Furthermore, $P[[X]] = \{f = \sum_{i=0}^{\infty} a_iX^i \in R[[X]] : c(f) \subseteq P\}$ is an ideal of $R[[X]]$.

Lemma 2.13. *Let R be a G-graded ring and P be a graded ideal of R. Then $P[X]$ is a graded ideal of $R[X]$.*

Proof. Let $f(X) \in P[X]$. Then $f(X) = a_0 + a_1X + ... + a_nX^n$ for some $a_0, a_1, ..., a_n \in P$. Since P is a graded ideal, we have $(a_k)_g \in P$ for all $k = 0, 1, ..., n$ and $g \in G$, and then $(f(X))_g = (a_0)_g + (a_1)_gX + ... + (a_n)_gX^n \in P[X]$ for all $g \in G$. Hence, $P[X]$ is a graded ideal of $R[X]$. □

Proposition 2.14. *Let R be a G-graded ring and P be a proper graded ideal of R. If $P[X]$ is a graded strongly quasi primary ideal of $R[X]$, then P is a graded strongly quasi primary ideal of R.*

Proof. Consider the graded homomorphism $f : R \to R[X]$ defined by $f(x) = x$ for each $x \in R$. Then by Proposition 2.10 (2), $f^{-1}(P[X]) = P$ is a graded strongly quasi primary ideal of R. □

Proposition 2.15. *Let R be a G-graded ring and P be a proper graded ideal of R. If $P[[X]]$ is a graded strongly quasi primary ideal of $R[[X]]$, then P is a graded strongly quasi primary ideal of R.*

Proof. Choose $S = P[[X]]$ and apply Corollary 2.11.

□

Corollary 2.16. *If P is a graded strongly quasi primary ideal of R, then P_e is a strongly quasi primary ideal of R_e.*

Proof. Since R_e is a subring of R and $R_e \nsubseteq P$ as $1 \in R_e - P$, we have by Corollary 2.11 (2), $P_e = R_e \bigcap P$ is a strongly quasi primary ideal of R_e. □

Let $S \subseteq h(R)$ be a multiplicative closed set. Then $S^{-1}R$ is a graded ring with $(S^{-1}R)_g = \left\{\frac{a}{s}, a \in R_h, s \in S \cap R_{hg^{-1}}\right\}$. For a graded ideal P of R, we set $HZd(P) = \{r \in h(R) : rs \in P \text{ for some } s \in h(R) - P\}$.

Proposition 2.17. *Let R be a G-graded ring, S be a multiplicatively closed subset of $h(R)$ and P a proper graded ideal of R.*

1. *If P is a graded strongly quasi primary ideal of R with $P \bigcap S = \emptyset$, then $S^{-1}P$ is a graded strongly quasi primary ideal of $S^{-1}R$.*

2. *If $S^{-1}P$ is a graded strongly quasi primary ideal of $S^{-1}R$ with $S \bigcap HZd(P) = \emptyset$, then P is a graded strongly quasi primary ideal of R.*

Proof.
1. Let $\frac{x}{s}.\frac{y}{t} \in S^{-1}P$ for some $x, y \in h(R), s, t \in S$. Then there exists $u \in S$ such that $(ux)y \in P$. Since P is a graded strongly quasi primary ideal, we have either $(ux)^2 \in P$ or $y^n \in P$ for some $n \in \mathbb{N}$, which implies that $(\frac{x}{s})^2 = \frac{x^2}{s^2} = \frac{u^2x^2}{u^2s^2} \in S^{-1}P$ or $(\frac{y}{t})^n = \frac{y^n}{t^n} \in S^{-1}P$. Hence, $S^{-1}P$ is a graded strongly quasi primary ideal of $S^{-1}R$.

2. Let $xy \in P$ for some $x, y \in h(R)$. Then $\frac{x}{1}.\frac{y}{1} \in S^{-1}P$, so that $(\frac{x}{1})^2 = \frac{x^2}{1} \in S^{-1}P$ or $(\frac{y}{1})^n = \frac{y^n}{1} \in S^{-1}P$ for some $n \in \mathbb{N}$, which implies that $ux^2 \in P$ or $ty^n \in P$ for some $u, t \in S$. Since $S \bigcap HZd(P) = \emptyset$, we have either $x^2 \in P$ or $y^n \in P$. Hence, P is a graded strongly quasi primary ideal of R.

□

Proposition 2.18. *Let $P_1, P_2, ..., P_n$ be graded strongly quasi primary ideals of R with $Grad(P_k) = Q$ for each $k = 1, 2, ..., n$. Then $P = \bigcap_{k=1}^{n} P_k$ is a graded strongly quasi primary ideal of R.*

Proof. Let $xy \in P$ for some $x, y \in h(R)$. Suppose that $y^i \notin P$ for all $i \in \mathbb{N}$, that is, $y \notin Q$. Since $xy \in P_k$ and $y^i \notin P_k$ for all $i \in \mathbb{N}$, we have $x^2 \in P_k$, and so $x^2 \in P$. Hence, P is a graded strongly quasi primary ideal of R. □

Lemma 2.19. *Let P be an ideal of a G-graded ring R and K be an ideal of a G-graded ring S. Then $P \times K$ is a graded ideal of $R \times S$ if and only if P is a graded ideal of R and K is a graded ideal of S.*

Proof. Suppose that P is a graded ideal of R and K is a graded ideal of S. Clearly, $P \times K$ is an ideal of $R \times S$. Let $(x, y) \in P \times K$. Then $x \in P$ and $y \in K$, and since P, K are graded, $x_g \in P$ and $y_g \in K$ for all $g \in G$, which implies that $(x, y)_g = (x_g, y_g) \in P \times K$ for all $g \in G$. Hence, $P \times K$ is a graded ideal of $R \times S$. Conversely, let $x \in P$. Then $(x, 0_S) \in P \times K$, and since $P \times K$ is graded, $(x_g, 0_S) = (x, 0_S)_g \in P \times K$ for all $g \in G$, which implies that $x_g \in P$ for all $g \in G$. Hence, P is a graded ideal of R. Similarly, K is a graded ideal of S. □

Proposition 2.20. *Let R and S be G-graded rings, P be an ideal of R, K be an ideal of S, $T = R \times S$ and $Q = P \times K$. Then the following statements are equivalent.*

1. *Q is a graded strongly quasi primary ideal of T.*
2. *$P = R$ and K is a graded strongly quasi primary ideal of S or $K = S$ and P is a graded strongly quasi primary ideal of R.*

Proof. $(1) \Rightarrow (2)$: By Proposition 2.2, $Grad(Q) = Grad(P) \times Grad(K)$ is a graded prime ideal of T, and so $P = R$ or $K = S$. Without loss of generality, we may assume that $P = R$. We show that K is a graded strongly quasi primary ideal of S. By Lemma 2.19, K is a graded ideal of S. Let $xy \in K$ for some $x, y \in h(S)$. Then $(0, x), (0, y) \in h(T)$ such that $(0, x)(0, y) \in Q$. Since Q is a graded strongly quasi primary ideal of T, $(0, x)^2 = (0, x^2) \in Q$ or $(0, y)^n = (0, y^n) \in Q$ for some $n \in \mathbb{N}$, and then we obtain that $x^2 \in K$ or $y^n \in K$. Consequently, K is a graded strongly quasi primary ideal of S.

$(2) \Rightarrow (1)$: Suppose that $P = R$ and K is a graded strongly quasi primary ideal of S. By Lemma 2.19, Q is a graded ideal of T. Let $(a, b)(x, y) \in Q$ for some $(a, b), (x, y) \in h(T)$. Then $a, x \in h(R)$ and $b, y \in h(S)$ such that $by \in K$, and so either $b^2 \in K$ or $y^n \in K$ for some $n \in \mathbb{N}$, which implies that $(a, b)^2 \in Q$ or $(x, y)^n \in Q$. Hence, Q is a graded strongly quasi primary ideal of T. In other case, one can similarly show that Q is a graded strongly quasi primary ideal of T. □

Corollary 2.21. *Let $R_1, R_2, ..., R_n$ be G-graded rings, $R = R_1 \times R_2 \times ... \times R_n$ and $P = P_1 \times P_2 \times ... \times P_n$, where P_i is an ideal of R_i for all $i = 1, 2, ..., n$. Then the following statements are equivalent.*

1. *P is a graded strongly quasi primary ideal of R.*
2. *P_k is a graded strongly quasi primary ideal of R_k for some $k \in \{1, 2, ..., n\}$ and $P_j = R_j$ for all $j \neq k$.*

Proof. The result holds by induction on n applying Proposition 2.20. □

Recall that a graded ideal is called *principal* if it is generated by a single homogeneous element. A graded ring R is called a *graded divided ring* if every graded prime ideal P of R and every $a \in h(R) - P$ implies that a divides p for every $p \in P$.

Theorem 2.22. [6, Theorem 2.2] Let R be a G-graded domain. Then the following assertions are equivalent.

(i) R is a graded divided ring.

(ii) Every proper graded principal ideal is graded strongly quasi primary ideal.

(iii) Every proper graded ideal is graded strongly quasi primary ideal.

(iv) For any $x, y \in h(R)$, either y divides x or x divides y^n for some $n \in \mathbb{N}$.

Proof. $(i) \Rightarrow (ii)$: Let R be a graded domain. Suppose that (a) is a proper graded ideal of R where $a \in h(R)$. Let $xy \in (a)$ for $x, y \in h(R)$ with $y \notin Grad((a))$. Then there exists a graded prime ideal P of R containing (a) such that $y \in h(R) - P$. Since R is graded divided ring, we have $P \subseteq (y)$. Since $xy \in (a) \subseteq P$ and P is a graded prime ideal of R, we get $x \in P$, which implies that $x^2 \in xP \subseteq (x)(y) \subseteq (a)$. Therefore, (a) is a graded strongly quasi primary ideal of R.

$(ii) \Rightarrow (iii)$: Let P be a proper graded ideal of R and $xy \in P$ for some $x, y \in h(R)$. Then, $xy \in (xy) \subseteq P$ and it gives $x^2 \in (xy) \subseteq P$ or $y^n \in (xy) \subseteq P$ for some $n \in \mathbb{N}$ since (xy) is a graded strongly quasi primary ideal by the assumption.

$(iii) \Rightarrow (i)$: Assume that P is a graded prime ideal of R and take an element $x \in h(R) - P$. Let $p \in P$. Since every proper graded ideal of R is graded strogly quasi primary, by Proposition 2.22, for any $a, b \in h(R)$, either b divides a or a divides b^n for some $n \in \mathbb{N}$. Since $x \in h(R) - P$ and P is a graded prime ideal, we get $x^n \notin P$ so that p does not divide x^n, which implies that x divides p. Therefore, R is a graded divided ring.

$(iii) \Longleftrightarrow (iv)$: Suppose that every proper graded ideal of R is a graded strongly quasi primary ideal of R. Let $x, y \in h(R)$. Assume that x and y are not units. Let $P = (xy)$. Since P is a graded strongly quasi primary ideal and $xy \in P$, we have $x^2 \in (xy)$ or $y^n \in (xy)$ for some $n \in \mathbb{N}$. If $x^2 \in (xy)$, then $x^2 = xyz$ for some $z \in R$, and so $x = yz$, so that y divides x. If $y^n \in (xy)$, then $y^n = xyw$ for some $w \in R$, and so $y^{n-1} = xw$, so that x divides y^{n-1}. Conversely, let P be a proper graded ideal of R. Suppose that $x, y \in h(R)$ such that $xy \in P$. By assumption, either y divides x or x divides y^n for some $n \in \mathbb{N}$. If y divides x, then $x = \alpha y$ for some $\alpha \in R$, and then $x^2 = \alpha(xy) \in P$. Assume that x divides y^n for some $n \in \mathbb{N}$. Then $y^n = \beta x$ for some $\beta \in R$, and then $y^{n+1} = \beta(xy) \in P$. Hence, P is a graded strongly quasi primary ideal of R. □

Assume that M is an R-module. Then M is said to be G-graded if $M = \bigoplus_{g \in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$ where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. It is clear that M_g is an R_e-submodule of M for all $g \in G$. We assume that $h(M) = \bigcup_{g \in G} M_g$. Let N be an R-submodule of a graded R-module M. Then N is said to be a graded R-submodule if $N = \bigoplus_{g \in G} (N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g \in G} x_g$ where $x_g \in N$ for all $g \in G$. It is known that an R-submodule of a graded R-module need not be graded.

Let M be an R-module. The idealization $R(+)M = \{(r, m) : r \in R \text{ and } m \in M\}$ of M is a commutative ring with componentwise addition and multiplication; $(x, m_1) + (y, m_2) = (x + y, m_1 + m_2)$ and $(x, m_1)(y, m_2) = (xy, xm_2 + ym_1)$ for each $x, y \in R$ and $m_1, m_2 \in M$. Let G be an abelian group and M be a G-graded R-module. Then $X = R(+)M$ is G-graded by $X_g = R_g(+)M_g$ for all $g \in G$. Note that, X_g is an additive subgroup of X for all $g \in G$. Also, for $g, h \in G$, $X_g X_h = (R_g(+)M_g)(R_h(+)M_h) = (R_g R_h, R_g M_h + R_h M_g) \subseteq (R_{gh}, M_{gh} + M_{hg}) \subseteq (R_{gh}, M_{gh}) = X_{gh}$ as G is abelian [10]. The authors in [10] determined the certain classes of graded ideals such as graded maximal ideal, graded prime ideals, graded primary ideals, graded quasi primary ideals, graded 2-absorbing ideals and graded 2-absorbing quasi primary ideals of graded idealization $R(+)M$. Now, we investigate the graded strongly quasi primary ideals in $R(+)M$.

Lemma 2.23. *[10, Proposition 3.3] Let G be an abelian group, M be a G-graded R-module, P be an ideal of R and N be an R-submodule of M such that $PM \subseteq N$. Then $P(+)N$ is a graded ideal of $R(+)M$ if and only if P is a graded ideal of R and N is a graded R-submodule of M.*

In ([10], Corollary 3.5), graded radical of $P(+)N$ is characterized as follows:

$$Grad(P(+)N) = Grad(P)(+)M.$$

Proposition 2.24. *Let G be an abelian group, M be a G-graded R-module, P be an ideal of R and N be an R-submodule of M such that $PM \subseteq N$. If $P(+)N$ is a graded strongly quasi primary ideal of $R(+)M$, then P is a graded strongly quasi primary ideal of R.*

Proof. By Lemma 2.23, P is a graded ideal of R. Let $xy \in P$ for some $x, y \in h(R)$. Then $(x,0),(y,0) \in h(R(+)M)$ with $(x,0)(y,0) = (xy,0) \in P(+)N$. Since $P(+)N$ is a graded strongly quasi primary ideal, $((x,0))^2 = (x^2,0) \in P(+)N$ or $((y,0))^n = (y^n,0) \in P(+)N$ for some $n \in \mathbb{N}$, which implies that $x^2 \in P$ or $y^n \in P$. Hence, P is a graded strongly quasi primary ideal of R. □

Proposition 2.25. *Let G be an abelian group, M be a G-graded R-module, P be a graded strongly quasi primary ideal of R and N be a graded R-submodule of M such that $Grad(P)M \subseteq N$. Then $P(+)N$ is a graded strongly quasi primary ideal of $R(+)M$.*

Proof. By Lemma 2.23, $P(+)N$ is a graded ideal of $R(+)M$. Let $(x,m)(y,t) = (xy, xt + ym) \in P(+)N$ for some $x, y \in h(R)$, $m, t \in h(M)$. Then $xy \in P$, and so $x^2 \in P$ or $y^n \in P$ for some $n \in \mathbb{N}$. If $y^n \in P$, then $y^n t \in PM \subseteq N$, and so $((y,t))^{n+1} = (y^{n+1}, (n+1)y^n t) \in P(+)N$. Otherwise, we should have $x^2 \in P$, and so $xm \in Grad(P)M \subseteq N$, which implies that $((x,m))^2 = (x^2, 2xm) \in P(+)N$, and hence $P(+)N$ is a graded strongly quasi primary ideal of $R(+)M$. □

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