

## PERFORMANCE COMPARISON OF BOX-COX TRANSFORMATION AND WEIGHTED VARIANCE METHODS WITH WEIBULL DISTRIBUTION

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### **ABSTRACT**

*A standard process capability index is calculated based on the assumption that the quality characteristic of the process follows the normal distribution. But there are many cases in which the quality characteristic comes from a non-normal distribution. This paper studies Box-Cox transformation method and Weighted Variance method to calculate process capability indices for Weibull distributed quality characteristic and compares performances of these methods. Weibull distribution is extensively used as a lifetime distribution model because of its flexible shape. The data sets used in performance comparison are randomly generated from Weibull distribution for two different shape and scale parameters through a simulation study. The results indicate that Box-Cox transformation method produces better estimates for process capability than Weighted Variance method.*

**Keywords:** Process Capability Index, Box-Cox Transformation Method, Weighted Variance Method.

### **BOX-COX DÜNYÜMÜ VE AĞIRLIKLI VARYANS YÖNTEMLERİNİN WEIBULL DAĞILIMI İLE PERFORMANSLARININ KARŞILAŞTIRILMASI**

### **ÖZET**

*Standart proses yetenek indeksi, prosesin kalite özelliğinin normal dağıldığı varsayımına dayanarak hesaplanır. Fakat, kalite özelliğinin dağılımının normal olmadığı pek çok durum vardır. Bu makalede, Weibull dağılım kalite özelliği için proses yetenek indekslerinin hesaplanmasında Box-Cox dönüşümü yöntemi ve Ağırlıklı Varyans yöntemi kullanılmakta ve performansları karşılaştırılmaktadır. Weibull dağılımı esnek şekli sebebiyle yaşam süresi dağılımının modellenmesinde yaygın olarak kullanılmaktadır. Performans karşılaştırmasında kullanılan veri setleri bir benzetim çalışması vasıtasıyla Weibull dağılımından iki farklı şekil ve ölçek parametresi için üretilmiştir. Sonuçlar, proses yeteneğini Box-Cox dönüşümü yönteminin Ağırlıklı Varyans yönteminden daha iyi tahmin ettiğini göstermiştir.*

**Anahtar Kelimeler:** Proses Yetenek İndeksi, Box-Cox Dönüşümü Yöntemi, Ağırlıklı Varyans Yöntemi.

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## 1. INTRODUCTION

Process capability is a performance measure to compare process variation with the product specifications. Process capability indices (PCIs) are widely used in industry to measure the ability of the process of the firm or its supplier to manufacture product that meets quality specifications.

Several PCIs including  $C_p$ ,  $C_{pu}$ ,  $C_{pl}$ ,  $C_{pk}$ , and  $C_{pm}$  (Equation (1)) have been used in the manufacturing industry to provide common quantitative measures on process potential and performance [1].

$$\begin{aligned}
 C_p &= \frac{USL - LSL}{6\sigma} \\
 C_{pu} &= \frac{USL - \mu}{3\sigma} \\
 C_{pl} &= \frac{\mu - LSL}{3\sigma} \\
 C_{pk} &= \min\left\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\} \\
 C_{pm} &= \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}
 \end{aligned} \tag{1}$$

where USL is the upper specification limit, LSL is the lower specification limit,  $\mu$  is the process mean,  $\sigma$  is the process standard deviation (overall process variation), and  $T$  is the target value. Product specification limits are set with respect to product design (customer) requirements, while the process variation is a function of the process, materials, equipment, tooling, operation methods, and so forth. Hence, capability indices link the product design related specifications to the process related results [2]. A process is called inadequate if process capability index (either  $C_{pu}$  or  $C_{pl}$ )  $PCI < 1.00$ , capable if  $1.00 \leq PCI < 1.33$ , satisfactory if  $1.33 \leq PCI < 1.50$ , excellent if  $1.50 \leq PCI < 2.00$ , and super if  $PCI \geq 2.00$  [1]. The assumptions of stability (statistical control) of the process and a normal distribution of process output are essential to the correct interpretation of any process capability index. But there are many cases in which the quality characteristic comes from a non-normal distribution. If the distribution is non-normal, the estimate of process capability is unlikely to be correct [3]. Therefore, several methods have been proposed to deal with non-normal distributions [4]-[5].

There are two approaches to deal with non-normal quality characteristics in order to obtain reliable estimates of process capability indices:

1. Transform non-normal data to normal data and use normally-based process capability indices.
2. Use the process capability indices defined for non-normal data.

In this study, Box-Cox transformation (BCT) method is used to calculate process capability indices corresponding to the first approach and Weighted Variance (WV) method corresponding to the second approach. The data sets used in performance comparison of the methods are randomly generated from Weibull distribution for two different shape and scale parameters through a simulation study. Due to its flexible shape, Weibull distribution is extensively used as a lifetime distribution model and hence, it is preferred as the distribution of quality characteristic in this study. Simulations and computations are performed using Minitab 16 and MS Excel 2010 software packages.

## 2. BOX-COX TRANSFORMATION (BCT) METHOD

Box and Cox [6] proposed a family of power transformations of a necessarily positive response variable  $X$ . If there are negative values, a constant value can be added in order to make the values positive. BCT uses the parameter  $\lambda$  (Equation (2)). In order to transform the data as closely as possible to normality, the best possible transformation should be performed by selecting the most appropriate value of  $\lambda$ .

$$X^{(\lambda)} = \begin{cases} \frac{X^\lambda - 1}{\lambda} & , \text{ for } \lambda \neq 0 \\ \ln X & , \text{ for } \lambda = 0 \end{cases} \tag{2}$$

The maximum likelihood estimator of  $\lambda$  is obtained as the value of  $\lambda$  that maximizes log-likelihood function  $L_{\max}$  (Equation (3)) after evaluating several values of  $\lambda$  within a pre-assigned range.

$$\begin{aligned}
 L_{\max} &= -\frac{1}{2} \ln \hat{\sigma}^2 + \ln J(\lambda, X) \\
 &= -\frac{1}{2} \ln \hat{\sigma}^2 + (\lambda - 1) \sum_{i=1}^n \ln X_i
 \end{aligned} \tag{3}$$

where  $J(\lambda, X) = \prod_{i=1}^n \frac{\partial W_i}{\partial X_i} = \prod_{i=1}^n X_i^{\lambda-1}$  for all  $\lambda$ . The

estimate of  $\sigma^2$  for fixed  $\lambda$  is  $\hat{\sigma}^2 = S(\lambda)/n$ , where  $S(\lambda)$  is the residual sum of squares in the analysis of variance of  $X^{(\lambda)}$ .

When the optimum value of  $\lambda$  is obtained, the quality characteristic  $X$ , upper and lower specification limits are transformed using Equation (2). After the transformation, process capability is evaluated using normally-based capability indices.

### 3. WEIGHTED VARIANCE (WV) METHOD

The weighted variance method was first introduced by Choobineh and Ballard [7] to construct control charts when the underlying population is skewed and afterwards it was utilized by Bai and Choi [8] to adjust capability index values in order to account for the degree of skewness of non-normal process data [4]. Wu et al. [9] have modified the original WV method used by Bai and Choi. However, the main idea of both WV methods is to divide a skewed or asymmetric (non-normal) distribution into two normal distributions from its mean.

For a non-normal distribution with the mean of  $\mu$  and a standard deviation of  $\sigma$ , there are  $n_1$  observations out of  $n$  total observations which are less than or equal to  $\mu$ . And there are  $n_2$  observations out of  $n$  total observations which are greater than  $\mu$ . The two new distributions have the same mean ( $\mu$ ) but different sample sizes ( $n_1$  and  $n_2$ ) and different standard deviations ( $\sigma_1$  and  $\sigma_2$ ). The sample size for each new distribution is determined by values of the skewness and kurtosis of the non-normal distribution.

$\mu$ ,  $\sigma_1^2$ , and  $\sigma_2^2$  are estimated by  $\bar{X}$ ,  $S_1^2$ , and  $S_2^2$ , respectively (Equation (4)).

$$\begin{aligned} \bar{X} &= \sum_{i=1}^n \frac{X_i}{n} \\ S_1^2 &= \frac{2 \sum_{i=1}^{n_1} (X_i - \bar{X})^2}{2n_1 - 1} \\ S_2^2 &= \frac{2 \sum_{i=1}^{n_2} (X_i - \bar{X})^2}{2n_2 - 1} \end{aligned} \quad (4)$$

The modified WV method defines  $C_p$ ,  $C_{pu}$ ,  $C_{pl}$ , and  $C_{pk}$  indices as in Equation (5) [9].

$$\begin{aligned} \hat{C}_p &= \frac{USL - LSL}{3(S_1 + S_2)} \\ \hat{C}_{pu} &= \frac{USL - \bar{X}}{3S_2} \\ \hat{C}_{pl} &= \frac{\bar{X} - LSL}{3S_1} \\ \hat{C}_{pk} &= \min \left\{ \frac{USL - \bar{X}}{3S_2}, \frac{\bar{X} - LSL}{3S_1} \right\} \end{aligned} \quad (5)$$

### 4. SIMULATION STUDY

50 data sets ( $r=50$ ) each having a sample size of 100 ( $n=100$ ) with subgroup size of 1 are randomly generated from Weibull distributions with shape and scale parameters of (1,1), (1,2), (2,1), and (2,2). Figure 1 shows the probability density functions (PDFs) of these distributions. The cumulative distribution function (CDF) of a Weibull distribution having shape parameter  $\alpha$  and scale parameter  $\beta$  is expressed as in Equation (6).

$$F(x; \alpha, \beta) = 1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}, \quad x \geq 0 \quad (6)$$

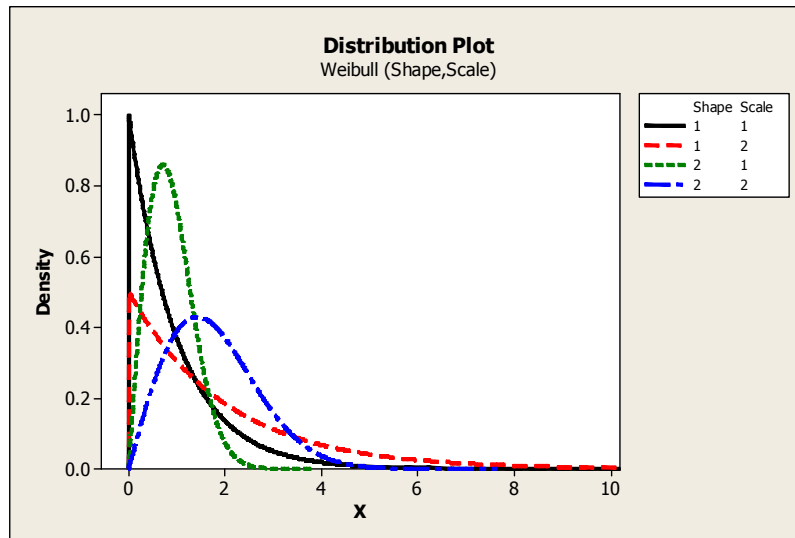


Figure 1. PDFs of Weibull distributions.

Weibull (1,1) and (1,2) distributions with their shape parameter values of 1 are at the same time Exponential distributions. When its shape parameter is equal to 1, the Weibull distribution reduces to the Exponential distribution with its parameter equal to the reciprocal of the scale parameter of the Weibull distribution. The skewness and kurtosis values give information about tail behavior of a distribution. The average values of skewness and kurtosis calculated from 50 data sets generated from Weibull distribution with specified parameters are given in Table 1.

In this study one-sided specification interval with an upper specification limit is considered. USL is calculated through Equation (7) for the targeted  $Cpu$  values of 1.0 and 1.5 and theoretical quantiles of the Weibull distribution with the specified shape and scale parameters.

**Table 1.** Skewness and kurtosis values.

Weibull ( $\alpha,\beta$ )	Skewness	Kurtosis
Weibull (1,1)	1.676698	3.413711
Weibull (1,2)	1.747334	3.982865
Weibull (2,1)	0.562298	0.104820
Weibull (2,2)	0.590805	0.212021

$$C_{pu} = \frac{USL - x_{0.50}}{x_{0.99865} - x_{0.50}} \quad (7)$$

where  $x_{0.99865}$  and  $x_{0.50}$  quantiles correspond to 0.99865 and 0.50 cumulative probabilities of the Weibull distribution, respectively.

## 5. RESULTS AND DISCUSSION

In order to compare performances of BCT and WV methods, box plots of estimated process capability indices (estimated  $Cpu$ ) corresponding to the targeted  $Cpu$  values of 1.0 and 1.5 are used. A box plot is used to show the shape of a distribution, its

central value (median= $x_{0.50}$ ), variability (interquartile range= $x_{0.75}-x_{0.25}$ ), and outliers by star symbols if exist.

Based on the box plots for targeted  $Cpu$  values of 1.0 and 1.5 (Figure 2 and Figure 3) it is observed that while BCT method underestimates the targeted values, WV method overestimates them. It is also observed that BCT method provides more accurate estimates and less variability than WV method. The worst estimates of WV method are observed when Weibull distribution is the same with Exponential distribution. These results can also be confirmed with descriptive statistics presented in Table 2.

The root-mean-square deviation ( $RMSD$ ) is used to measure the differences between the targeted  $Cpu$  values and the estimates obtained by BCT and WV methods (Equation (8)).

$$RMSD = \sqrt{\frac{\sum_{i=1}^r (\text{Estimated } Cpu_i - \text{Targeted } Cpu_i)^2}{r}} \quad (8)$$

where  $r$  is the number of data sets generated randomly for each Weibull distribution with specified parameters.

The results in Table 3 indicate that the higher target value ( $Cpu = 1.5$ ) corresponds to worse estimates for WV method. The Weibull distributions (1,1) and (1,2) with near values of skewness and kurtosis (Table 1) have similar tail behaviors and as it can be observed in the radar chart (Figure 4), WV method produces much higher  $RMSD$  values for these distributions than the Weibull distributions (2,1) and (2,2), particularly when the targeted  $Cpu$  value is higher. This result indicates that the effect of tail behavior is more significant when the process is more capable for WV method, whereas this is not the case for BCT method.

Performance Comparison of Box-Cox Transformation and Weighted Variance Methods with Weibull Distribution

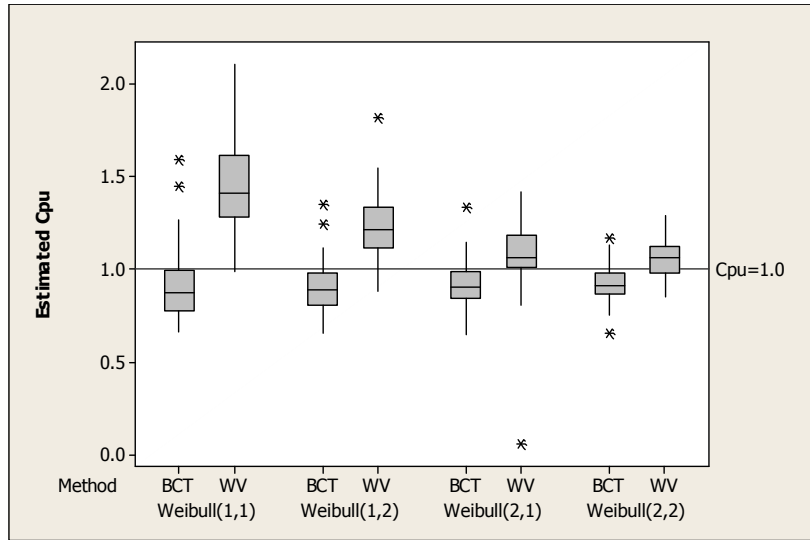


Figure 2. Box plots of BCT and WV methods for targeted  $C_{pu}=1.0$ .

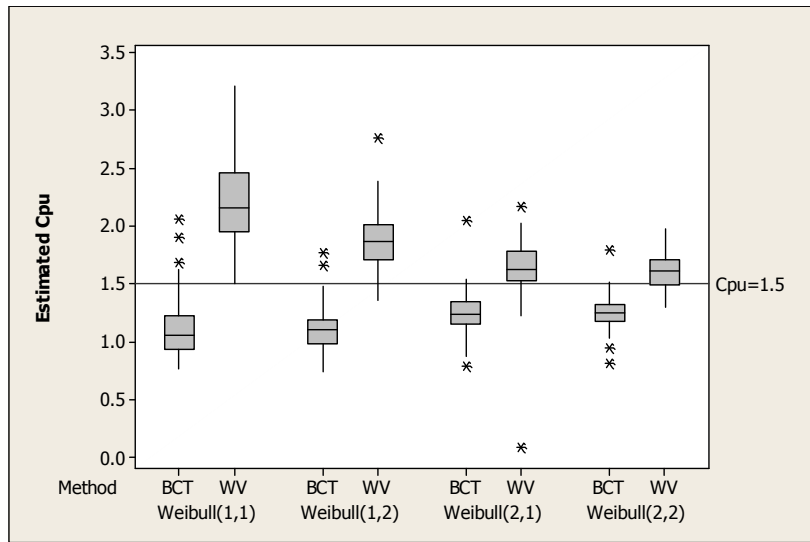


Figure 3. Box plots of BCT and WV methods for targeted  $C_{pu}=1.5$ .

Table 2. Descriptive statistics for WV and BCT methods.

Target $C_{pu}$	Statistics	Method	Weibull (1,1)	Weibull (1,2)	Weibull (2,1)	Weibull (2,2)
1.0	Mean	WV	1.4494	1.2338	1.0646	1.0659
		BCT	0.9155	0.9016	0.9111	0.9159
	Standard Deviation	WV	0.2336	0.1779	0.1945	0.1079
		BCT	0.1904	0.1368	0.1129	0.1001
1.5	Mean	WV	2.2111	1.8863	1.6149	1.6164
		BCT	1.1214	1.1023	1.2411	1.2453
	Standard Deviation	WV	0.3486	0.2697	0.2969	0.1609
		BCT	0.2718	0.1976	0.1799	0.1543

Table 3. The root-mean-square deviations for WV and BCT methods.

Target $C_{pu}$	Method	Weibull (1,1)	Weibull (1,2)	Weibull (2,1)	Weibull (2,2)
1.0	WV	0.505	0.293	0.203	0.126
	BCT	0.614	0.614	0.599	0.592
1.5	WV	1.259	0.926	0.682	0.637
	BCT	0.464	0.443	0.314	0.297

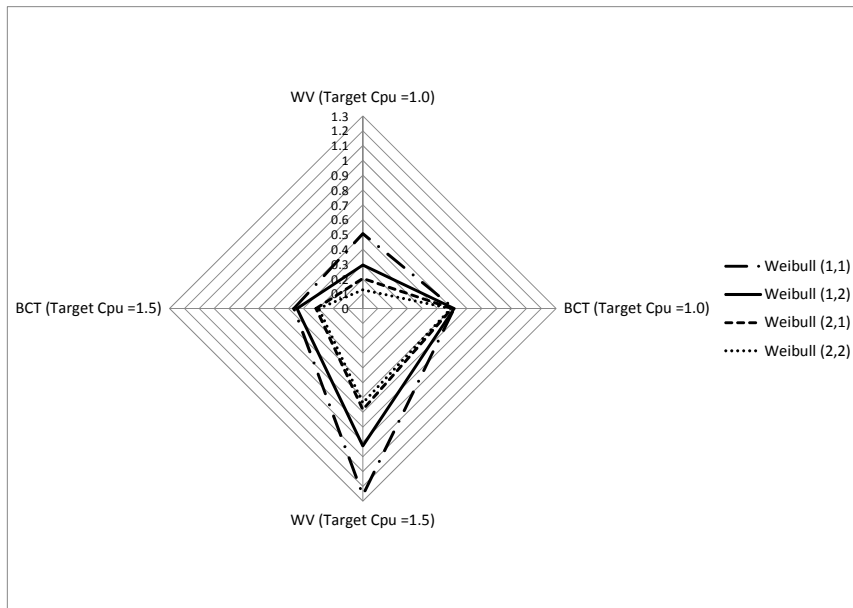


Figure 4. Radar chart for RMSD.

## 6. CONCLUSION

This study compares performances of Box-Cox transformation method and Weighted Variance method for process capability estimation when quality characteristic has Weibull distribution. Performance comparison of methods is made in terms of box plots, descriptive statistics, the root-mean-square deviation, and a radar chart. The results indicate that BCT method produces better estimates for process capability than WV method when quality characteristic is Weibull distributed. It is also observed that WV method is more sensitive to tail behavior than the BCT method. Weibull distributions are known to have significantly different tail behaviors, which greatly affects the process capability. Weibull distribution is extensively used as a lifetime distribution model because of its flexible shape. When the distribution of a quality characteristic is non-normal, normally-based PCIs would give unreliable and misleading results as well as incorrect assessment of process capability. Incorrect assessment of process capability can lead incorrect decision making, waste of resources, money, time, and so on. These findings would be helpful for selecting appropriate methods in process capability assessments with non-normal processes, especially with Weibull or Exponential distributed quality characteristics, which have been used extensively in quality and reliability applications in various industries including aerospace industry.

## 7. REFERENCES

- [1] Pearn, W.L., Chen, K.S., "One-sided capability indices Cpu and Cpl: decision making with sample information", *International Journal of Quality & Reliability Management*, 19(3), 221-245,2002.
- [2] Kolarik, W.J., "Creating Quality: Concepts, Systems, Strategies, and Tools", McGraw-Hill, New York, 1995.
- [3] Montgomery, D.C., "Statistical Quality Control: A Modern Introduction", 6th ed., Wiley, New York, 2009.
- [4] Pearn, W.L., Kotz, S., "Encyclopedia and Handbook of Process Capability Indices: A Comprehensive Exposition of Quality Control Measures", World Scientific Publishing Company, Singapore, 2006.
- [5] Tang, L.C., Than, S.E., Ang, B.W., "Computing Process Capability Indices for Non-normal Data: A Review and Comparative Study", in *Six Sigma: Advanced Tools for Black Belts and Master Black Belts* (Eds. L.C. Tang, T.N. Goh, H.S. Yam, T. Yoap), John Wiley & Sons, 107-130, 2006.
- [6] Box, G.E.P., Cox, D.R., "An analysis of transformations", *Journal of the Royal Statistical Society: Series B*, 26, 211-252, 1964.
- [7] Choobineh, F., Ballard, J.L., "Control-Limits of QC Charts for Skewed Distributions Using Weighted-Variance", *IEEE Transactions on Reliability*, 36(4), 473-477, 1987.
- [8] Bai, D.S., Choi, I.S., " $\bar{X}$  and R Control charts for skewed populations", *Journal of Quality Technology*, 27, 120-131, 1995.
- [9] Wu, H.-H., Swain, J.J., Farrington, P.A., Messimer S.L., "A weighted variance capability index for general non-normal processes", *Quality and Reliability Engineering International*, 15(5), 397-402, 1999.

## VITAE

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