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RESEARCH PAPER

# Annihilator Condition on Modules

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**Abstract**

Let $R$ be a commutative ring with $1 \neq 0$ and $M$ a unital $R$-module. $M$ is said to satisfy *Property* $(A)$ if for each finitely generated ideal $J$ of $R$ contained in $Z_R(M)$, there exists $0 \neq m \in M$ such that $Jm = (0)$. Also $M$ is said to satisfy *Property* $(T)$ if for each finitely generated submodule $N$ of $M$ contained in $T(M)$, there exists $0 \neq a \in R$ such that $aN = (0)$. In this article, we study certain annihilator conditions on modules such as Property $(A)$ and Property $(T)$. In addition to give many properties of modules satisfying Property $(A)$ (Property $(T)$), we characterize these classes of modules in terms of $r$-submodules and $sr$-submodules. Also, we give a method to construct non Noetherian rings in which every ideal satisfies Property $(A)$.

**Keywords** Annihilator condition · Mccoy modules · $r$-submodules · Special $r$-submodules

**Mathematical Subject Classifications** 13C13 · 16D80 · 13A99

## 1 Introduction

This paper concerns with certain annihilator conditions on modules as Property $(A)$ and Property $(T)$. Throughout this paper, we focus only on commutative rings with nonzero identity and nonzero unital modules. Let $R$ always denote such a ring and $M$ denote such an $R$-module. A ring $R$ is said to satisfy *Property* $(A)$ if for each finitely generated ideal $I$ consisting of entirely of zero divisors of $R$, then there exists $0 \neq a \in R$ such that $aI = (0)$ (Huckaba 1988).The class of rings satisfying Property $(A)$ is quite large including Noetherian rings and nontrivial $\mathbb{Z}$-graded rings (See, [Huckaba (1988), Theorem 2.5] and (Kaplansky 1970, p 63)). Darani, in his paper Darani (2010), initiated to the study of Property $(A)$ on modules and he used the name F-McCoy instead of Property $(A)$. Afterwards, Anderson and Chun introduced another extension of Property $(A)$ in rings to modules which is called Property $(T)$. Our aim in this paper is to investigate the further properties of Property $(A)$ and Property $(T)$ for modules and to characterize these two conditions in terms of certain class of submodules such as $r$-submodules and $sr$-submodules. For the sake of completeness, now we shall give some notions and notations which will be frequently used troughout this paper.

Let $M$ be an $R$-module. An element $a \in R$ is said to be a zero divisor on $M$ if $ann_M(a) = \{m \in M : am = 0\} \neq (0)$. Also, an element $m \in M$ is called a torsion element if $ann_R(m) = \{r \in R : rm = 0\}$ is not equal to the zero ideal of $R$. Here, the set of all zero divisors on $M$ and all torsion elements of $M$ are denoted in the following:

$$Z_R(M) = \{r \in R : ann_M(r) \neq 0\}$$
$$T(M) = \{m \in M : ann_R(m) \neq 0\}.$$

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In particular, we denote the set of all zero divisor on $R$ by $zd(R)$. An $R$-module $M$ is said to be a *torsion-free module* if the torsion subset $T(M)=0$ (Sharp 2000). For any $R$-module $M$, the annihilator of $M$ is denoted by $ann(M)=\{x\in R: xM=(0)\}$. Also, $M$ is said to be a *faithful module* if $ann(M)=(0)$ (Sharp 2000).Recall from Barnard (1981) that an $R$-module $M$ is said to be a *multiplication module* if its each submodule $P$ of $M$ has the form $IM$ for some ideal $I$ of $R$.It is clear that $M$ is a multiplication module if and only $P=(P:M)M$ for each submodule $P$ of $M$, where $(P:M)$ is the annihilator of $M/P$ (El-Bast and Smith 1988).Further information on multiplication modules, we refer El-Bast and Smith (1988) and Barnard (1981) to the reader.

Anderson and Chun trasferred the Property $(A)$ in rings to modules in two different ways. According to their definition, an $R$-module $M$is said to satisfy *Property ( A)* if for each finitely generated ideal $I$of $R$ contained in $Z_R(M)$, there exists $0\neq m\in M$ such that $Im=(0)$. Also, $M$ satisfies *Property ( T)* if for each finitely generated submodule $N$ of $M$with $N\subseteq T(M)$, there exists $0\neq r\in R$ such that $rN=(0)$ (Anderson and Chun 2017a). The authors in Anderson and Chun (2017a), gave many properties and characterizations of Property $(A)$ and Property $(T)$. Recently, the notions of Property $(A)$ and Property $(T)$ on modules have been widely studied in many papers. See, for example, Bouchiba et al. (2019)– Bouchiba and El-Arabi (2019). Koc and Tekir in their paper (2018), introduced two classes of submodules which are called $r$-submodules and $sr$-submodules. Recall from Koc and Tekir (2018), a proper submodule $P$of $M$is said to be an *r-submodule* if whenever $am\in P$ for some $a\in R, m\in M$, then either $a\in Z_R(M)$or $m\in P$. Also, $P$ is called a *special r- submodule* (briefly, *sr-submodule*) if $am\in P$for some $a\in R, m\in M$ implies either $a\in(P:M)$or $m\in T(M)$. The authors in Koc and Tekir (2018) investigated the properties of these two classes of submodules and characterized torsion free modules in terms of $sr$-submodules. Among other results in this paper, we give characterization of Property $(T)$ and Property $(A)$ on modules in terms of $r$-submodules and $sr$-submodules. Also, we provide some methods to construct non-Noetherian rings in which every ideal satisfies Property $(A)$. These facts produce many examples of non-Noetherian rings satisfying Property $(A)$.

## 2 Modules Satisfying Property (A) and Property (T)

In this section, $R$ will always denote a commutative ring with nonzero identity and $M$ will always denote a nonzero unital module.

**Theorem 1** *Let $M$ be a multiplication $R$-module. The following statements are equivalent.*

(i) $M$ satisfies Property $(A)$.

(ii) $R/ann(M)$ satisfies Property $(A)$.

***Proof***

$(i)\Rightarrow(ii)$ Suppose that $M$ satisfies Property $(A)$. Let $\overline{a_1},\overline{a_2},\dots,\overline{a_n}\in R/ann(M)$ for some $a_i\in R$ and assume that $J=(\overline{a_1},\overline{a_2},\dots,\overline{a_n})\subseteq zd(R/ann(M))$. Now, put $I=(a_1,a_2,\dots,a_n)$. Let $x\in I$. Then $\overline{x}\in J\subseteq zd(R/ann(M))$ and so $\overline{xy}=\overline{0}$ for some $\overline{0}\neq\overline{y}\in R/ann(M)$. This implies that $xyM=0$. As $\overline{0}\neq\overline{y}$, we have $y\notin ann(M)$ and so there exists $m'\in M$ such that $ym'\neq 0$. As $xyM=0$, we get $x(ym')=0$ and so $x\in Z_R(M)$. Thus we conclude that $I=(a_1,a_2,\dots,a_n)\subseteq Z_R(M)$. As $M$ satisfies Property $(A)$, there exists $0\neq m\in M$ such that $Im=(0)$. Thus we conclude that $I(Rm:M)\subseteq ann(M)$. Since $M$ is multiplication and $m\neq 0$, there exists $b\in(Rm:M)-ann(M)$ and so $Ib\subseteq ann(M)$. Then we obtain $J\overline{b}=\overline{0}$ and $\overline{b}\neq\overline{0}$. Hence, $R/ann(M)$ satisfies Property $(A)$.

$(ii)\Rightarrow(i)$ Suppose that $R/ann(M)$ satisfies Property $(A)$. Let $I$ be a finitely generated ideal contained in $Z_R(M)$. Then we can write $I=Ra_1+Ra_2+\cdots+Ra_n$ for some $a_i\in R$. Now, put $J=(\overline{a_1},\overline{a_2},\dots,\overline{a_n})$. Take an element $\overline{x}\in J=(\overline{a_1},\overline{a_2},\dots,\overline{a_n})$ for some $x\in R$. Then there exists $y\in ann(M)$ and $r\in I$ such that $x=r+y$. As $r\in I\subseteq Z_R(M)$, there exists $0\neq m\in M$ such that $rm=0$ and so $xm=(r+y)m=0$. This yields that $xRm=x(Rm:M)M=0$ and hence $x(Rm:M)\subseteq ann(M)$. As $M$ is multiplication and $m\neq 0$, there exists $b\in(Rm:M)-ann(M)$ and so $xb\in ann(M)$ and this implies that $\overline{x}\overline{b}=\overline{0}$ for some $\overline{b}\neq\overline{0}$. Thus we conclude that $J=(\overline{a_1},\overline{a_2},\dots,\overline{a_n})\subseteq zd(R/ann(M))$. As $R/ann(M)$ satisfies Property $(A)$, there exists $\overline{0}\neq\overline{c}\in R/ann(M)$ such that $J\overline{c}=(\overline{0})$ and so $a_ic\in ann(M)$ with $c\notin ann(M)$. Thus we have $cm'\neq 0$ for some $m'\in M$ and so $a_i(cm')=c(a_im')=0$. Hence, we get $I(cm')=0$ and so $M$ satisfies Property $(A)$.

□

In Anderson and Chun (2017b), Anderson and Chun extended the McCoy Theorem and Dedekind-Mertens Lemma to modules for studying the McCoy Property on modules. The following remark and theorem are well-known and found in Anderson and Chun (2017b). Now, we remind it here to use them in the sequel.

Recall that an $R$-module $M$ is said to be a *dual McCoy* if for each $f(X) \in R[X]$ such that $ann_{M[X]}(f(X)) \neq 0$, there exists $0 \neq m \in M$ such that $f(X)m = 0$. Also, $M$ is called a *McCoy module* if for each $m(X) \in M[X]$ such that $ann_{R[X]}(m(X)) \neq 0$, there exists $0 \neq r \in R$ such that $rm(X) = 0$ (Anderson and Chun 2017b).

Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$ and $m(X) = m_0 + m_1X + \cdots + m_kX^k \in M[X]$. Then the contents of $f(X)$ and $m(X)$ are denoted by $c(f(X)) = (a_0, a_1, \ldots, a_n)$ and $c(m(X)) = (m_0, m_1, \ldots, m_k)$. Here, $c(f(X))$ is an ideal generated by the coefficients of $f(X)$ and $c(m(X))$ is a submodule generated by the coefficients of $m(X)$.

**Theorem 2** (*Anderson and Chun* 2017*b*, *Theorem* 3.5) For any commutative ring $R$, every $R$-module is dual McCoy.

**Theorem 3** (*Anderson and Chun* 2017*b*, *Theorem* 3.4) Let $R$ be a commutative ring and $M$ be an $R$-module. Let $f(X) \in R[X]$ and $m(X) \in M[X]$. Suppose that $m(X)$ has $n + 1$ nonzero terms. Then $c(f(X))^{n+1}c(m(X)) = c(f(X))^n c(f(X)m(X))$.

Let $M$ be an $R$-module and $S = R - Z_R(M)$. Then $S$ is a multiplicatively closed subset of $R$. Here, we denote the quotient module $S^{-1}M$ of $M$ by $q(M)$ and we will call it *total quotient module* of $M$.

**Theorem 4** *Let $M$ be an $R$-module. The following statements are equivalent.*

(i) $M$ satisfies Property $(A)$.

(ii) $q(M)$ satisfies Property $(A)$.

(iii) For each $f(X) \in R[X]$, $ann_{M[X]}(f(X)) = 0$ if and only if $c(f(X)) \nsubseteq Z_R(M)$.

***Proof*** $(i) \Leftrightarrow (ii):$ Follows from (Anderson and Chun 2017a, Theorem 2.1). $(i) \Rightarrow (iii):$ Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$ such that $ann_{M[X]}(f(X)) = 0$. Now, we will show $c(f(X)) \nsubseteq Z_R(M)$. Suppose to the contrary. Then $c(f(X)) = (a_0, a_1, \ldots, a_n) \subseteq Z_R(M)$. As $M$ satisfies Property $(A)$ and $c(f(X))$ is a finitely generated ideal contained in $Z_R(M)$, there exists $0 \neq m \in M$ such that $c(f(X))m = 0$ and so $f(X)m = 0$ which is a contradiction. Thus $c(f(X)) \nsubseteq Z_R(M)$. Conversely, assume that $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$ such that $c(f(X)) \nsubseteq Z_R(M)$. Now, we will show that $ann_{M[X]}(f(X)) = 0$. Suppose to the contrary. Then $ann_{M[X]}(f(X)) \neq 0$. As $M$ is a dual McCoy module, there exists $0 \neq m \in M$ such that $f(X)m = 0$ and so $a_im = 0$ for each $i = 0, 1, 2, \ldots, n$. This implies that $m \in ann_M(a_0, a_1, \ldots, a_n)$. As $c(f(X)) \nsubseteq Z_R(M)$, there exists $a \in R - Z_R(M)$ such that $a \in c(f(X))$ and so $a = r_0a_0 + r_1a_1 + \cdots + r_na_n$ for some $r_i \in R$. Since $m \in ann_M(a_0, a_1, \ldots, a_n)$, we have $am = 0$ and so $a \in Z_R(M)$ which is a contradiction. Hence, $ann_{M[X]}(f(X)) = 0$. $(iii) \Rightarrow (i):$ Suppose that $I$ is a finitely generated ideal contained in $Z_R(M)$. Then we can write $I = Rb_0 + Rb_1 + \cdots + Rb_n$ for some $b_i \in R$. Put $f(X) = b_0 + b_1X + \cdots + b_nX^n \in R[X]$. Then $c(f(X)) = I \subseteq Z_R(M)$. By (iii), $ann_{M[X]}(f(X)) \neq 0$. As $M$ is a dual McCoy module, there exists $0 \neq m \in M$ such that $f(X)m = 0$ and so $b_im = 0$ for each $i = 0, 1, 2, \ldots, n$. This implies that $Im = 0$. Hence $M$ satisfies Property $(A)$. ☐

Recall that a proper submodule $N$ of $M$ is said to be an $r$-submodule if for $a \in R$ and $m \in M$, $am \in N$ with $ann_M(a) = 0$ then $m \in N$ (Koc and Tekir 2018).

**Lemma 1** *Let $M$ be an $R$-module and $N$ a submodule of $M$. If $N[X]$ is an $r$-submodule of $R[X]$-module $M[X]$, then $N$ is an $r$-submodule of $M$.*

***Proof*** It is straightforward. ☐

Recall from Lu (1984) that a proper submodule $P$ of $M$ is said to be a *prime submodule* if $am \in P$ for some $a \in R$ and $m \in M$, then $a \in (P : M)$ or $m \in P$. In this case $\mathfrak{p} = (P : M)$ is a prime ideal of $R$. Note that an ideal $\mathfrak{p}$ of $R$ is said to be a prime ideal if it is a prime submodule of $R$-module $R$.

**Proposition 5** *(i) Let $M$ be a finitely generated R-module. Assume that $N$ is an $r$-submodule of $M$ if and only if $N[X]$ is an r-submodule of R[X]-module $M[X]$. Then $M$ satisfies Property (A).*

*(ii) Let $M$ be an R-module satisfying Property (A) and $N$ be a submodule of $M$. Then $N$ is an r-submodule of $M$ if and only if $N[X]$ is an r-submodule of R[X]-module M[X].*

***Proof*** $(i):$ Suppose that $N$ is an $r$-submodule of $M$ if and only if $N[X]$ is an $r$-submodule of $R[X]$-module $M[X]$ for each submodule $N$ of $M$. Now, we will show that $M$ satisfies Property $(A)$. Suppose to the contrary. By Theorem 4, there exists $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$ such that $ann_{M[X]}(f(X)) = 0$ and $c(f(X)) \subseteq Z_R(M)$. Put $S = R - Z_R(M)$. Note that $S$ is a multiplicatively closed subset and $c(f(X)) \cap S = \emptyset$. Then there exists a prime ideal $P$ of $R$ containing $c(f(X))$ and disjoint from $S$, that is $c(f(X)) \subseteq P \subseteq Z_R(M)$ (Sharp 2000, Theorem 3.44). This implies that $c(f(X)) \subseteq P + ann(M) \subseteq Z_R(M)$. Similarly, there exists a prime ideal $P^*$ of $R$ such that $c(f(X)) \subseteq P + ann(M) \subseteq P^* \subseteq Z_R(M)$. As $M$ is a finitely generated module, by Lu (1984), $(P^*M : M) = P^*$. Then by

(McCasland and Moore 1992, Theorem 3.3), there exists a prime submodule $N$ of $M$ such that $(N:M)=P^*$ and $P^*M\subseteq N$. As $(N:M)=P^*\subseteq Z_R(M)$, by (Koc and Tekir 2018, Proposition 2), $N$ is an $r$-submodule of $M$. Since $c(f(X))\subseteq P^*=(N:M)$, we have $f(X)\in(N:M)[X]$ and so $(f(X))M[X]\subseteq(N:M)[X]M[X]\subseteq N[X]$ and thus $f(X)\in(N[X]:_{R[X]}M[X])$. Since $ann_{M[X]}(f(X))=0$, we get $(N[X]:_{R[X]}M[X])\nsubseteq Z_R(M[X])$ and so by (Koc and Tekir 2018, Proposition 2), $N[X]$ is not an $r$-submodule of $R[X]$-module $M[X]$ which contradicts with the assumption. Hence, $M$ satisfies Property $(A)$.

$(ii):$ Suppose that $M$ satisfies Property $(A)$. Let $N$ be a submodule of $M$. If $N[X]$ is an $r$-submodule of $R[X]$-module $M[X]$, then by Lemma 1, $N$ is an $r$-submodule of $M$. Conversely, assume that $N$ is an $r$-submodule of $M$. Now, we will show that $N[X]$ is an $r$-submodule of $R[X]$-module $M[X]$. Let $f(X)m(X)\in N[X]$ with $ann_{M[X]}(f(X))=0$, where $f(X)=a_0+a_1X+\cdots+a_nX^n\in R[X]$ and $m(X)=m_0+m_1X+\cdots+m_kX^k\in M[X]$. Then $c(f(X)m(X))\subseteq N$. Since $M$ satisfies Property $(A)$, by Theorem 4, $c(f(X))\nsubseteq Z_R(M)$ so there exists $a\in c(f(X))$ such that $ann_M(a)=0$. First note that $ann_M(a^{k+1})=0$. Also by Theorem 3,
$c(f(X))^{k+1}c(m(X))=c(f(X))^kc(f(X)m(X))\subseteq N$. Then
$(a^{k+1})c(m(X))\subseteq c(f(X))^{k+1}c(m(X))\subseteq N$. As $N$ is an $r$-submodule, by (Koc and Tekir 2018, Theorem 1), $c(m(X))\subseteq N$ and thus $m(X)\in N[X]$. Hence, $N[X]$ is an $r$-submodule of $R[X]$-module $M[X]$. ☐

Now, we give a characterization of Property $(A)$ on modules in terms of $r$-submodules.

**Theorem 6** *Let $M$ be a finitely generated $R$-module. Then the following statements are equivalent.*

(i) $M$ satisfies Property $(A)$.
(ii) $N$ is an r-submodule of $M$ if and only if $N[X]$ is an r-submodule of $R[X]$-module $M[X]$ for each submodule $N$ of $M$.

***Proof*** Follows from Proposition 5 (i) and (ii). ☐

Recall that an ideal $I$ of $R$ is an *irreducible ideal* if whenever $J\cap K=I$ for some ideals $J,K$ of $R$, then $J=I$ or $K=I$ (Sharp 2000). Recall from Anderson and Chun (2017a) that an $R$-module $M$ satisfies *strong Property* $(T)$ if whenever each subset $N\subseteq T(M)$ with $|N|\le m$ for some $m\in\mathbb{N}$, then $0\neq a\in R$ such that $aN=(0)$. The authors in Anderson and Chun (2017a) showed that if the zero ideal is an irreducible ideal of a ring $R$, then each $R$-module satisfies strong Propert $(T)$. Now, we prove the converse is also true.

**Theorem 7** *Let $R$ be a ring. The following statements are equivalent.*

(i) Zero ideal is an irreducible ideal of $R$.
(ii) Every $R$-module satisfies strong Property $(T)$.

***Proof*** $(i)\Rightarrow(ii):$ Follows from (Anderson and Chun 2017a, Theorem 3.1).

$(ii)\Rightarrow(i):$ Suppose that every $R$-module satisfies Property $(T)$. Now, we will show that zero ideal is irreducible. Let $I\cap J=0$ for some ideals $I$ and $J$ of $R$. Assume that $I\nsubseteq J$ and $J\nsubseteq I$. Put $M=R/I\oplus R/J$. Then $M$ is an $R$-module. As $I\nsubseteq J$ and $J\nsubseteq I$, choose $a\in I-J$ and $b\in J-I$. Consider the subset $N=\{(\overline{a},\overline{1}),(\overline{1},\overline{b})\}$. Then note that $b(\overline{a},\overline{1})=0_M$ and $a(\overline{1},\overline{b})=0_M$ and this yields $N\subseteq T(M)$. As $M$ satisfies strong Property $(T)$, there exists a nonzero element $r\in R$ such that $rN=(0)$. This gives $r(\overline{a},\overline{1})=0_M=r(\overline{1},\overline{b})$ and this yields $r\in I\cap J=0$ which is a contradiction. Hence, we get either $I\subseteq J$ or $J\subseteq I$ and so that $I=0$ or $J=0$. Therefore, zero ideal is irreducible. ☐

**Theorem 8** *Let $M$ be a faithful $R$-module. The following statements are equivalent.*

(i) $M$ satisfies Property $(T)$ as an $R$-module.
(ii) $q(M)$ satisfies Property $(T)$ as a $q(R)$-module.

***Proof*** $(i)\Rightarrow(ii):$ Suppose $M$ satisfies Property $(T)$ as an $R$ -module. Suppose that $W=(\frac{m_1}{s_1},\frac{m_2}{s_2},\cdots,\frac{m_n}{s_n})\subseteq T_{q(R)}(q(M))$ for some $\frac{m_i}{s_i}\in q(M)$. Now, put $N=(m_1,m_2,\ldots,m_n)$ and choose $x\in N$. Then there exist $r_1,r_2,\ldots,r_n\in R$ such that $x=r_1m_1+r_2m_2+\cdots+r_nm_n$. Let $s=s_1s_2\cdots s_n$. Then note that $\frac{x}{s}=\frac{r_1}{s_2s_3\cdots s_n}\frac{m_1}{s_1}+\frac{r_2}{s_1s_3\cdots s_n}\frac{m_2}{s_2}+\cdots+\frac{r_n}{s_1s_2\cdots s_{n-1}}\frac{m_n}{s_n}\in W$. Then there exists $0\neq\frac{a}{t}\in q(R)$ such that $\frac{a}{t}\frac{x}{s}=0$. This implies that $uax=0$ for some $u\in R-Z_R(M)$. Thus we have $ax=0$, that is, $x\in T_R(M)$. As $M$ satisfies Property $(T)$ and $N\subseteq T_R(M)$, there exists $0\neq y\in R$ such that $yN=0$ and so $\frac{y}{1}W=0$. Now, we will show that $\frac{y}{1}\neq0$. If $\frac{y}{1}=0$, then there exists $ty=0$ for some $t\in R-Z_R(M)$. Then we get $t(yM)=0$ and so $yM=0$. Since $M$ is faithful, we conclude that $y=0$, a contradiction. Therefore, $q(M)$ satisfies Property $(T)$ as a $q(R)$-module.

$(ii)\Rightarrow(i):$ Suppose that $q(M)$ satisfies Property $(T)$ as a $q(R)$-module and take a finitely generated submodule $N$ of $M$ that is contained in $T_R(M)$. Then we can write $N=(m_1,m_2,\ldots,m_n)$ for some $m_i\in M$. Now, put $W=S^{-1}N=(\frac{m_1}{1},\frac{m_2}{1},\cdots,\frac{m_n}{1})$. A similar argument in (i) shows that $W\subseteq T_{q(R)}(q(M))$ since $M$ is faithful. As $q(M)$ satisfies Property $(T)$, there exists $0\neq\frac{a}{t}\in q(R)$ such that $\frac{a}{t}W=0$. As $\frac{m_i}{1}\in W$, we have $\frac{a}{t}\frac{m_i}{1}=0$ and so $uam_i=0$ for some $u \in R - Z_R(M)$. This implies that $am_i = 0$. Also note that $a \neq 0$ and $aN = 0$. Thus, $M$ satisfies Property $(T)$ as $R$-module. □

Recall from Anderson and Chun (2017b) that an $R$-module $M$ is said to be a *Gaussian* if $f(X)m(X) = c_0 + c_1X + \cdots + c_{m+n}X^{m+n}$ for $f(X) = a_0 + a_1X + \cdots + a_mX^m \in R[X]$ and $m(X) = m_0 + m_1X + \cdots + m_nX^n \in M[X]$, then $\sum_{i=0}^{m}\sum_{j=0}^{n} Ra_iRm_j = \sum_{i=0}^{m+n} Rc_i$. Note that $M$ is Gaussian if and only if $c(f(X)m(X)) = c(f(X))c(m(X))$. Also, note that every Gaussian module is McCoy.

**Theorem 9** *Let R be a ring and M be a McCoy R-module. Then, the following statements are equivalent.*

(i) $M$ satisfies Property $(T)$.

(ii) For each $m(X) \in M[X]$, $ann_{R[X]}(m(X)) = 0$ if and only if $c(m(X)) \nsubseteq T(M)$.

***Proof*** $(i) \Rightarrow (ii)$ : Assume that $M$ satisfies Property $(T)$. Let $m(X) = m_0 + m_1X + \cdots + m_nX^n \in M[X]$ with $ann_{R[X]}(m(X)) = 0$. Suppose that $c(m(X)) \subseteq T(M)$. Since $M$ satisfies Property $(T)$, there exists $0 \neq r \in R$ such that $rc(m(X)) = 0$ where $c(m(X)) = (m_0, m_1, \ldots, m_n)$. This implies that $rm_i = 0$ for all $i = 0, 1, \ldots, n$. So, we get $rm(X) = 0$ which is a contradiction. Thus, $c(m(X)) \nsubseteq T(M)$. On the other hand, assume that $c(m(X)) / \subseteq T(M)$ but $ann_{R[X]}(m(X)) \neq 0$. Then, there exists $0 \neq f(X) \in R[X]$ such that $f(X)m(X) = 0$. As $M$ is a McCoy $R$-module, $rm(x) = 0$ for some $0 \neq r \in R$. This gives that $rm_i = 0$ for all $i = 0, 1, \ldots, n$. So, $r \in ann_R((m_0, m_1, \cdots, m_n))$. Since $c(m(X)) \nsubseteq T(M)$, there exists $m \in M - T(M)$ such that $m \in c(m(X)) = (m_0, m_1, \cdots, m_n)$. Hence, $m = r_0m_0 + r_1m_1 + \cdots + r_nm_n$ for some $r_0, r_1, \ldots, r_n \in R$. This implies that $rm = 0$ and so, $m \in T(M)$, a contradiction. Therefore, $ann_{R[X]}(m(X)) = 0$, as desired.

$(ii) \Rightarrow (i)$ : Now, suppose that $N$ is a finitely generated submodule of $M$ such that $N \subseteq T(M)$. Then, $N = Rm_0 + Rm_1 + \cdots + Rm_k$ for some $m_i \in M$. Take $m(X) = m_0 + m_1X + \cdots + m_kX^k \in M[X]$. So, $c(m(X)) \subseteq T(M)$ and we get $ann_{R[X]}(m(X)) \neq 0$, by the assumption. Then, there exists $0 \neq f(X) \in R[X]$ such that $f(X)m(X) = 0$. As $M$ is a McCoy $R$-module, we have $rm(X) = 0$ for some $0 \neq r \in R$. This means that $rm_i = 0$ for each $i = 0, 1, \ldots, k$. Hence, it shows $rN = 0$ which completes the proof. □

**Lemma 2** *Let M be an R-module and N a submodule of M. If* $N[X]$ *is an sr-submodule of* $M[X]$, *then N is an sr-submodule of M.*

***Proof*** Assume that $N[X]$ is an $sr$-submodule of $M[X]$. Suppose $am \in N$ with $ann_R(m) = 0$, where $a \in R$ and $m \in M$. Then note that $ann_{R[X]}(m) = 0$. This implies that $am \in N \subseteq N[X]$ with $ann_{R[X]}(m) = 0$. As $N[X]$ is an $sr$-submodule of $M[X]$, $a \in (N[X] :_{R[X]} M[X])$. This means $a \in (N :_R M)$ proving that $N$ is an $sr$-submodule of $M$. □

**Proposition 10** *Let M be a finitely generated multiplication McCoy R-module. Assume that N is an sr-submodule of M if and only if N[X] is an sr-submodule of M[X]. Then, M satisfies Property (T).*

***Proof*** Assume that $N$ is an $sr$-submodule of $M$ if and only if $N[X]$ is an $sr$-submodule of $M[X]$. Now, we will show that $M$ satisfies Property $(T)$. Suppose to the contrary. Then, by Theorem 9, there exists $m(X) \in M[X]$ such that $ann_{R[X]}(m(X)) = 0$ and $c(m(X)) \subseteq T(M)$. First, we will show that $S = R - (T(M) : M)$ is a multiplicatively closed set. To see this, choose $a, b \in S$. Then there exist $m_1, m_2 \in M$ such that $am_1, bm_2 \notin T(M)$, which implies that $ann_R(am_1) = ann_R(bm_2) = 0$. If $ab \in (T(M) : M)$, then we have $abm \in T(M)$ for each $m \in M$. Then $abm_1 \in T(M)$, which implies that $ann_R(abm_1) \neq 0$. Since $ann_R(am_1) = 0$, we have $ann_R(abm_1) = ann_R(b) \neq 0$. Then there exists $0 \neq x \in R$ such that $xb = 0$, which implies that $x \in ann_R(bm_2) = 0.$ , a contradiction. Thus, $S = R - (T(M) : M)$ is a multiplicatively closed set. Since $c(m(X)) \subseteq T(M)$, we have $(c(m(X)) : M) \cap S = \emptyset$. Then there exists a prime ideal $\mathfrak{p}$ of $R$ such that $(c(m(X)) : M) \subseteq \mathfrak{p} \subseteq (T(M) : M)$. Then by (McCasland and Moore 1992, Theorem 3.3), there exists a prime submodule $P^*$ of $M$ such that $c(m(X)) \subseteq P^*$ and also $(P^* : M) = \mathfrak{p}$. As $(P^* : M) = \mathfrak{p} \subseteq (T(M) : M)$ and $M$ is a multiplication module, we get $c(m(X)) \subseteq P^* \subseteq T(M)$. Then by (Koc and Tekir 2018, Proposition 10), $P^*$ is an $sr$-submodule of $M$. So by the assumption, $P^*[X]$ is an $sr$-submodule of $R[X]$-module $M[X]$. Since $c(m(X)) \subseteq P^*$, we have $m(X) \subseteq P^*[X]$. As $ann_{R[X]}(m(X)) = 0$, we get $P^*[X] \nsubseteq T_{R[X]}(M[X])$ which is a contradiction. Therefore, $M$ satisfies Property $(T)$. □

**Proposition 11** *Let M be a Gaussian R-module satisfying Property (T) and N be a submodule of M. Then, N[X] is an sr-submodule of M[X] if and only if N is an sr-submodule of M.*

***Proof*** Assume that $M$ satisfies Property $(T)$. If $N[X]$ is an $sr$-submodule of $M[X]$, then $N$ is an $sr$-submodule of $M$ by Lemma 2. For the other direction, suppose that $N$ is an $sr$-submodule of $M$. Take $f(X)m(X) \in N[X]$ with $ann_{R[X]}(m(X)) = 0$, where $f(X) \in R[X]$ and $m(X) \in M[X]$. As $M$ satisfies Property $(T)$ and $M$ is McCoy module, we have $c(m(X)) \nsubseteq T(M)$, by Theorem 9. Then, there exists $m' \in c(m(X))$ such that $ann_R(m') = 0$. Since $M$ is Gaussian, $c(f(X)m(X)) = c(f(X))c(m(X))$. Hence, $c(f(x))(m') \subseteq c(f(X))c(m(X)) = c(f(X)m(X)) \subseteq N$. Since $N$ is an $sr$-submodule of $M$ and $ann_R(m') = 0$, we conclude that $c(f(x)) \subseteq (N :_R M)$, by (Koc and Tekir 2018, Theorem 10). It implies that $f(X) \in (N :_R M)[X] \subseteq (N[X] :_{R[X]} M[X])$, as needed. □

Recall from Ansari-Toroghy and Farshadifar (2007) that an $R$-module $M$ is said to be a *comultiplication module* if each submodule $N$ of $M$ has the form $N = (0 :_M I)$ for some ideal $I$ of $R$.

**Theorem 12** *Every comultiplication module satisfies Property* ($T$).

***Proof*** Let $M$ be a comultiplication $R$-module. Take $N$ be a finitely generated submodule of $M$ such that $N \subseteq T(M)$. Since $M$ is a comultiplication module, there exists an ideal $I$ of $R$ such that $N = ann_M(I)$. Then, $aN = 0$ for every $a \in I$. This shows that $M$ satisfies Property $T$. □

Recall from Yassemi (2001) that a nonzero submodule $N$ of $M$ is said to be *second submodule* if for each $a \in R$, the homothety $N \xrightarrow{a} N$ is either zero or surjective, or equivalently, $aN = N$ or $aN = (0)$.

**Theorem 13** *Every second submodule of $M$ is an $r$-submodule.*

***Proof*** Assume that $N$ is a second submodule of $M$. Then for all $a \in R$, we have either $aN = N$ or $aN = 0$. If we choose $a \in R - Z_R(M)$, then $aN = N$. It is clear that $N \subseteq (N :_M a)$. For the converse, take $m \in (N :_M a)$. Then, $am \in N = aN$. This implies that $am = an$ for some $n \in N$ and so $a(m - n) = 0$. As $a \notin Z_R(M)$, we get $m = n \in N$ which gives $(N :_M a) \subseteq N$. Hence, we conclude that $(N :_M a) = N$. By (Koc and Tekir 2018, Proposition 4), $N$ is an $r$-submodule of $M$. □

Next example shows that the converse of Theorem 13 may not be true in general.

***Example 1*** Consider $M = \mathbb{Z}_{36}$ as a $\mathbb{Z}$-module. Every submodule of $M$ is an $r$-submodule. Take $N = (\overline{6})$ and $a = 3$. But neither $aN = N$ nor $aN = (\overline{0})$. So, $N$ is not a second submodule of $M$.

Recall from Anderson and Chun (2017a) that an $R$-module $M$ satisfies strong Property ($A$) if for each finite subset $X$ of $R$ such that $X \subseteq Z_R(M)$, then $Xm = (0)$ for some $0 \neq m \in M$. The following result was given in Anderson and Chun (2017a) without a proof. Here, again we give the proof of this result for the sake of completeness.

**Theorem 14** *$M$ satisfies strong Property* ($A$) *if and only if $M$ satisfies Property* ($A$) *and $Z_R(M)$ is an ideal of $R$.*

***Proof*** Assume that $M$ satisfies Property $A$ and $Z_R(M)$ is an ideal of $R$. Suppose $X = \{a_1, a_2, \ldots, a_n\} \subseteq Z_R(M)$. Since $Z_R(M)$ is an ideal of $R$, $(X) = (a_1, a_2, \ldots, a_n) \subseteq Z_R(M)$. As $M$ satisfies Property ($A$), there exists $0 \neq m \in M$ such that $Xm \subseteq (X)m = (0)$. Thus $M$ satisfies strong Property ($A$). On the other hand, assume that $M$ satisfies strong Property ($A$) and choose $a, b \in Z_R(M)$. Since $X = \{a, b\} \subseteq Z_R(M)$ and $M$ satisfies strong Property ($A$), there exists $0 \neq m \in M$ such that $m\{a, b\} = 0$. So, $m(a - b) = 0$ showing that $a - b \in Z_R(M)$. Also, it is clear that $ra \in Z_R(M)$ for all $a \in Z_R(M)$ and $r \in R$. Therefore, $Z_R(M)$ is an ideal of $R$. The rest is clear. □

Recall that a proper ideal $P$ of $R$ is said to be an $r$-ideal if $xy \in P$ with $ann(x) = 0$, then $y \in P$ (Mohamadian 2015). Note $r$-ideals of $R$ are exactly the $r$-submodule of $R$-module $R$.

**Proposition 15** *If $M$ is a faithful multiplication module satisfying strong Property (A), then $Z_R(M)$ is an $r$-ideal of $R$.*

***Proof*** By Theorem 14, $Z_R(M)$ is an ideal of $R$. Now, take $ab \in Z_R(M)$ for some $a, b \in R$ with $ann(a) = 0$. Then, we have $abm = 0$ for some $0 \neq m \in M$. Since $M$ is a multiplication module and $Rm \neq (0)$, choose $0 \neq x \in (Rm : M)$. As $abRm = (0)$, we have $abxM \subseteq abRm = (0)$. As $M$ is a faitful module, we conclude that $abx = 0$. Since $ann(a) = 0$, we get $bx \in ann(a) = 0$. As $M$ is faithful and $x \neq 0$, there exists $m' \in M$ such that $xm' \neq 0$. Then we have $bxm' = b(xm') = 0$, which implies that $b \in Z_R(M)$. Therefore, $Z_R(M)$ is an $r$-ideal of $R$. □

**Proposition 16** *If $M$ satisfies strong Property (T), then T(M) is both an $r$-submodule of $M$ and $sr$-submodule of $M$.*

***Proof*** It is clear that $T(M)$ is a submodule of $M$ by [Anderson and Chun (2017a), Theorem 3.1]. Now, choose $am \in T(M)$ with $ann_M(a) = 0$. Since $am \in T(M)$, we have $r(am) = a(rm) = 0$ for some $0 \neq r \in R$. So, it gives $rm = 0$ showing that $m \in T(M)$. Second part follows from (Koc and Tekir 2018, Proposition 10), as $T(M)$ is a prime submodule of $M$. □

We say that an $R$-module $M$ is a *torsion module* if $T(M) = M$. Otherwise, we say that $M$ is a *non-torsion module*, that is, $M$ is non-torsion if there exists $m \in M - T(M)$.

**Proposition 17** *Let $M$ be a non-torsion $R$-module. Then, every prime $sr$-submodule of $M$ is an $r$-submodule of $M$.*

***Proof*** Let $N$ be an $sr$-submodule of $M$. Then, $N \subseteq T(M)$ by (Koc and Tekir 2018, Lemma 3). This implies that $(N : M) \subseteq (T(M) : M)$. Take $a \in (T(M) : M)$. This gives $aM \subseteq T(M)$. Since $T(M) \neq M$, there exists $m \in M$ such that $ann_R(m) = 0$. Then we have $am \in T(M)$ which gives $ann_R(am) = ann(a) \neq 0$. This means that there exists a nonzero $r \in R$ such that $ra = 0$. Note that if $M$ is non-torsion, then it is a faithful module. As $M$ is faithful, $rm'$ /

$= 0$ for some $m' \in M$. Thus we have $a(rm') = (ra)m' = 0$ showing that $a \in Z_R(M)$. So we get $(N : M) \subseteq (T(M) : M) \subseteq Z_R(M)$. Since $N$ is prime, we conclude that $N$ is an $r$-submodule of $M$ by (Koc and Tekir 2018, Proposition 2). ☐

**Proposition 18** *Let $M$ be a multiplication $R$-module. Then, every prime $r$-submodule of $M$ is an $sr$-submodule of $M$.*

***Proof*** Let $N$ be a prime $r$-submodule of $M$. Then, $(N : M) \subseteq Z_R(M)$, by (Koc and Tekir 2018, Lemma 1). Now, we will show that $Z_R(M) \subseteq (T(M) : M)$. Suppose to the contrary. Choose $r \in Z_R(M) - (T(M) : M)$. Then there exists $0 \neq m \in M$ such that $rm = 0$. Since $M$ is a multiplication module, $(Rm : M) \neq 0$ and so there exists $0 \neq a \in (Rm : M)$. This gives that $raM \subseteq rRm = 0$ and so $raM = 0$. Also, since $r \notin (T(M) : M)$, there exists $m' \in M$ such that $rm' \notin T(M)$ which means $ann_R(rm') = 0$. As $raM = 0$, we have $ram' = 0$ implying $a \in ann_R(rm') = 0$, a contradiction. Hence, we conclude that $(N : M) \subseteq Z_R(M) \subseteq (T(M) : M)$ implying $(N : M)M \subseteq Z_R(M)M \subseteq (T(M) : M)M \subseteq T(M)$. Since $M$ is a multiplication module, $N \subseteq T(M)$ and this proves that $N$ is an $sr$-submodule of $M$ by (Koc and Tekir 2018, Proposition 10). ☐

**Corollary 1** *Let $M$ be a non-torsion multiplication $R$-module and $N$ be a prime submodule of $M$. Then, $N$ is an $r$-submodule of $M$ if and only if $N$ is an $sr$-submodule of $M$.*

***Proof*** It is an immediate consequence of Proposition 17 and Proposition 18. ☐

**Corollary 2** *Let $M$ be a non-torsion multiplication $R$-module. Then, the following assertions hold.*

(i) Every maximal $r$-submodule of $M$ is an $sr$-submodule.
(ii) Every maximal $sr$-submodule of $M$ is an $r$-submodule.

***Proof*** It follows from Proposition 17, Proposition 18, (Koc and Tekir 2018, Proposition 7) and (Koc and Tekir 2018, Proposition 14). ☐

Recall from Anderson and Chun (2017a) that an $R$-module $M$ satisfies *Property total-A* if for each ideal $J$ of $R$ with $J \subseteq Z_R(M)$, there exists $0 \neq m \in M$ such that $Jm = (0)$. Also, $M$ is said to satisfy *Property total-T* if for each submodule $N$ of $M$ with $N \subseteq T(M)$, there exists $0 \neq x \in R$ such that $xN = (0)$.

Now, we will give some results about $r$-submodule and $sr$-submodule on modules satisfying Property $T$, Property $A$, Property total-$A$ and Property total-$T$.

**Theorem 19**

(i) If $M$ satisfies Property ($T$), then $ann_R(N) \neq 0$ for every finitely generated $sr$-submodule $N$ of $M$.
(ii) If $M$ satisfies Property total-$T$, then $ann_R(N) \neq 0$ for every $sr$-submodule $N$ of $M$.
(iii) If $M$ is a multiplication module satisfying Property $(A)$, then $ann_R(N) \neq 0$ for every finitely generated $r$-submodule $N$ of $M$.
(iv) If $M$ is a multiplication module satisfying Property total-$A$, then $ann_R(N) \neq 0$ for every $r$-submodule $N$ of $M$.

***Proof***

(i): By [Koc and Tekir (2018), Lemma 3], note that $N \subseteq T(M)$ for every $sr$-submodule $N$ of $M$. The rest is clear.
(ii): It is similar to $(ii)$.
(iii): Suppose that $M$ is a multiplication module satisfying Property $(A)$ and $N$ is a finitely generated $r$-submodule of $M$. Since $M$ is multiplication and $N$ is a finitely generated submodule of $M$, we can write $N = JM$ for some finitely generated ideal $J$ of $R$. As $N$ is an $r$-submdule of $M$, by [Koc and Tekir (2018), Lemma 1], $J \subseteq (N : M) \subseteq Z_R(M)$. Since $M$ satisfies Property $(A)$, there exists $0 \neq m \in M$ such that $Jm = (0)$. Since $M$ is multiplication and $m \neq 0$, there exists $0 \neq x \in (Rm : M)$. Then we have $xN = xJM = J(xM) \subseteq Jm = (0)$, which completes the proof.
(iv): It is similar to $(iii)$.

☐

Let $A$ be a ring, $M$ an $A$-module and $R = A \ltimes M$, the set of pairs $(a, m)$ with $a \in A$ and $m \in M$, under componentwise addition and under an adjusted multiplication defined by $(a, m)(a', m') = (aa', am' + a'm)$, for all $a, a' \in A$; $m, m' \in M$. Then $R$ is called *the trivial ring extension of $A$ by $M$*. We define similarly $J := I \ltimes N$, where $I$ is an ideal of $A$ and $N$ is an $A$-submodule of $M$ such that $IM \subseteq N$. Trivial ring extensions have been studied extensively; the work is summarized in Glaz (1989) and Huckaba (1988). These extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance Kabbaj and Mahdou (2004) and Mahdou and Hassani (2012).

Let $R$ be a ring and $I$ an ideal of $R$. We say that $I$ satisfies Property $(A)$ if it satisfies Property $(A)$ as an $R$-module. It is clear that if $R$ is a Noetherian ring, then its every ideal satisfies Property $(A)$. Now, we give some methods to construct non-Noetherian rings in which every ideal satisfies Property $(A)$.

**Proposition 20** *Let* $(R, \mathfrak{m})$ *be a local ring such that* $\mathfrak{m}^2 = 0$, *where* $\mathfrak{m}$ *is a maximal ideal of* $R$. *Then every proper ideal of* $R$ *satisfies Property* $(A)$.

***Proof*** The proof is elementary. ☐

***Example 2*** Consider the trivial ring extension $R = K \ltimes E$ of a field $K$ by an infinite $K$-vector space $E$. Then,

(i) Every proper ideal of $R$ satisfies Property $(A)$ by Proposition 20.
(ii) $R$ is not coherent by (Kabbaj and Mahdou 2004, Theorem 2.6 (2)). In particular, $R$ is not Noetherian.

**Proposition 21** *Consider the trivial ring extension* $R = A \ltimes K$ *of a principal ideal domain* $A$ *by* $K := qf(A)$ *the quotient field of* $A$. *Then,*

(i) Every proper ideal of $R$ satisfies Property $(A)$.
(ii) $R$ is not coherent. In particular, $R$ is not Noetherian.

***Proof*** $(i):$ Let $J$ be a nonzero proper ideal of $R$. Two cases are then possible:

**Case 1:** $J \subseteq 0 \ltimes K$.In this case, we have $Z_R(J) = \{(a, e) \in R : (a, e)(0, f) = (0, af) = (0, 0)$ for some nonzero $(0, f) \in J\} = 0 \ltimes K$ since $af = 0$ and $f \neq 0$. We claim that $J$ satisfies Property $(A)$. Indeed, let $H$ be a finitely generated ideal of $R$ contained in $Z_R(J) (= 0 \ltimes K)$. Then, for every $(0, 0) \neq (0, m) \in J \subseteq 0 \ltimes K$, we have $(0, m)H = (0, 0)$, as desired.

**Case 2:** $J \nsubseteq 0 \ltimes K$. In this case, there exists $(a, m) \in J$ such that $a \neq 0$. Hence, $J_0 = \{b \in A : (b, m') \in J$ for some $m' \in K\}$ is a proper ideal of $A$ which is a principal ideal domain. Therefore, $J_0 = Ab$ for some $b \in A$ and so $J = R(b, 0)$, where $0 \neq b \in A$. Thus, $Z_R(J) = 0 \ltimes K$ since $J = R(b, 0) = Ab \ltimes K \supseteq 0 \ltimes K$. We claim that $J$ satisfies Property $(A)$. Indeed, let $H$ be a finitely generated proper ideal of $R$ contained in $Z_R(J) = 0 \ltimes K$. Then, for each $(0, 0) \neq (0, f) \in J$, we have $(0, f)H \subseteq (0, f)(0 \ltimes K) = (0, 0)$, as desired.

$(ii):$ $R$ is not coherent by (Kabbaj and Mahdou 2004, Theorem 2.8 (1)). In particular, $R$ is not Noetherian. ☐

**Proposition 22** *Consider the trivial ring extension* $R = A \ltimes M$ *of a local integral domain* $(A, \mathfrak{m})$ *by an* $A$-*module* $E$ *such that* $\mathfrak{m}E = 0$. *Then, every proper ideal of* $R$ *satisfies Property* $(A)$.

***Proof*** Let $J$ be a proper ideal of $R$. Two cases are then possible:

**Case 1:** $(0 \ltimes E) \cap J \neq (0, 0)$.In this case, there exists $(0, 0) \neq (0, e) \in J$. It is clear that $J$ satisfies Property $(A)$ since $(0, e)(\mathfrak{m} \ltimes E) = (0, 0)$ and $Z_R(J) = M \ltimes E$.

**Case 2:** $(0 \ltimes E) \cap J = (0, 0)$.

In this case, for each $(a, e) \in J$, we have $a \neq 0$. Hence, $Z_R(J) = 0 \ltimes E$. It is clear that $J$ satisfies Property $(A)$ since $J \subseteq \mathfrak{m} \ltimes E$ and $(\mathfrak{m} \ltimes E) Z_R(J) = (\mathfrak{m} \ltimes E)(0 \ltimes E) = (0, 0)$. ☐

***Example 3*** Consider the trivial ring extension $R = A \ltimes E$ of a local integral domain $(A, \mathfrak{m})$ by an infinite $(A/\mathfrak{m})$-vector space $E$. Then,

(i) Every proper ideal of $R$ satisfies Property $(A)$ by Proposition 22.
(ii) $R$ is not coherent by using (Mahdou and Hassani 2012, Theorem 2.1). In particular, $R$ is not Noetherian.

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