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MEROMORPHIC UNIVALENT FUNCTION WITH NEGATIVE COEFFICIENT

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ABSTRACT. Let M_n be the classes of regular functions $f(z) = z^{-1} + a_0 + a_1 z + \cdots$ defined in the annulus $0 < |z| < 1$ and satisfying $\mathrm{Re}\frac{I^{n+1}f(z)}{I^{n+1}f(z)} > 0$, $(n \in \mathbf{N}_0)$, where $I^0 f(z) = f(z)$, $If(z) = (z^{-1} - z(z-1)^{-2}) * f(z)$, $I^n f(z) = I(I^{n-1}f(z))$, and $*$ is the Hadamard convolution. We denote by $\Gamma_n = M_n \cup \Gamma$, where Γ denotes the class of functions of the form $f(z) = z^{-1} + \sum_{k=1}^{\infty} |a_k| z^k$. We obtained that relates the modulus of the coefficients to starlikeness for the classes M_n and Γ_n, and coefficent inequalities for the classes Γ_n.

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1. INTRODUCTION

Let $\sum$ denote the class of function of the form $f(z) = z^{-1} + a_0 + a_1 + \ldots$ that are regular in $0 < |z| < 1$ with a simple pole at $z = 0$. In [1] Dernek defined the classes M_n of functions $f \in \sum$ and satisfying the condition

$$\mathrm{Re}\,\frac{I^{n+1}f(z)}{I^n f(z)} > 0 \qquad (|z| < 1,\, n \in \mathbf{N}_0) \tag{1.1}$$

where $I^0 f(z) = f(z)$, $If(z) = (z^{-1} - z(z-1)^{-2}) * f(z) = -zf'(z)$ and $I^n f(z) = I(I^{n-1}f(z)) = z^{-1} + (-1)^n \sum_{k=1}^{\infty} k^n a_k z^k$. M_0 and M_1 are known classes of univalent functions that are meromorfically starlike and convex respectively. He proved that $M_{n+1} \subset M_n$ for each $n \in \mathbf{N}_0$. Since $M_0 = \sum^*$, tle element of M_n are univalent and starlike. Further $\Gamma_n = M_n \cap \Gamma$, where Γ denotes the subclass of $\sum$ consisting of functions of the form

$$f(z) = z^{-1} - \sum_{k=1}^{\infty} |a_k|\, z^k.$$

In section 2 coefficient inequalities are obtained for the classes M_n and Γ_n, similar problems were treated in [2] and [4].

2. COEFFICIENT INEQUALITIES

We begin with a theorem that relates the modulus of the coefficents to starlikeness. Our results are generalizations of the results obtained by Pommerenke in [3].

THEOREM 1. Let $f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k$. If $\sum_{k=1}^{\infty} k^{n+1}|a_k| < 1$, then $f \in M_n$, $(n \in \mathbf{N}_0)$.

PROOF. We define $w(z)$ in $0 < |z| < 1$ by

$$\frac{I^{n+1}f(z)}{I^n f(z)} = \frac{1 - w(z)}{1 + w(z)}. \tag{2.1}$$

It sufficies to show that $|w(z)| < 1$. We have from (2.1)

$$\begin{aligned} |w(z)| &= \left| \frac{I^n f(z) - I^{n+1} f(z)}{I^n f(z) + I^{n+1} f(z)} \right| \\ &= \left| \frac{(-1)^n \sum_{k=1}^{\infty} (k+1) k^n a_k z^{k+1}}{2 - (-1)^n \sum_{k=1}^{\infty} (k-1) k^n a_k z^{k+1}} \right| \\ &\leq \frac{\sum_{k=1}^{\infty} (k+1) k^n |a_k|}{2 - \sum_{k=1}^{\infty} (k-1) k^n |a_k|}. \end{aligned}$$

The last expression is bounded by 1 if

$$\sum_{k=1}^{\infty} (k+1) k^n |a_k| < 2 - \sum_{k=1}^{\infty} (k-1) k^n |a_k|$$

which reduces to

$$\sum_{k=1}^{\infty} k^{n+1} |a_{k+1}| \leq 1. \tag{2.2}$$

But (2.2) is true by hypotesis. Hence $|w(z)| < 1$ and the theorem is proved.

Special cases of Theorem 1 have been proved by Pommerenke [3, p. 274]:

COROLLARY 1: If we substitute $n = 0$ in the above theorem, then we have $f \in \sum$ and $\sum_{k=1}^{\infty} k|a_k| \leq 1$, therefore f is starlike univalent in $0 < |z| < 1$.

COROLLARY 2: If we substitute $n = 1$ in the above theorem, then we have $f \in \sum$ and $\sum_{k=1}^{\infty} k^2|a_k| \leq 1$, therefore f is convex univalent in $0 < |z| < 1$.

THEOREM 2: A function $f(z) = \frac{1}{z} - \sum_{k=1}^{\infty} |a_k| z^k$ is in Γ_n if and only if

$$\sum_{k=1}^{\infty} k^{n+1} |a_k| < 1, \qquad (n \in \mathbf{N}_0).$$

PROOF: In view of Theorem 1, it sufficies to show that the only if part. Assume that $f \in \Gamma_n$. Let z be complex numbers. If $\operatorname{Re}(z) > 0$ then $\operatorname{Re}(1/z) > 0$. Thus from (1.1) we obtain

$$\begin{aligned} 0 < \operatorname{Re}\left\{ \frac{I^n f(z)}{I^{n+1} f(z)} \right\} &\leq \left| \frac{I^n f(z)}{I^{n+1} f(z)} \right| \\ &= \left| \frac{1 - (-1)^n \sum_{k=1}^{\infty} k^n |a_k| z^{k+1}}{1 - (-1)^{n+1} \sum_{k=1}^{\infty} k^{n+1} |a_k| z^{k+1}} \right| \\ &\leq \frac{1 + \sum_{k=1}^{\infty} k^n |a_k|}{1 - \sum_{k=1}^{\infty} k^{n+1} |a_k|}. \end{aligned}$$

Hence $\sum_{k=1}^{\infty} k^{n+1} |a_k| \leq 1$ and the proof is complete.

This result is thus generalization of the result obtained by Pommerenke [3, p. 275].

COROLLARY 3: If $f \in \Gamma_n$, then $|a_k| \leq \frac{1}{k^{n+1}}$, $(n \in \mathbf{N}_0)$, with equality for $f_k(z) = \frac{1}{z} - \frac{1}{k^{n+1}} z^k$, $(n \in \mathbf{N}_0)$.

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