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Theory and applications on a new generalized Laplace-type integral transform

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In this paper, several theorems and relations are examined by using a generalized Laplace-type integral transform. A generalization of the harmonic oscillator in a non-resisting and resisting medium problems, initial-boundary problems, and integral equations is solved via this integral transform. Furthermore, the well-known series entitled as Basel problem is obtained in a similar way. Moreover, we compare numerically classical and newly introduced by us integral transform.

KEYWORDS
convolution, differential equations, integral equations, Laplace transform, partial differential equations

MSC CLASSIFICATION
44A10, 44A35, 44A30, 44A15, 33D10

1 | INTRODUCTION

One of the methods to obtain analytical solutions of differential equations (ordinary, partial, and fractional) and integral equations is integral transforms. Integral transforms are used in many disciplines and have a variety of applications. The importance of the integral transform is that it converts a difficult equation into a relatively easy problem that can be easily solved. The integral transform of a function $f(t)$ defined on $a \leq t \leq b$ is denoted by $I\{f(t)\} = I(s)$ and is defined by

$$I(s) = I\{f(t); s\} = \int_a^b K(t,s) f(t) dt, \tag{1}$$

where $K(t,s)$ is the kernel of the transform, s is the transform parameter, and t is the transform variable.

The integral transform has different classifications and names as its kernel and limits change. For example, if we put $a = 0, b = \infty$ and $K(t,s) = e^{-st}$, then $I = \mathcal{L}$ is called the Laplace transform.[1] For the same a and b values, if we set $K(t,s) = te^{-s^2t^2}$ and $K(t,s) = t^{s-1}$ in (1), respectively, then we get $I = \mathcal{L}_2$, which is called the $\mathcal{L}_2$ transform,[2] and $I = \mathcal{M}$, which is called the Mellin transform.[1] Each newly defined integral transform either solves a new type of differential equation or facilitates the solution of a differential equation previously solved by another method, such as Bessel and Hermite's differential equations are solved by the $\mathcal{L}_2$ transform method in Yürekli and Wilson.[3,4] On the other hand, any redefined integral transform can also be used to evaluate many generalized integrals via Parseval-Goldstein type theorems under certain conditions.[2,5]

After an integral transform is defined, the first thing to be done is to analyze the class of functions for which the transform exists. This analysis leads to an existence theorem for the transform. Then, the properties of the transform are

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examined based on this function class. Through the theorems obtained with these properties, many application areas are illuminated.

In this paper, we study a new integral transform, the generalized Laplace transform, denoted $\mathcal{L}_{\alpha,\mu}$, since it is an exponential function in its kernel.[6] The existence theorem of transform and some properties are proved from a different point of view. Then, a generalization of the harmonic oscillator in a non-resisting and resisting (internal mechanical resistance or external air resistance) medium problems, a special case of an initial-value problem of the form

$$a(t)u_t(x,t)+b(x)u_x(x,y)=f(x), \qquad x>0,\, t>0,$$

with the initial and boundary conditions

$$u(x,0)=0 \qquad x>0, \qquad u(0,t)=0 \qquad t>0,$$

a different kind of initial-value problem and an integral equation are solved via new generalized Laplace-type integral transform. Finally, the well-known

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6} \tag{2}$$

series equation was proved in a different way using the $\mathcal{L}_{\alpha,\mu}$ transform.

Let's start with basic definitions and properties of some special functions that will be used throughout the article.

The Bessel function of the first kind of order ν is defined by

$$J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}, \tag{3}$$

where $\nu\in\mathbb{C}$, $\mathrm{Re}\nu>-1$.[7] The Mittag-Leffler function is defined by

$$E_\mu(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu n+1)}, \tag{4}$$

where $\mu,z\in\mathbb{C}$, $\mathrm{Re}\mu>0$.[8] The Wiman function is defined by

$$E_{\mu,\nu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu n+\nu)}, \tag{5}$$

where $\mu,\nu,z\in\mathbb{C}$, $\mathrm{Re}\mu>0$, $\mathrm{Re}\nu>0$.[8] The function $E^{\gamma}_{\mu,\nu}(z)$ is defined by

$$E^{\gamma}_{\mu,\nu}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\mu n+\nu)\,n!}, \tag{6}$$

where $\mu,\nu,\gamma,z\in\mathbb{C}$, $\mathrm{Re}\mu>0$, $\mathrm{Re}\nu>0$, $\mathrm{Re}\gamma>0$.[8]

The generalized hypergeometric function ${}_pF_q\left(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;z\right)$ is defined by[9] means of a hypergeometric series as

$${}_pF_q\left[\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix}\middle|\, z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\ldots\left(a_p\right)_n}{(b_1)_n(b_2)_n\ldots\left(b_q\right)_n}\frac{z^n}{n!},$$

where $(a)_n$ is defined by

$$(a)_n:=a(a+1)(a+2)\ldots(a+n-1),\ n=1,2,3,\ldots \text{ and } (a)_0:=1.$$

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The Fox-Wright function is a generalisation of the generalized hypergeometric function ${}_pF_q(z)$ and defined by[10]

$$ {}_p\Psi_q\left[\left.\begin{matrix}(a_1,\alpha_1),\ldots\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots\ldots,(b_q,\beta_q)\end{matrix}\right| z\right]=\sum_{n=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 n)\ldots\Gamma\left(a_p+\alpha_p n\right)}{\Gamma(b_1+\beta_1 n)\ldots\Gamma\left(b_q+\beta_q n\right)}\frac{z^n}{n!}, $$

where

$$ \Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}dt \qquad Re(z)>0. $$

For integers m,n,p,q such that $0\leq m\leq q$ and $0\leq n\leq p$ and the parameters $a_i,b_j\in\mathbb{C},\alpha_i,\beta_j\in\mathbb{R}^+$ $(i=1,2,\ldots,p,\ j=1,2,\ldots,q)$, the H function in terms of Mellin-Barnes type integral is as follows[11–13]:

$$ H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ \ldots\ ,(a_p,\alpha_p)\\(b_1,\beta_1),\ \ldots\ ,(b_q,\beta_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\mathcal{H}_{p,q}^{m,n}(s)z^{-s}ds, \tag{7} $$

where

$$ \mathcal{H}_{p,q}^{m,n}(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j+\beta_j s\right)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma\left(1-b_j-\beta_j s\right)}. $$

The contour L is chosen in such a way to separate the poles of the two factors in the numerator. The following sufficient conditions for the absolute convergence of the defining integral for the H-function given by Equation (7) have been given by

$$ \zeta=\sum_{j=1}^{m}\left|\beta_j\right|+\sum_{j=1}^{n}\left|\alpha_j\right|-\sum_{j=m+1}^{p}\left|\beta_j\right|-\sum_{j=n+1}^{q}\left|\alpha_j\right|>0 \text{ and } |\arg(z)|<\frac{\pi}{2}\zeta. $$

In this paper, we consider a different generalization of Laplace transform[6] over the set of functions

$$ A=\left\{f(t)|\exists K,M,a\in\mathbb{R},|t^{\alpha-\mu}f(t)|\leq Ke^{at^{\mu}} \text{ for all } t\geq M,K>0\right\}, \tag{8} $$

which is defined by

$$ F(y)=\mathcal{L}_{\alpha,\mu}\{f(t);y\}=\int_0^{\infty}t^{\alpha-1}e^{-y^{\mu}t^{\mu}}f(t)\,dt, \tag{9} $$

where $\alpha,y\in\mathbb{C}$, $\mu\in\mathbb{R}$, $\mathrm{Re}\alpha>\mu>0$, $\mathrm{Re}y>0$. Setting $\alpha=\mu$ and $y=\left(\frac{y}{\mu}\right)^{\frac{1}{\mu}}$ in (9), we obtain a special case of $\mathcal{L}_{\alpha,\mu}$-transform:

$$ G(y)=\mathcal{L}_{\mu}\{f(t);y\}=\int_0^{\infty}t^{\mu-1}e^{-y\frac{t^{\mu}}{\mu}}f(t)dt, \tag{10} $$

which is defined in Jarad and Abdeljawad.[14]

It is clear that the following relations hold true[6]:

$$ \mathcal{L}_{\alpha,\mu}\{f(t);y\}=\frac{1}{\mu}\mathcal{L}\left\{t^{\alpha/\mu-1}f\left(t^{1/\mu}\right);y^{\mu}\right\}, \tag{11} $$

$$ \mathcal{L}_{\alpha,\mu}\{t^{\mu-\alpha}f(t);y\}=\mathcal{L}_{\mu}\left\{f(t);\left(\frac{y}{\mu}\right)^{\frac{1}{\mu}}\right\}, \tag{12} $$

$$ \mathcal{L}_{\alpha,\mu}\{f(at);y\}=\frac{1}{a^{\alpha}}\mathcal{L}_{\alpha,\mu}\left\{f(t);\frac{y}{a^{1/\mu}}\right\}, \tag{13} $$

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$$\mathcal{L}_{\alpha,\mu}\left\{e^{-a^\mu t^\mu} f(t); y\right\} = \mathcal{L}_{\alpha,\mu}\left\{f(t); (a^\mu + y^\mu)^{1/\mu}\right\}, \tag{14}$$

$$\mathcal{L}_{\alpha,\mu}\left\{t^{\mu-\alpha}\sqrt[\mu]{(t^\mu - z^\mu)^{\alpha-\mu}}\ f\left(\sqrt[\mu]{t^\mu - z^\mu}\right) H(t-z); y\right\} = e^{-z^\mu y^\mu}\mathcal{L}_{\alpha,\mu}\{f(t); y\}, \tag{15}$$

where Re$a > 0$ and $H(t)$ is the Heaviside unit step function.

By the relation (11) and the definition of the inverse Laplace transform formula,[1] the inverse $\mathcal{L}_{\alpha,\mu}^{-1}$ transform[6] can be defined as follows:

$$\mathcal{L}_{\alpha,\mu}^{-1}\{F(y); t\} = \frac{\mu t^{\mu-\alpha}}{2\pi i}\int_C e^{yt^\mu} F\left(y^{1/\mu}\right) dy, \tag{16}$$

where C is chosen so that all the singularities of the integrand of (16) lie to the left of the modified contour C.

2 | BASIC PROPERTIES AND MAIN THEOREMS OF $L\alpha,\mu$ TRANSFORM

In this section, we will first give an existence theorem for the $\mathcal{L}_{\alpha,\mu}$ transform. The existence theorem was proved in Albayrak et al.[6] Here, we will prove this theorem for different classes of functions. Then, according to this theorem, we will evaluate the $\mathcal{L}_{\alpha,\mu}$ transform for some elementary and special functions. Then, we will prove some basic properties and the convolution theorem of the $\mathcal{L}_{\alpha,\mu}$ transform. Finally, we will define a new derivative operator to solve problems.

Theorem 1 (Existence Theorem for $\mathcal{L}_{\alpha,\mu}$-transform)**.** *If a function $f(t)$ is continuous or piecewise continuous in every finite interval* $(0, M)$ *and satisfies the following property for* $K, a, M, \mu \in \mathbb{R}$, $\alpha \in \mathbb{C}$, $K > 0$, *Re*$\alpha \geq \mu > 0$,

$$|t^{\alpha-\mu} f(t)| \leq K e^{at^\mu} \ \textit{for all}\ \ t \geq M,$$

then the $\mathcal{L}_{\alpha,\mu}$–transform of $f(t)$,

$$F(y) = \mathcal{L}_{\alpha,\mu}\{f(t); y\} = \int_0^\infty t^{\alpha-1} e^{-y^\mu t^\mu} f(t)\, dt, \tag{17}$$

exists for all y provided Re$y^\mu > a$.

Proof. To prove the theorem, first of all, it should be shown that the generalized Laplace type transform defined in (17) converges for Re$y^\mu > a$. For this purpose, if we split the integral into two parts, then we get

$$\int_0^\infty t^{\alpha-1} e^{-y^\mu t^\mu} f(t)dt = \int_0^M t^{\alpha-1} e^{-y^\mu t^\mu} f(t)dt + \int_M^\infty t^{\alpha-1} e^{-y^\mu t^\mu} f(t)dt. \tag{18}$$

The first integral on the right-hand side of (18) exists by the first hypothesis of the theorem. By the second hypothesis of the theorem, we have for $t \geq M$,

$$\left|\int_M^\infty t^{\alpha-1} e^{-y^\mu t^\mu} f(t)dt\right| \leq \int_M^\infty \left|t^{\mu-1} e^{-y^\mu t^\mu} t^{\alpha-\mu} f(t)\right| dt \leq K\int_M^\infty t^{\mu-1} e^{-(y^\mu - a)t^\mu} dt = \frac{Ke^{-(y^\mu-\alpha)M}}{\mu(y^\mu - a)},$$

where Re$y^\mu > a$. Thus, the second integral on the right-hand side of (18) converges and the proof is completed. ☐

Now, using the definition of $\mathcal{L}_{\alpha,\mu}$ transform, we could evaluate the $\mathcal{L}_{\alpha,\mu}$ transforms of some elementary functions.

Example 1. Using the relation (11) and the Laplace table, it is quite easy to obtain the following formula:

$$\mathcal{L}_{\alpha,\mu}\left\{t^{\beta-1}; y\right\} = \frac{y^{1-\alpha-\beta}}{\mu}\Gamma\left(\frac{\alpha+\beta-1}{\mu}\right). \qquad \text{Re}(\alpha+\beta) > 1. \tag{19}$$

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Example 2. Setting $f(t) = t^{\delta-1} H_{p,q}^{m,n}\left[x^{\lambda} t^{\gamma} \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right]$ in (17), we have

$$\mathcal{L}_{\alpha,\mu}\left\{ f(t); y \right\} = \frac{1}{2\pi i} \int\limits_{L} \mathcal{H}_{p,q}^{m,n}(s) x^{-\lambda s} \mathcal{L}_{\alpha,\mu}\left\{ t^{\delta-\gamma s-1}; y \right\} dx.$$

Using the formula (19) and the definition of H-function, we obtain

$$\mathcal{L}_{\alpha,\mu}\left\{ f(t); y \right\} = \frac{1}{\mu y^{\alpha+\delta-1}} H_{p+1,q}^{m,n+1}\left[\frac{x^{\lambda}}{y^{\gamma}} \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \left(1-\frac{\alpha+\delta-1}{\mu}; \frac{\gamma}{\mu}\right) \\ (b_j,\beta_j)_{1,q} \end{matrix} \right]. \tag{20}$$

As a result of this example, many generalized integrals that have not been evaluated or evaluated before can be re-evaluated by special cases of parameters. The following two remarks are examples of this. It is possible to increase these results.

Remark 1. If we use special cases of H-function which are given in Mathai et al[15] and the formula (20), we obtain the following results:

1. $\mathcal{L}_{\alpha,\mu}\left\{ t^{\delta-1} e^{-x^{\lambda} t^{\gamma}}; y \right\} = \frac{1}{\mu y^{\alpha+\delta-1}} H_{1,1}^{1,1}\left[\frac{x^{\lambda}}{y^{\gamma}} \,\middle|\, \begin{matrix} \left(1-\frac{\alpha+\delta-1}{\mu}; \frac{\gamma}{\mu}\right) \\ (0,1) \end{matrix} \right]$, where $\left| \arg\left(\frac{x^{\lambda}}{y^{\gamma}}\right) \right| < \frac{\pi}{2}\left(1 + \left|\frac{\gamma}{\mu}\right|\right)$.
2. $\mathcal{L}_{\alpha,\mu}\left\{ t^{\delta-1} \mathcal{J}_{\nu}\left(2x^{\lambda/2} t^{\gamma/2}\right); y \right\} = \frac{1}{\mu y^{\alpha+\delta-1}} H_{1,2}^{1,1}\left[\frac{x^{\lambda}}{y^{\gamma}} \,\middle|\, \begin{matrix} \left(1-\frac{\alpha+\delta-1}{\mu}; \frac{\gamma}{\mu}\right) \\ \left(\frac{\nu}{2},1\right),\left(-\frac{\nu}{2},1\right) \end{matrix} \right]$, where $\left| \arg\left(\frac{x^{\lambda}}{y^{\gamma}}\right) \right| < \frac{\pi}{2}\left|\frac{\gamma}{\mu}\right|$.
3. $\mathcal{L}_{\alpha,\mu}\left\{ t^{\delta-1} E_{\nu,\beta}\left(x^{\lambda} t^{\gamma}\right); y \right\} = \frac{1}{\mu y^{\alpha+\delta-1}} H_{2,2}^{1,2}\left[-\frac{x^{\lambda}}{y^{\gamma}} \,\middle|\, \begin{matrix} (0,1),\left(1-\frac{\alpha+\delta-1}{\mu}; \frac{\gamma}{\mu}\right) \\ (0,1),(1-\beta,\nu) \end{matrix} \right]$, where $\left| \arg\left(\frac{x^{\lambda}}{y^{\gamma}}\right) \right| < \frac{\pi}{2}\left(2 + \left|\frac{\gamma}{\mu}\right| - |\nu|\right)$.
4. $\mathcal{L}_{\alpha,\mu}\left\{ t^{\delta-1} E_{\nu}\left(x^{\lambda} t^{\gamma}\right); y \right\} = \frac{1}{\mu y^{\alpha+\delta-1}} H_{2,2}^{1,2}\left[-\frac{x^{\lambda}}{y^{\gamma}} \,\middle|\, \begin{matrix} (0,1),\left(1-\frac{\alpha+\delta-1}{\mu}; \frac{\gamma}{\mu}\right) \\ (0,1),(0,\nu) \end{matrix} \right]$, where $\left| \arg\left(\frac{x^{\lambda}}{y^{\gamma}}\right) \right| < \frac{\pi}{2}\left(2 + \left|\frac{\gamma}{\mu}\right| - |\nu|\right)$.

Remark 2. If we set $\gamma = \mu$ in Remark 1 and use special cases of H-function which are given in Mathai et al,[15] we find the following special cases:

1. $\mathcal{L}_{\alpha,\mu}\left\{ t^{\delta-1} e^{-x^{\lambda} t^{\mu}}; y \right\} = \frac{\Gamma\left(\frac{\alpha+\delta-1}{\mu}\right)}{\mu\left(y^{\mu}+x^{\lambda}\right)^{\frac{\alpha+\delta-1}{\mu}}}$, where $\left| \arg\left(\frac{x^{\lambda}}{y^{\gamma}}\right) \right| < \pi$.
2. $\mathcal{L}_{\alpha,\mu}\left\{ t^{\delta-1} \mathcal{J}_{\nu}\left(2x^{\lambda/2} t^{\mu/2}\right); y \right\} = \frac{x^{\frac{\lambda\nu}{2}}}{\mu y^{\alpha+\delta+\frac{\mu\nu}{2}-1}} \frac{\Gamma\left(\frac{\alpha+\delta-1}{\mu}\right)}{\Gamma(1+\nu)}\, {}_1F_1\left(\frac{\alpha+\delta-1}{\mu} + \frac{\nu}{2}; 1+\nu; \frac{x^{\lambda}}{y^{\gamma}}\right)$, where $\left| \arg\left(\frac{x^{\lambda}}{y^{\gamma}}\right) \right| < \frac{\pi}{2}$.
3. $\mathcal{L}_{\alpha,\mu}\left\{ t^{\delta-1} E_{\nu,\beta}\left(x^{\lambda} t^{\mu}\right); y \right\} = \frac{\Gamma\left(\frac{\alpha+\delta-1}{\mu}\right)}{\mu y^{\alpha+\delta-1}} E_{\nu,\beta}^{\frac{\alpha+\delta-1}{\mu}}\left(\frac{x^{\lambda}}{y^{\mu}}\right)$, where $\left| \arg\left(\frac{x^{\lambda}}{y^{\gamma}}\right) \right| < \frac{\pi}{2}(3 - |\nu|)$.
4. $\mathcal{L}_{\alpha,\mu}\left\{ t^{\delta-1} E_{\nu}\left(x^{\lambda} t^{\mu}\right); y \right\} = \frac{\Gamma\left(\frac{\alpha+\delta-1}{\mu}\right)}{\mu y^{\alpha+\delta-1}} E_{\nu,1}^{\frac{\alpha+\delta-1}{\mu}}\left(\frac{x^{\lambda}}{y^{\mu}}\right)$, where $\mathrm{Re}(\alpha+\delta) > 1$ and $\left| \arg\left(\frac{x^{\lambda}}{y^{\gamma}}\right) \right| < \frac{\pi}{2}(3 - |\nu|)$.

Now, we will define a new derivative operator and consider the $\mathcal{L}_{\alpha,\mu}$ transform of this derivative of a function to solve differential equations.

Definition 1. A differentiation operator $\delta_{\alpha,\mu}$, which is called $\delta_{\alpha,\mu}$ derivative, is defined as

$$\delta_{\alpha,\mu}^{n} f(t) = \left(\delta_{\alpha} \circ \underbrace{\delta_{\mu} \circ \ \ldots \ \circ \delta_{\mu}}_{(n-1)\,\text{times}} \right) f(t), \qquad (\forall n \in \mathbb{N}), \tag{21}$$

where δ_{μ} derivative is defined as

$$\delta_{\mu} f(t) = \frac{1}{t^{\mu-1}} \frac{d}{dt} f(t). \tag{22}$$

When $\mu \to 1$ in (22), the δ_{μ} derivative and the classical derivative coincide. Now, we will derive $\mathcal{L}_{\alpha,\mu}$ transform of the $\delta_{\alpha,\mu}$ derivative of a function.

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Theorem 2. *If $f(t)$ and $\delta_{\alpha,\mu}f(t)$ satisfy the conditions of Theorem 1, then $\mathcal{L}_{\alpha,\mu}$ transform of the $\delta_{\alpha,\mu}$ derivative of a function is given as follows:*

$$\mathcal{L}_{\alpha,\mu}\left\{\delta_{\alpha,\mu}f(t);y\right\} = \mu y^{\mu}\mathcal{L}_{\mu,\mu}\left\{f(t);y\right\} - f(0). \tag{23}$$

Proof. Using the definitions of the $\mathcal{L}_{\alpha,\mu}$ transform and the $\delta_{\alpha,\mu}$ derivative and then using integration by parts, we obtain (23). ☐

Remark 3. Similarly, if $\delta^{k}_{\alpha,\mu}f(t)$, $(k = 0, 1, \dots, n)$ satisfy the conditions of Theorem 1, then $\mathcal{L}_{\alpha,\mu}$ transform of the $\delta^{n}_{\alpha,\mu}$ derivative of a function is given as

$$\mathcal{L}_{\alpha,\mu}\left\{\delta^{n}_{\alpha,\mu}f(t);y\right\} = \mu^{n}y^{n\mu}\mathcal{L}_{\mu,\mu}\left\{f(t);y\right\} - \sum_{k=0}^{n-1}\mu^{(n-k-1)}y^{(n-k-1)\mu}\delta^{k}_{\mu}f(0). \tag{24}$$

The Convolution Theorem for $\mathcal{L}_{\alpha,\mu}$ transform was given in Albayrak et al.[6] Now, we will prove it to solve differential and integral equations.

Theorem 3 (The Convolution Theorem for $\mathcal{L}_{\alpha,\mu}$ transform). *If $F(y) = \mathcal{L}_{\alpha,\mu}\{f(t);y\}$ and $G(y) = \mathcal{L}_{\alpha,\mu}\{g(t);y\}$, then*

$$\mathcal{L}_{\alpha,\mu}\left\{(f * g)(t);y\right\} = F(y)G(y), \tag{25}$$

where $f(t) * g(t)$ is the convolution of $f(t)$ and $g(t)$ and it is defined by the following integral:

$$f(t) * g(t) = t^{\mu-\alpha}\int_{0}^{t}\xi^{\alpha-1}\sqrt[\mu]{(t^{\mu}-\xi^{\mu})^{\alpha-\mu}}\, f\left(\sqrt[\mu]{t^{\mu}-\xi^{\mu}}\right)g(\xi)d\xi. \tag{26}$$

Proof. Using the definitions (9), (26), and changing the order of integration, which is permissible by absolute convergence of the integrals involved, we obtain

$$\mathcal{L}_{\alpha,\mu}\left\{f(t) * g(t);y\right\} = \int_{0}^{\infty}\xi^{\alpha-1}g(\xi)\left(\int_{t=\xi}^{\infty}t^{\mu-1}e^{-y^{\mu}t^{\mu}}\sqrt[\mu]{(t^{\mu}-\xi^{\mu})^{\alpha-\mu}}\, f\left(\sqrt[\mu]{t^{\mu}-\xi^{\mu}}\right)dt\right)d\xi.$$

Now, making the change of variable $t^{\mu} - \xi^{\mu} = u^{\mu}$ on the right-hand side of integration, we obtain (25). ☐

Theorem 4. *The δ_{μ}-derivatives of the $\mathcal{L}_{\alpha,\mu}$-transform is as follow:*

$$\delta_{\mu}\mathcal{L}_{\alpha,\mu}\left\{f(t);y\right\} = -\mu\mathcal{L}_{\alpha+\mu,\mu}\left\{f(t);y\right\} \tag{27}$$

Proof. Using the definition of $\mathcal{L}_{\alpha,\mu}$ transform and changing the order of derivative and integral operators, we get

$$\frac{1}{y^{\mu-1}}\frac{d}{dy}\mathcal{L}_{\alpha,\mu}\left\{f(t);y\right\} = -\mu\int_{0}^{\infty}t^{\alpha+\mu-1}e^{-y^{\mu}t^{\mu}}f(t)\,dt.$$

Using the definition of $\mathcal{L}_{\alpha,\mu}$ transform once again, we obtain (27). ☐

By induction, we obtain

$$\delta^{n}_{\mu}\mathcal{L}_{\alpha,\mu}\left\{f(t);y\right\} = (-1)^{n}\mu^{n}\mathcal{L}_{\alpha,\mu}\left\{t^{\mu n}f(t);y\right\}. \tag{28}$$

Now, we will give $\mathcal{L}_{\alpha,\mu}$ transform of a definite integral and example.

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Theorem 5. *The following identity holds true:*

$$\mathcal{L}_{\alpha,\mu}\left\{t^{\mu-\alpha}\int_{0}^{t}x^{\alpha-1}f(x)dx;y\right\}=\frac{1}{\mu y^{\mu}}\mathcal{L}_{\alpha,\mu}\left\{f(x);y\right\} \tag{29}$$

provided that the integrals involved converge absolutely.

Proof. Using the definition of $\mathcal{L}_{\alpha,\mu}$ transform and changing the order of integration, which is permissible by absolute convergence of the integrals involved, we get

$$\mathcal{L}_{\alpha,\mu}\left\{t^{\mu-\alpha}\int_{0}^{t}x^{\alpha-1}f(x)dx;y\right\}=\int_{0}^{\infty}t^{\mu-1}e^{-y^{\mu}t^{\mu}}\left(\int_{0}^{t}x^{\alpha-1}f(x)dx\right)dt=\int_{0}^{\infty}\left(\int_{x}^{\infty}t^{\mu-1}e^{-y^{\mu}t^{\mu}}dt\right)x^{\alpha-1}f(x)dx.$$

Now, evaluating the inner integral, we obtain (29). ☐

Remark 4. If we set $f(x)=\sqrt[\mu]{(t^{\mu}-x^{\mu})^{\alpha-\mu}}f\left(\sqrt[\mu]{t^{\mu}-x^{\mu}}\right)g(x)$ in (29) and use the relation (25), we obtain

$$\int_{0}^{\infty}x^{\alpha-1}\sqrt[\mu]{(t^{\mu}-x^{\mu})^{\alpha-\mu}}f\left(\sqrt[\mu]{t^{\mu}-x^{\mu}}\right)g(x)dx=\mu y^{\alpha}\mathcal{L}_{\alpha,\mu}\left\{f(x);y\right\}\mathcal{L}_{\alpha,\mu}\left\{g(x);y\right\}. \tag{30}$$

Now, we will give some integrals of $\mathcal{L}_{\alpha,\mu}$ transform and examples.

Theorem 6. *The following identity holds true:*

$$\int_{y}^{\infty}y^{\mu-1}\mathcal{L}_{\alpha,\mu}\left\{f(t);y\right\}dy=\frac{1}{\mu}\mathcal{L}_{\alpha,\mu}\left\{\frac{f(t)}{t^{\mu}};y\right\} \tag{31}$$

provided that the integrals involved converge absolutely.

Proof. Starting the proof with steps similar to the proof of Theorem 5, we obtain

$$\int_{y}^{\infty}y^{\mu-1}\mathcal{L}_{\alpha,\mu}\left\{f(t);y\right\}dy=\int_{y}^{\infty}y^{\mu-1}\left(\int_{0}^{\infty}t^{\alpha-1}e^{-y^{\mu}t^{\mu}}f(t)dt\right)dy=\int_{0}^{\infty}t^{\alpha-1}f(t)\left(\int_{y}^{\infty}y^{\mu-1}e^{-y^{\mu}t^{\mu}}dy\right)dt.$$

Finally, evaluating the inner integral and using the definition of $\mathcal{L}_{\alpha,\mu}$ transform, we reach (31). ☐

Theorem 7. *The following identity holds true:*

$$\int_{0}^{\infty}\mathcal{L}_{\alpha,\mu}\left\{f(t);y\right\}dy=\frac{1}{\mu}\Gamma\left(\frac{1}{\mu}\right)\mathcal{M}\left\{f(t);\alpha-1\right\}, \tag{32}$$

where $\mathcal{M}\left\{f(t);y\right\}$ is the Mellin transform.

Proof. Using (9) and changing the order of integration, which is permissible by absolute convergence of the integrals involved, we have

$$\int_{0}^{\infty}\mathcal{L}_{\alpha,\mu}\left\{f(t);y\right\}dy=\int_{0}^{\infty}t^{\alpha-1}f(t)\left(\int_{0}^{\infty}e^{-y^{\mu}t^{\mu}}dy\right)dt.$$

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Now, using (9) for $\alpha = 1$, the formula (19) and then using the definition of Mellin transform, we obtain (32). □

Using the relations (31) and (32), which are called Parseval-Goldstein relations, many new generalized integrals can be evaluated.

3 | APPLICATIONS

In this section, we will solve a generalization of the harmonic oscillator problem in a non-resisting and resisting (internal mechanical resistance or external air resistance) medium problems, some initial boundary value problems, and some integral equations. Finally, we obtain the known series using again the $\mathcal{L}_{\alpha,\mu}$ transform.

Example 3. (A generalization of The Harmonic Oscillator in a Non-Resisting Medium Problem). We consider the differential equation of a generalization of the oscillator in the presence of an external driving force $Ft^{2\mu-2}f(t)$ is

$$\frac{d^2x}{dt^2} - \frac{\mu-1}{t}\frac{dx}{dt} + w^2t^{2\mu-2}x = Ft^{2\mu-2}f(t), \tag{33}$$

where w is the frequency and F is a constant with initial conditions

$$x(t=0) = a \text{ and } \delta_{\alpha,\mu}x(t=0) = U, \tag{34}$$

where a and U are constants.[1]

Equation (33) with initial conditions (34) corresponds to that of a harmonic oscillator in a non-resisting (internal mechanical resistance or external air resistance) medium problem as a special when $\mu = 1$.

If the differential equation (33) is multiplied by $t^{-\alpha-\mu+2}$, we get

$$\frac{1}{t^{\alpha+\mu-2}}\frac{d^2x}{dt^2} - \frac{\mu-1}{t^{\alpha+\mu-1}}\frac{dx}{dt} + w^2t^{\mu-\alpha}x = Ft^{\mu-\alpha}f(t). \tag{35}$$

Using the definition of $\delta_{\alpha,\mu}$ derivative (21) in (35), then we have

$$\delta^2_{\alpha,\mu}x(t) + w^2t^{\mu-\alpha}x = Ft^{\mu-\alpha}f(t). \tag{36}$$

Now, applying the $\mathcal{L}_{\alpha,\mu}$ transform to both sides of (36) and using the identity (24), then we obtain

$$(\mu^2s^{2\mu} + w^2)\mathcal{L}_{\mu,\mu}\{x(t);s\} = a\mu s^{\mu} + U + F\mathcal{L}_{\mu,\mu}\{f(t);y\}, \tag{37}$$

or

$$\mathcal{L}_{\mu,\mu}\{x(t);s\} = \frac{a\mu s^{\mu}}{\mu^2s^{2\mu}+w^2} + \frac{U}{\mu^2s^{2\mu}+w^2} + \frac{F}{\mu^2s^{2\mu}+w^2}\mathcal{L}_{\mu,\mu}\{f(t);y\}. \tag{38}$$

Applying the $\mathcal{L}^{-1}_{\alpha,\mu}$ transform to both sides of (38) for $\alpha = \mu$ and using the convolution theorem (25), we get

$$x(t) = \mathcal{L}^{-1}_{\mu,\mu}\left\{\frac{a\mu s^{\mu}}{\mu^2s^{2\mu}+w^2};t\right\} + \mathcal{L}^{-1}_{\mu,\mu}\left\{\frac{U}{\mu^2s^{2\mu}+w^2};t\right\} + \mathcal{L}^{-1}_{\mu,\mu}\left\{\frac{F}{\mu^2s^{2\mu}+w^2};t\right\} * f(t). \tag{39}$$

By the definition of $\mathcal{L}^{-1}_{\alpha,\mu}$ transform for $\alpha = \mu$, we have

$$\begin{aligned}\mathcal{L}^{-1}_{\alpha,\mu}\left\{\frac{1}{\mu^2s^{2\mu}+w^2};t\right\} &= \frac{\mu}{2\pi i}\int_C \frac{e^{st^{\mu}}}{\mu^2s^{2\mu}+w^2}ds\\ &= \text{Res}\left\{\frac{e^{st^{\mu}}}{\mu^2s^{2\mu}+w^2};-\left(\frac{w}{\mu}\right)^{\frac{1}{\mu}}\right\} + \text{Res}\left\{\frac{e^{st^{\mu}}}{\mu^2s^{2\mu}+w^2};\left(\frac{w}{\mu}\right)^{\frac{1}{\mu}}\right\}\\ &= \frac{1}{w}\sin\left(\frac{w}{\mu}t^{\mu}\right)\end{aligned} \tag{40}$$

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and

$$\begin{aligned}\mathcal{L}^{-1}_{\alpha,\mu}\left\{\frac{s^{\mu}}{\mu^2 s^{2\mu}+w^2};t\right\} &= \frac{\mu}{2\pi i}\int\limits_{C}\frac{e^{st^{\mu}}s^{\mu}}{\mu^2 s^{2\mu}+w^2}ds \\ &= \operatorname{Res}\left\{\frac{e^{st^{\mu}}s^{\mu}}{\mu^2 s^{2\mu}+w^2};-\left(\frac{w}{\mu}\right)^{\frac{1}{\mu}}\right\}+\operatorname{Res}\left\{\frac{e^{st^{\mu}}s^{\mu}}{\mu^2 s^{2\mu}+w^2};\left(\frac{w}{\mu}\right)^{\frac{1}{\mu}}\right\} \\ &= \frac{1}{\mu}\cos\left(\frac{w}{\mu}t^{\mu}\right).\end{aligned} \tag{41}$$

Using the results (40) and (41) into (39) and then using the definition (26) for $\alpha = \mu$, we obtain the solution of (33) is

$$x(t) = a\cos\left(\frac{w}{\mu}t^{\mu}\right)+\frac{U}{w}\sin\left(\frac{w}{\mu}t^{\mu}\right)+\frac{F}{w}\int\limits_{0}^{t}\xi^{\mu-1}\sin\left(\frac{w}{\mu}\xi^{\mu}\right)f\left(\sqrt[\mu]{t^{\mu}-\xi^{\mu}}\right)d\xi,$$

or

$$x(t) = A\cos\left(\frac{wt^{\mu}}{\mu}+\phi\right)+\frac{F}{w}\int\limits_{0}^{t}\xi^{\mu-1}\sin\left(\frac{w\xi^{\mu}}{\mu}\right)f\left(\sqrt[\mu]{t^{\mu}-\xi^{\mu}}\right)d\xi, \tag{42}$$

where $A = \sqrt{a^2+\frac{U^2}{w^2}}$ and $\phi = \arctan\left(\frac{U}{wa}\right)$. There are two terms in this solution. The first term reflects the oscillator's response to the initial data and describes free oscillations with amplitude A, phase ϕ, and frequency ω, which is known as the oscillator's natural frequency. Forced oscillations are represented by the second term, which originates in reaction to an external force. We analyze the following specific situations[1] to study some interesting aspects of the solution (42). In Figures 1 and 2, we graphically compare these specific situations of the solution (42):

- **Zero Forcing Function:** ($\mathbf{f(t) = 0}$). In this situation, (33) reduces to

  $$\frac{d^2x}{dt^2}-\frac{\mu-1}{t}\frac{dx}{dt}+w^2t^{2\mu-2}x = 0,\ x(0) = a \text{ and } \delta_{\alpha,\mu}x(0) = U, \tag{43}$$

  where a and U are constants, and solution (42) reduces to

  $$x(t) = A\cos\left(\frac{wt^{\mu}}{\mu}+\phi\right). \tag{44}$$

  With amplitude A, frequency w, and phase ϕ, this is a simple harmonic motion. Obviously, it is an oscillating motion.[1]
- **Steady Forcing Function:** ($\mathbf{f(t) = 1}$). In this case, (33) reduces to

  $$\frac{d^2x}{dt^2}-\frac{\mu-1}{t}\frac{dx}{dt}+w^2t^{2\mu-2}x = Ft^{2\mu-2},\ x(0) = a \text{ and } \delta_{\alpha,\mu}x(0) = U, \tag{45}$$

  where a and U are constants, and solution (42) reduces to

  $$x(t) = A\cos\left(\frac{wt^{\mu}}{\mu}+\phi\right)-\frac{F}{w^2}\cos\left(\frac{wt^{\mu}}{\mu}\right)+\frac{F}{w^2}. \tag{46}$$

  In particular, when the particle is released from rest, $U = 0$, it takes the form

  $$x-\frac{F}{w^2} = \left(a-\frac{F}{w^2}\right)\cos\left(\frac{wt^{\mu}}{\mu}\right).$$

  This corresponds to free oscillations with natural frequency ω and shows a shift of the equilibrium position from the origin to the point $\frac{F}{\omega^2}$.[1]

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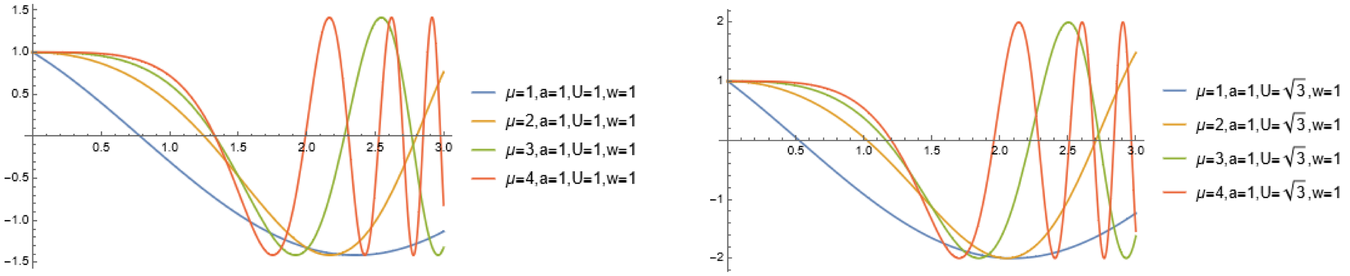
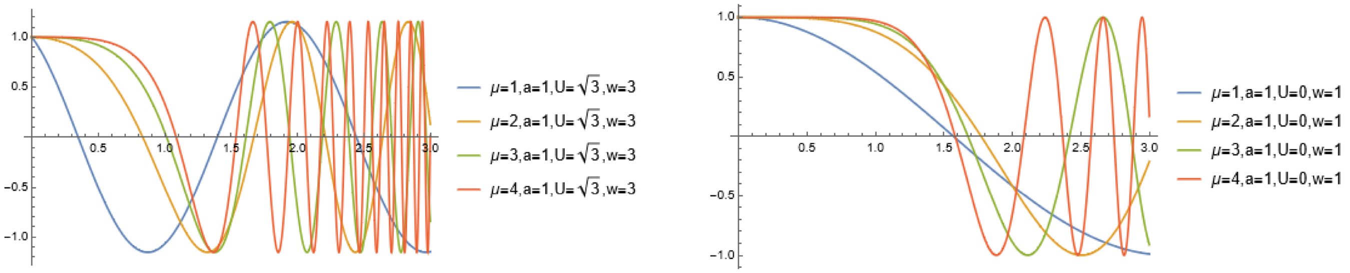


(A) Amplitude $A = \sqrt{2}$, frequency $w = 1$ and phase $\phi = \frac{\pi}{4}$

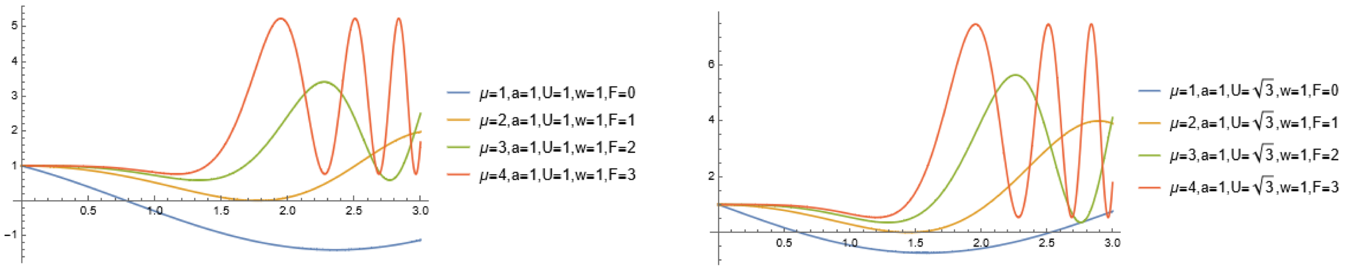
(B) Amplitude $A = 2$, frequency $w = 1$ and phase $\phi = \frac{\pi}{3}$



(C) Amplitude $A = \frac{2}{\sqrt{3}}$, frequency $w = 3$ and phase $\phi = \frac{\pi}{6}$

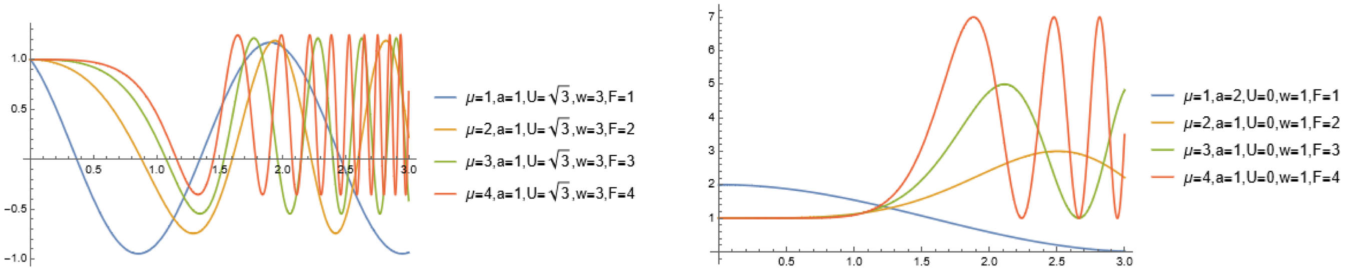
(D) Amplitude $A = 1$, frequency $w = 1$ and phase $\phi = 0$

FIGURE 1 Zero zorcing function: $(\mathbf{f(t) = 0})$ [Colour figure can be viewed at wileyonlinelibrary.com]



(A) Amplitude $A = \sqrt{2}$, frequency $w = 1$ and phase $\phi = \frac{\pi}{4}$

(B) Amplitude $A = 2$, frequency $w = 1$ and phase $\phi = \frac{\pi}{3}$



(C) Amplitude $A = \frac{2}{\sqrt{3}}$, frequency $w = 3$ and phase $\phi = \frac{\pi}{6}$

(D) Amplitude $A = 1$, frequency $w = 1$ and phase $\phi = 0$

FIGURE 2 Steady forcing function: $(\mathbf{f(t) = 1})$ [Colour figure can be viewed at wileyonlinelibrary.com]

Example 4. (A generalization of The Harmonic Oscillator in a Resisting Medium Problem). We consider the differential equation of a generalization of the oscillator in the presence of an external driving force $Ft^{2\mu-2}f(t)$ is

$$\frac{d^2x}{dt^2} - \left[\frac{(\mu-1)-2kt^{\mu}}{t}\right]\frac{dx}{dt} + w^2t^{2\mu-2}x = Ft^{2\mu-2}f(t), \tag{47}$$

where w is the frequency and F is a constant with initial conditions

$$x(t=0) = a \ \text{and} \ \delta_{\alpha,\mu}x(t=0) = U, \tag{48}$$

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where a and U are constants.[1]

Equation (47) with initial conditions (48) corresponds to that of a harmonic oscillator in a resisting (internal mechanical resistance or external air resistance) medium problem as a special when $\mu = 1$.

If the differential equation (47) is multiplied by $t^{-\alpha-\mu+2}$, we get

$$\frac{1}{t^{\alpha+\mu-2}}\frac{d^2x}{dt^2} - \frac{\mu-1-2kt^{\mu}}{t^{\alpha+\mu-1}}\frac{dx}{dt} + w^2t^{\mu-\alpha}x = Ft^{\mu-\alpha}f(t). \tag{49}$$

Using the definition of $\delta_{\alpha,\mu}$ derivative (21) in (49), then we have

$$\delta^2_{\alpha,\mu}x(t) + 2k\delta_{\alpha,\mu}x(t) + w^2t^{\mu-\alpha}x(t) = Ft^{\mu-\alpha}f(t). \tag{50}$$

Now, applying the $\mathcal{L}_{\alpha,\mu}$ transform to both sides of (50) and using the identity (24), then we obtain

$$(\mu^2s^{2\mu} + 2k\mu s^{\mu} + w^2)\mathcal{L}_{\mu,\mu}\{x(t);s\} = a\mu s^{\mu} + 2ka + U + F\mathcal{L}_{\mu,\mu}\{f(t);y\},$$

or

$$\mathcal{L}_{\mu,\mu}\{x(t);s\} = \frac{a\mu s^{\mu} + ak}{(\mu s^{\mu}+k)^2 + l^2} + \frac{U + ak}{(\mu s^{\mu}+k)^2 + l^2} + \frac{F}{(\mu s^{\mu}+k)^2 + l^2}\mathcal{L}_{\mu,\mu}\{f(t);y\}. \tag{51}$$

where $l^2 = w^2 - k^2$. Applying the $\mathcal{L}^{-1}_{\alpha,\mu}$ transform to both sides of (51) for $\alpha = \mu$ and using the convolution theorem (25), we get

$$x(t) = \mathcal{L}^{-1}_{\mu,\mu}\left\{\frac{a(\mu s^{\mu}+k)}{(\mu s^{\mu}+k)^2 + l^2};t\right\} + \mathcal{L}^{-1}_{\mu,\mu}\left\{\frac{U+ak}{(\mu s^{\mu}+k)^2 + l^2};t\right\} + \mathcal{L}^{-1}_{\mu,\mu}\left\{\frac{F}{(\mu s^{\mu}+k)^2 + l^2};t\right\} * f(t), \tag{52}$$

where $l^2 = w^2 - k^2$. Three possible cases deserve attention:

- **Small Damping:** (**k** $<$ **w**) By the (14) and the formulas (40) and (41), and then using the definition (26) for $\alpha = \mu$, we obtain the solution of (47) is[1]

$$x(t) = ae^{-\frac{k}{\mu}t^{\mu}}\cos\left(\frac{l}{\mu}t^{\mu}\right) + \frac{U+ak}{l}e^{-\frac{k}{\mu}t^{\mu}}\sin\left(\frac{l}{\mu}t^{\mu}\right) + \frac{F}{l}\int_0^t \xi^{\mu-1}\sin\left(\frac{l}{\mu}\xi^{\mu}\right) f\left(\sqrt[\mu]{t^{\mu}-\xi^{\mu}}\right)d\xi. \tag{53}$$

- **Critical Damping:** (**k = w**) Suppose $l^2 = 0$. By (52), the solution can be given the form[1]

$$x(t) = ae^{-\frac{k}{\mu}t^{\mu}} + \frac{U+ak}{\mu}t^{\mu}e^{-\frac{k}{\mu}t^{\mu}} + \frac{F}{\mu}\int_0^t \xi^{2\mu-1}e^{-\frac{k}{\mu}\xi^{\mu}} f\left(\sqrt[\mu]{t^{\mu}-\xi^{\mu}}\right)d\xi. \tag{54}$$

- **Large Damping:** (**k** $>$ **w**) If we put $l = il$ in (53) and use the relationship between trigonometric functions and hyperbolic functions, we get the solution in the following form[1]

$$x(t) = ae^{-\frac{k}{\mu}t^{\mu}}\cosh\left(\frac{l}{\mu}t^{\mu}\right) + \frac{U+ak}{l}e^{-\frac{k}{\mu}t^{\mu}}\sinh\left(\frac{l}{\mu}t^{\mu}\right) + \frac{F}{l}\int_0^t \xi^{\mu-1}\sinh\left(\frac{l}{\mu}\xi^{\mu}\right) f\left(\sqrt[\mu]{t^{\mu}-\xi^{\mu}}\right)d\xi. \tag{55}$$

If we set the term of the external force $f(t) = 0$ in this problem, then the solution is

$$x(t) = Ae^{-\frac{k}{\mu}t^{\mu}}\cos\left(\frac{l}{\mu}t^{\mu} + \phi\right),\ (0 < k < \omega), \tag{56}$$

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where $A = \left(a^2 + \frac{(U+ak)^2}{l^2}\right)^{1/2}$ and $\phi = \arctan\left(\frac{U+ak}{al}\right)$. This means that, if the damping is small, the motion is changed by the small resistance and decays to zero as $t \to \infty$.

$$x(t) = ae^{-\frac{k}{\mu}t^\mu} + \frac{U+ak}{\mu}t^\mu e^{-\frac{k}{\mu}t^\mu},\ (k = \omega). \tag{57}$$

Thus, when damping is critical, the motion has no oscillation and decays very rapidly as $t \to \infty$.

$$x(t) = Ae^{-\left(\frac{k}{\mu}-\frac{l}{\mu}\right)t^\mu} + Be^{-\left(\frac{k}{\mu}+\frac{l}{\mu}\right)t^\mu}, \tag{58}$$

where $A = \frac{1}{2}\left(a + \frac{U+ak}{l}\right)$ and $B = \frac{1}{2}\left(a - \frac{U+ak}{l}\right)$. This solution shows that the motion is no longer oscilatory and decays very rapidly as $t \to \infty$.

Now, we will compare these three possible cases graphically in Figure 3 for the external force $f(t) = 0$.

Now, we will solve a special case of an initial-value problem of the form

$$a(t)u_t(x,t) + b(x)u_x(x,y) = f(x), \qquad x > 0,\ t > 0,$$

with the initial and boundary conditions

$$u(x,0) = 0 \qquad x > 0, \qquad u(0,t) = 0 \qquad t > 0,$$

via the $\mathcal{L}_{\alpha,\mu}$ transform.

Example 5. If we set $a(t) = t^{1-\alpha}$, $b(x) = x^{\mu-\alpha+1}$, then we obtain the first-order initial-boundary value problem

$$\frac{u_t}{t^{\alpha-1}} + x^{\mu-\alpha+1}u_x = x^{\mu-\alpha+1}, \qquad x > 0,\ t > 0,\ \mathrm{Re}\alpha \geq \mu \geq 0, \tag{59}$$

with the initial and boundary conditions

$$u(x,0) = 0 \qquad x > 0, \tag{60}$$

$$u(0,t) = 0 \qquad t > 0. \tag{61}$$

Using the definition of $\delta_{\alpha,\mu}$ derivative (21) in (59), then we have

$$\delta_{\alpha,\mu}u + x^{\mu-\alpha+1}u_x = x^{\mu-\alpha+1}. \tag{62}$$

Applying the $\mathcal{L}_{\alpha,\mu}$ transform to both sides of (62) with respect to t, we obtain

$$\mu s^\mu U(x,s) - u(x,0) + x\frac{d}{dx}U(x,s) = \frac{x}{\mu s^\mu}, \qquad U(0,s) = 0,$$

or

$$\frac{d}{dx}U(x,s) + \frac{\mu s^\mu}{x}U(x,s) = \frac{1}{\mu s^\mu}, \qquad U(0,s) = 0, \tag{63}$$

where $U(x,s) = \mathcal{L}_{\mu,\mu}\left\{u(x,t)\right\}$. Multiplying the differential equation (63) by the integration factor $x^{\mu s^\mu}$, we get

$$\frac{d}{dx}\left(x^{\mu s^\mu}U(x,s)\right) = \frac{x^{\mu s^\mu}}{\mu s^\mu}, \qquad U(0,s) = 0. \tag{64}$$

Thus, the solution of (64) is

$$U(x,s) = \frac{x}{\mu s^\mu(\mu s^\mu + 1)} = x\left(\frac{1}{\mu s^\mu} - \frac{1}{\mu s^\mu + 1}\right). \tag{65}$$

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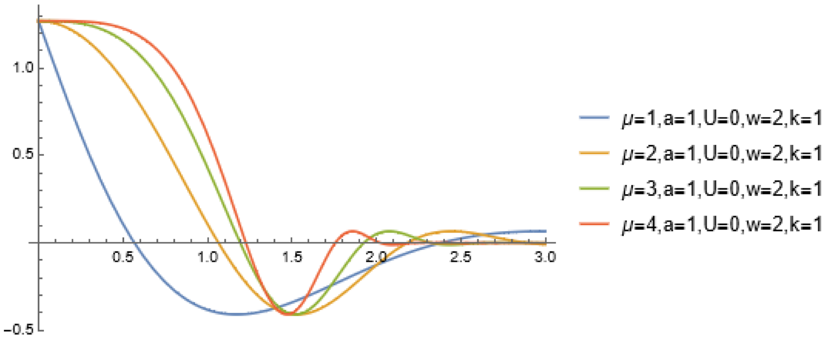
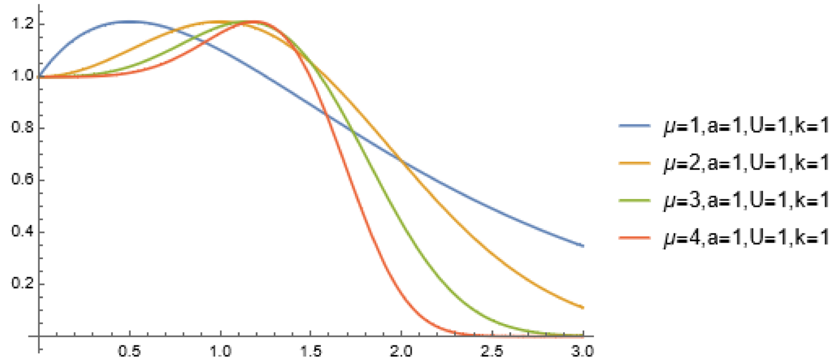
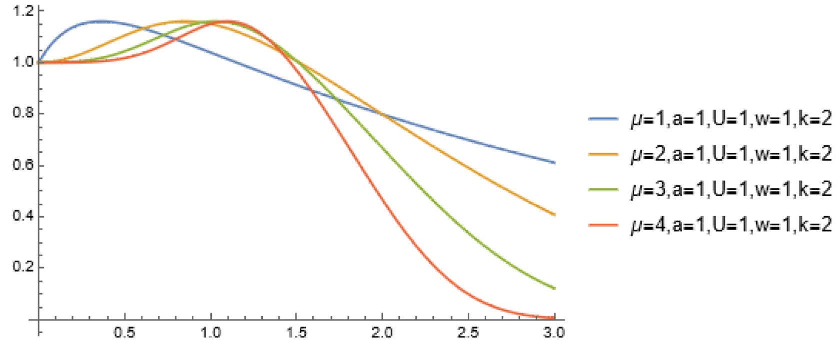
FIGURE 3 External force: $f(t) = 0$ [Colour figure can be viewed at wileyonlinelibrary.com]



(A) Small Damping: (k < w)



(B) Critical Damping: (k = w)



(C) Large Damping: (k > w)

Finally, if we apply the $\mathcal{L}^{-1}_{\alpha,\mu}$ transform to both sides of (65) for $\alpha = \mu$, we obtain the solution of (59) with initial and boundary conditions is

$$u(x,t) = x(1 - e^{-t^{\mu}}). \tag{66}$$

Now, we will solve a different type of an initial-value problem using $\mathcal{L}_{\alpha,\mu}$ transform.

Example 6. We consider the following initial-boundary value problem:

$$t^{2-2\mu}u_{tt} - t^{1-2\mu}u_t = c^2 u_{xx}, \qquad x > 0,\ t > 0,\ \mu \geq 0, \tag{67}$$

with the initial and boundary conditions

$$u(0,t) = f(t) \qquad t > 0, \tag{68}$$

$$u(x,0) = 0 \qquad x > 0, \tag{69}$$

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$$\lim_{x\to\infty} u(x,t) = 0, \tag{70}$$

$$\delta_{\alpha,\mu} u(x,0) = 0 \qquad x > 0. \tag{71}$$

Multiplying the initial-boundary value problem (67) by $t^{\mu-\alpha}$ ($\mathrm{Re}\alpha \geq \mu \geq 0$) and using the definition of $\delta_{\alpha,\mu}$ derivative (21) in (59) then, we have

$$\delta^2_{\alpha,\mu} u = c^2 t^{\mu-\alpha} u_{xx}. \tag{72}$$

Applying the $\mathcal{L}_{\alpha,\mu}$ transform to both sides of (72) with respect to t and initial and boundary conditions, we obtain

$$\frac{d^2}{dx^2} U(x,s) - \frac{\mu^2 s^{2\mu}}{c^2} U(x,s) = 0, \tag{73}$$

$$U(0,s) = F(s), \tag{74}$$

$$\lim_{x\to\infty} U(x,s) = 0, \tag{75}$$

where $U(x,s) = \mathcal{L}_{\mu,\mu}\{u(x,t);s\}$ and $F(s) = \mathcal{L}_{\mu,\mu}\{f(t);s\}$. If we solve this differantial equation under given conditions, then we have

$$U(x,s) = F(s) e^{-\frac{x\mu}{c} s^{\mu}}. \tag{76}$$

Finally, if we apply the $\mathcal{L}^{-1}_{\alpha,\mu}$ transform to both sides of (76) and use (13) for $\alpha = \mu$, we obtain the solution of (67) with initial and boundary conditions is

$$u(x,t) = f\left(\sqrt[\mu]{t^{\mu} - \frac{x\mu}{c}}\right) H\left(t - \left(\frac{x\mu}{c}\right)^{1/\mu}\right), \tag{77}$$

where $H(t)$ is the Heaviside unit step function.

Example 7. We consider the integral equation

$$f(t) = h(t) + \lambda t^{\mu-\alpha} \int_0^t \xi^{\alpha-1} \sqrt[\mu]{(t^{\mu} - \xi^{\mu})^{\alpha-\mu}}\, g\left(\sqrt[\mu]{t^{\mu} - \xi^{\mu}}\right) f(\xi) d\xi. \tag{78}$$

Applying the $\mathcal{L}_{\alpha,\mu}$ transform to both sides of (78), we obtain

$$F(y) = H(y) + \lambda F(y) G(y),$$

where $\mathcal{L}_{\alpha,\mu}\{f(t),y\} = F(y)$, $\mathcal{L}_{\alpha,\mu}\{g(t),y\} = G(y)$ and $\mathcal{L}_{\alpha,\mu}\{h(t),y\} = H(y)$. Hence, we get

$$F(y) = \frac{H(y)}{1 - \lambda G(y)}. \tag{79}$$

Applying the $\mathcal{L}^{-1}_{\alpha,\mu}$ transform to both sides of (79), we obtain the solution of (78) as follows:

$$f(t) = \frac{\mu t^{\mu-\alpha}}{2\pi i} \int_C e^{yt^{\mu}} \frac{H\left(y^{1/\mu}\right)}{1 - \lambda G\left(y^{1/\mu}\right)} dy = \mu t^{\mu-\alpha} \sum_k \mathrm{Res}\left\{ e^{yt^{\mu}} \frac{H\left(y^{1/\mu}\right)}{1 - \lambda G\left(y^{1/\mu}\right)}; y_k \right\}. \tag{80}$$

Remark 5. If we choose $h(t) = g(t) = t^{\mu-\alpha}$ in (78), then we obtain the following integral equation:

$$f(t) = t^{\mu-\alpha} + \lambda t^{\mu-\alpha} \int_0^t \xi^{\alpha-1} f(\xi) d\xi,$$

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and we have its solution as follows:

$$f(t) = t^{\mu-\alpha} e^{\lambda t^{\mu}/\mu}.$$

Now, we will show the well-known series (2) via the $\mathcal{L}_{\alpha,\mu}$ transform.

Example 8. Let's consider the following series

$$\sum_{n=1}^{\infty} a_n F\left(\sqrt[\mu]{n}\right), \tag{81}$$

where $F(n) = \mathcal{L}_{\alpha,\mu}\{f(t), n\}$. Using the definition of $\mathcal{L}_{\alpha,\mu}$ transform and interchanging summation and integration, we obtain

$$\sum_{n=1}^{\infty} a_n F\left(\sqrt[\mu]{n}\right) = \sum_{n=1}^{\infty} a_n \left\{ \int_0^{\infty} t^{\alpha-1} e^{-nt^{\mu}} f(t)\, dt \right\} = \int_0^{\infty} t^{\alpha-1} f(t) \left\{ \sum_{n=1}^{\infty} a_n e^{-nt^{\mu}} \right\} dt. \tag{82}$$

If we set $f(t) = \frac{t^{\delta-1} e^{-x^{\lambda} t^{\mu}}}{\Gamma\left(\frac{\alpha+\delta-1}{\mu}\right)}$ and $a_n = 1$ in (82) and use Remark 1, then we have

$$\frac{1}{\mu} \sum_{n=1}^{\infty} \frac{1}{\left(n + x^{\lambda}\right)^{(\alpha+\delta-1)/\mu}} = \frac{1}{\Gamma\left(\frac{\alpha+\delta-1}{\mu}\right)} \int_0^{\infty} \frac{t^{\alpha+\delta-2} e^{-x^{\lambda} t^{\mu}}}{1 - e^{t^{\mu}}} dt.$$

Making the change of variable $t = u^{1/\mu}$, we get

$$\sum_{n=1}^{\infty} \frac{1}{\left(n + x^{\lambda}\right)^{(\alpha+\delta-1)/\mu}} = \zeta\left(\frac{\alpha+\delta-1}{\mu}, x^{\lambda} + 1\right),$$

where ζ is the Hurwitz Zeta function. Finally, if we put $\frac{\alpha+\delta-1}{\mu} = 2$, $x = 0$ and use the value $\zeta(2,1) = \frac{\pi^2}{6}$, we obtain the following well-known series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{83}$$

4 | CONCLUSION

Laplace integral transform is used in a very large domain such as applied mathematics, physics, and engineering. In this paper, we considered a new Laplace-type integral transform and applied it to some important differential (partial and integral) equations and evaluated a well-known series: (83). Thus, by applying the new integral transform $\mathcal{L}_{\alpha,\mu}$, we can solve fractional differential equations. Also by using the $\mathcal{L}_{\alpha,\mu}$-integral transform, we can obtain Parseval-Goldstein type integral relations.

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CONFLICT OF INTEREST

The author declare no potential conflict of interest.

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