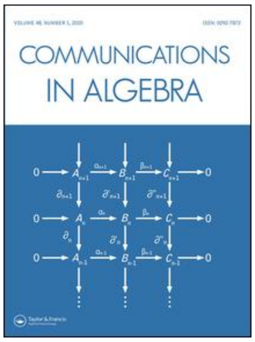

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S-principal ideal multiplication modules

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ABSTRACT

In this paper, we study S-Principal ideal multiplication modules. Let A be a commutative ring with $1 \neq 0$, $S \subseteq A$ a multiplicatively closed set and M an A-module. A submodule N of M is said to be an *S-multiple* of M if there exist $s \in S$ and a principal ideal I of A such that $sN \subseteq IM \subseteq N$. M is said to be an *S-principal ideal multiplication module* if every submodule N of M is an S-multiple of M. Various examples and properties of S-principal ideal multiplication modules are given. We investigate the conditions under which the trivial extension $A \ltimes M$ is an $S \ltimes 0$-principal ideal ring. Also, we prove Cohen type theorem for S-principal ideal multiplication modules in terms of S-prime submodules.

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1. Introduction

Throughout this study A denotes a commutative ring with a nonzero identity and M denotes a nonzero unital A-module. Also, the set of all prime ideals and maximal ideals of A will be denoted by $Spec(A)$ and $Max(A)$, respectively. In 1981, Barnard initiated the study of multiplication modules. Recall from [10] that an A-module M is said to be a *multiplication module* if every submodule N of M has the form IM for some ideal I of A. Afterwards, Z. A. El-Bast and P. F. Smith [11] in their excellent work, developed the theory of multiplication modules. They showed that an A-module M is a multiplication module if and only if $N = (N : M)M$ for each submodule N of M, where $(N : M) = ann(M/N) = \{x \in A : xM \subseteq N\}$. So far, there have been a great many publication addressing the structure of multiplication modules (See, for example, [4, 5, 19]). In a recent paper, Anderson et al. [1] defined the concept of S-multiplication modules which are a generalization of the concept of multiplication modules, where S is a multiplicatively closed set of A. They generalized the concept in the following approach by the help of a multiplicatively closed subset S of A. According to their point of view, in place of expecting $(N : M)M$ to contain all of N, containing only an s-multiple of N for some $s \in S$ will be enough, that is, $sN \subseteq (N : M)M$. Moreover, if all submodules of a module satisfy this condition, then it is called an *S-multiplication module*. It is clear that every multiplication module is also an S-multiplication module but the converse is not true in general (See, [1, Example 3]). Recall from [18] that a submodule P of M is said to be an *S-prime submodule* if $(P : M) \cap S = \emptyset$ and there exists a fixed $s \in S$ such that $am \in P$ for some $a \in A, m \in P$ implies $sa \in (P : M)$ or $sm \in P$. In particular, an ideal P of A is said to be an *S-prime ideal* if it is an S-prime submodule of the A-module A. Among many things in the paper [1], the authors examined some properties of S-prime submodules. In particular, they proved S-prime Avoidance Theorem in S-multiplication modules (See, [1, Theorem 2]). The concept of S-prime submodule is an important tool

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and has some applications to other areas such as General Topology. For instance, in a recent paper [21], Yildiz et al. constructed a topology on the set $Spec_S(A)$ of all S-prime ideals of a ring A and investigated the connections between algebraic properties of the ring A and topological properties of $Spec_S(A)$. For more information about S-prime ideals and S-prime submodules, see [13] and [20].

On the other hand, A. Azizi [7] introduced an important subclass of multiplication modules as follows: A submodule N of M will be called a multiple of M when $N = rM$ for some $r \in A$. If every submodule of M is a multiple of M, then he said that M is a principal ideal multiplication module, or briefly, PI-multiplication module. Also, he showed every cyclic module over a principal ideal ring is a PI-multiplication module. If A is a principal ideal ring, then A is a PI-multiplication A-module. As a fact, we can say that the concept "PI-multiplication module" for modules is similar to the concept "principal ideal ring" for rings. Hence, we conclude PI-multiplication modules are the module version of principal ideal rings.

This paper is inspired by the concepts of "PI-multiplication module" [7] and "S-multiplication module" [1]. We introduce the concept of S-Principal ideal multiplication module (briefly, S-PI-multiplication module) including the classes of PI-multiplication and S-multiplication modules. We call M an S-PI-multiplication module if for each submodule N of M, there exist a principal ideal I of A and $s \in S$ such that $sN \subseteq IM \subseteq N$. Among other things in this paper, some examples and characterizations of S-PI-multiplication modules and PI-multiplication modules are presented (See Example 1, 2, 3, Proposition 1, and Theorem 5). Also, we investigate the behavior of S-PI-multiplication modules under homomorphism, in factor modules, in localization of modules, in trivial extension, in Cartesian product of modules, under exact sequence of modules (See, Theorem 3, Corollary 1, and Theorems 1, 8–10). In particular, we investigate the conditions under which the trivial extension $A \ltimes M$ is an $S \ltimes 0$-PIR and $S \ltimes M$-PIR for an A-module M and a multiplicatively closed set S of A (See, Theorem 12). Recall from [18] that a ring A is said to be an S-integral domain if the zero ideal is an S-prime ideal of A. Also, an ideal I of A with $I \cap S = \emptyset$ is said to be an S-maximal if there exists $s \in S$ and whenever $I \subseteq J$ for some ideal J of A, then $sJ \subseteq I$ or $J \cap S = \emptyset$. Note that every S-maximal ideal is S-prime but the converse is not true in general (see, [21, Example 4]). It is well known that in a PID, every nonzero prime ideal is maximal. In Theorem 6, we extend this result to context of S-principal ideal domains. Finally, as another application of S-prime submodules, we prove Cohen type Theorem for S-PI-multiplication modules (See, Theorem 13).

2. Properties of S-PI-multiplication modules

Definition 1. Let M be an A-module and $S \subseteq A$ a multiplicatively closed set of A. A submodule N of M is said to be an S-multiple of M if there exist $s \in S$ and a principal ideal I of A such that $sN \subseteq IM \subseteq N$. Also M is said to be an S-PI-multiplication module if every submodule N of M is an S-multiple of M.

Note that by definition an A-module M is an S-PI-multiplication module if and only if for each submodule N of M, there exist $s \in S$ and $a \in A$ such that $sN \subseteq aM \subseteq N$.

Example 1. Every PI-multiplication A-module M is an S-PI-multiplication module for each multiplicatively closed set S of A. The converse also holds if $S \subseteq u(A)$, where $u(A)$ is the set of all units in A.

Example 2. **(S-PI-multiplication module which is not PI-multiplication)** Consider $\mathbb{Z}$-module $E(p) = \{\alpha = \frac{r}{p^t} + \mathbb{Z} : r \in \mathbb{Z}, t \in \mathbb{N} \cup \{0\}\}$, where p is a prime number. Then we know that all submodules of $E(p)$ have the form $G_n = \{\alpha = \frac{r}{p^n} + \mathbb{Z} : r \in \mathbb{Z}\}$ for some $n \in \mathbb{N} \cup \{0\}$. Note that $E(p)$ is not a multiplication module because for each $n \geq 1$, we have $G_n \neq (G_n : E(p))E(p) = (0)$. Thus, $E(p)$ is not a PI-multiplication module. On the other hand, take the multiplicatively closed set $S = \{p^n : n \in \mathbb{N} \cup \{0\}\}$. Let $s = p^n \in S$. Then we have $sG_n = (0) \subseteq 0(E(p)) \subseteq G_n$ which implies that $E(p)$ is an S-PI-multiplication module.

Example 3. **(S-multiplication module which is not S-PI-multiplication)** Suppose $A = \prod_{i=1}^{\infty} \mathbb{Z}_3$ and $S = u(A)$ is the set of units in A. Then S is a multiplicatively closed set of A and A is a von Neumann regular ring. Let $M = 0 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots$. Then clearly M is an S-multiplication module. Consider the submodule $N = \{(x_n)_{n\in\mathbb{N}} \in M : \exists n_0 \in \mathbb{N}$ such that $x_n = 0 \ (\forall n \geq n_0)\}$ of M. Now we will show that M is not an S-PI-multiplication module. Assume that there exist $s \in S$ and $a \in A$ such that $sN \subseteq aM \subseteq N$. Since s is a unit of A, we have $N = aM$ which gives $a = 0$. This is a contradiction.

Let S be a multiplicatively closed set of A. The *saturation* $S^\star$ of S is defined as $\{x \in A : ax \in S$ for some $a \in A\}$. Note that $S^\star$ is a multiplicatively closed set containing S. Now, we start our first result which gives the basic properties of S-PI-multiplication modules. Since its proof is elementary, we omit the proof.

Proposition 1. *Let M be an A-module and $S \subseteq A$ a multiplicatively closed set. The following statements are satisfied:*

(i) If M is an S-PI-multiplication module and $S \subseteq T$ are multiplicatively closed sets of A, then M is an T-PI-multiplication module.
(ii) M is an S-PI-multiplication module if and only if M is an $S^\star$-PI-multiplication module.
(iii) If M is an S-PI-multiplication module, then M_S is an PI-multiplication A_S-module.

The converse of Proposition 1(iii) is not always true. See the following example.

Example 4. Consider $\mathbb{Z}$-module $\mathbb{Q}$ and $S = reg(\mathbb{Z}) = \mathbb{Z} - \{0\}$. Then $\mathbb{Q}$ is not an S-PI-multiplication module. In fact, take $N = \frac{1}{2}\mathbb{Z}$ and assume that there exists $s \in S$ and $a \in \mathbb{Z}$ such that $sN \subseteq a\mathbb{Q} \subseteq N$. This gives $a = 0 = s$ which is a contradiction. However, $\mathbb{Z}_S$-module $\mathbb{Q}_S$ is isomorphic to $\mathbb{Q}$-module $\mathbb{Q}$. Hence, it is 1-dimensional vector space which is a PI-multiplication module.

Recall from [1] that a multiplicatively closed set S of A is said to *satisfy the maximal multiple condition* if there exists $s \in S$ such that $t|s$ for each $t \in S$. Note that all finite multiplicatively closed sets and the set of units in any ring satisfy the maximal multiple condition. Now we will investigate the conditions under which the converse of Proposition 1(iii) is satisfied.

Theorem 1. *Let M be an A-module and $S \subseteq A$ a multiplicatively closed set satisfying the maximal multiple condition. Then M is an S-PI-multiplication module if and only if M_S is a PI-multiplication A_S-module.*

Proof. $(\Rightarrow)$: Follows from Proposition 1.

$(\Leftarrow)$: Suppose that M_S is a PI-multiplication A_S-module. Let N be a submodule of M. Since M_S is a PI-multiplication A_S-module, there exists $x \in A$ such that $N_S = \frac{x}{1}M_S = (xM)_S$. Choose an element $m \in N$. As $\frac{m}{1} \in N_S = (xM)_S$, there exists $u \in S$ such that $um \in xM$. By maximal multiple condition, there exists $s \in S$ such that $t|s$ for each $t \in S$, namely, $s = ty$ for some $y \in A$. So we can write $s = uz$ for some $z \in A$, which implies that $sm = z(um) \in xM$. Thus, we have $sN \subseteq xM$. Likewise, we conclude that $s(xM) \subseteq N$. Now, put $s^\star = s^2 \in S$. Then note that $s^\star N = s^2 N \subseteq (sx)M \subseteq N$. Therefore, M is an S-PI-multiplication module. $\square$

Recall that a ring A is called a *Bézout ring* if its each finitely generated ideal is principal.

Theorem 2. *Let M be a module over a Bézout ring A. The following statements are equivalent:*

(i) M is a PI-multiplication module.
(ii) M is a $(A - q)$-PI-multiplication module for each $q \in Spec(A)$.
(iii) M is a $(A - q)$-PI-multiplication module for each $q \in Max(A)$.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$: Trivial.

$(iii) \Rightarrow (i)$: Take a maximal ideal $q \in Max(A)$ and a submodule N of M. Since M is an $(A-q)$-PI-multiplication module, there exists $s_q \notin q$ and $a_q \in (N:M)$ such that $s_qN \subseteq a_qM$. Now, consider the set $\mathcal{F} = \{x : \exists q \in Max(A), b_q \in (N:M)$ and $x \notin q$ such that $xN \subseteq b_qM\}$. Then note that $\mathcal{F}$ generates the whole ring A. This implies that $(x_{k_1}) + (x_{k_2}) + \cdots + (x_{k_m}) = A$ and $x_{k_i}N \subseteq a_{k_i}M$ for some $x_{k_i} \notin q_{k_i}$ and $a_{k_i} \in (N:M)$, where $q_{k_i} \in Max(A)$ and $1 \leq i \leq n$. Then we have $N = AN \subseteq \sum_{i=1}^{n}\left(x_{k_i}N\right) \subseteq \sum_{i=1}^{n}(a_{k_i}M) = IM$, where $I = \sum_{i=1}^{n}\left(a_{k_i}\right)$. As A is a Bézout ring and I is finitely generated, $I = (y)$ for some $y \in (N:M)$. Then we conclude that $N \subseteq yM \subseteq N$, that is $N = yM$ which is needed. □

Theorem 3. *Let $f : M \to M'$ be an epimorphism, where M and M' are A-modules. Suppose that $S \subseteq A$ is a multiplicatively closed set. If M is an S-PI-multiplication module, then M' is an S-PI-multiplication module. The converse is true if $sKer(f) = (0)$ for some $s \in S$.*

Proof. Suppose that M is an S-PI-multiplication module and $f : M \to M'$ is an epimorphism. Choose a submodule K of M'. Then we can write $K = f(L)$ for some submodule L of M. As M is an S-PI-multiplication module, there exist $s \in S$ and $x \in A$ such that $sL \subseteq xM \subseteq L$, which implies that $sK = sf(L) = f(sL) \subseteq f(xM) = xM' \subseteq f(L) = K$. Therefore, M' is an S-PI-multiplication module.

For the converse, assume that M' is an S-PI-multiplication module and $f : M \to M'$ is an epimorphism. Take a submodule N of M. As M' is an S-PI-multiplication module, there exist $t \in S$ and $x \in A$ such that $f(tN) = tf(N) \subseteq xM' = f(xM) \subseteq f(N)$. By getting contraction, we obtain that $tN + Ker(f) \subseteq xM + Ker(f) \subseteq N + Ker(f)$. Since $sKer(f) = (0)$, we have $stN \subseteq (sx)M \subseteq sN \subseteq N$. Therefore, M is an S-PI-multiplication module. □

As an immediate consequences of the previous theorem, we have the following explicit results.

Corollary 1. *(i) Let M be an S-PI-multiplication A-module and K be a submodule of M. Then M/K is an S-PI-multiplication A-module.*

(ii) Let M/K be an S-PI-multiplication A-module and $sK = (0)$ for some $s \in S$, where K is a submodule of M. Then M is an S-PI-multiplication A-module.

Recall from [2] that a submodule N of M is said to be an *S-finite* if there exist a finitely generated submodule K of M such that $sN \subseteq K \subseteq N$. In particular, M is said to be an *S-Noetherian module* if its each submodule is S-finite.

Theorem 4. *Every S-finite S-PI-multiplication module is an S-Noetherian module.*

Proof. Suppose that M is an S-finite and S-PI-multiplication A-module for a multiplicatively closed set S of A. Let N be a submodule of M. Since M is an S-PI-multiplication module, there exist $s \in S$ and $x \in A$ such that $sN \subseteq xM \subseteq N$. As M is an S-finite module, there exist $t \in S$ and $m_1, m_2, \ldots, m_n \in M$ such that $tM \subseteq \sum_{i=1}^{n} Am_i$. This implies that $stN \subseteq x(tM) \subseteq x\sum_{i=1}^{n} Am_i = \sum_{i=1}^{n} Axm_i \subseteq N$, that is, N is an S-finite submodule and so M is an S-Noetherian module. □

Recall from [2] that a ring R is said to be an *S-Principal ideal ring* (briefly, *S-PIR*) if for each ideal I of R there exist $s \in S$ and $a \in I$ such that $sI \subseteq (a) \subseteq I$.

Theorem 5. *(i) Let M be an S-multiplication A-module and $A/ann(M)$ a $\pi(S)$-PIR, where $\pi : A \to A/ann(M)$ is the canonical ring homomorphism. Then M is an S-PI-multiplication module.*

(ii) Let M be a finitely generated multiplication A-module. If M is an S-PI-multiplication module and $\pi : A \to A/ann(M)$ is the canonical ring homomorphism, then $A/ann(M)$ is a $\pi(S)$-PIR.

Proof. (i) : Suppose that M is an S-multiplication A-module and $A/ann(M)$ is a $\pi(S)$-PIR. Take a submodule N of M. As M is S-multiplication, there exists $s \in S$ such that $sN \subseteq (N : M)M$. Also, since $A/ann(M)$ is a $\pi(S)$-PIR, there exist $s' \in S$ and $x \in A$ such that

$$(s' + ann(M))((N : M)/ann(M)) \subseteq (x + ann(M)) \subseteq (N : M)/ann(M).$$

Then we obtain $s'(N : M) \subseteq (x) + ann(M) \subseteq (N : M)$. Now, let $s'' = ss' \in S$. Then we get $s''N \subseteq s'(N : M)M \subseteq xM \subseteq N$, which implies that M is an S-PI-multiplication module.

(ii) : Suppose that M is an S-PI-multiplication module and choose an ideal J of $A/ann(M)$. Then there exists an ideal I of A containing $ann(M)$ such that $J = I/ann(M)$. Now, put $N = IM$. Since M is an S-PI-multiplication module, there exist $s \in S$ and $x \in A$ such that $sIM \subseteq xM \subseteq IM$. As M is finitely generated multiplication, by [19, Corollary to Theorem 9], $sI \subseteq (x) + ann(M) \subseteq I$. This gives $(s + ann(M))J \subseteq (x + ann(M))A/ann(M) \subseteq J$ which completes the proof. □

Recall from [16] that a commutative ring R is said to be an S-integral domain if there exists an $s \in S$ such that $xy = 0$ implies $sx = 0$ or $sy = 0$. Also, recall from [21] that an ideal I of R with $I \cap S = \emptyset$ is said to be an S-maximal if there exists $s \in S$ and whenever $I \subseteq J$ for some ideal J of R, then $sJ \subseteq I$ or $J \cap S \neq \emptyset$. Note that every S-maximal ideal is S-prime but the converse is not true in general (see, [21, Example 4]). It is well known that in a principal ideal domain every nonzero prime ideal is maximal. Now, we generalize this fact to S-Principal ideal S-integral domains.

Theorem 6. *(i) Let A be an S-Principal ideal ring which is also an S-integral domain. Then every S-prime ideal P of A is either S-maximal or $sP = (0)$ for some $s \in S$.*

(ii) In Principal ideal domain every nonzero prime ideal is maximal.

Proof. (i) : Suppose that P is an S-prime ideal of A such that $sP \neq (0)$ for each $s \in S$. Then there exists $s \in S$ such that $(P : s)$ is a prime ideal and $(P : s') \subseteq (P : s)$ for each $s' \in S$ by [16]. As A is an S-integral domain, there exists $s^\star \in S$ such that $xy = 0$ for some $x, y \in A$ implies that $s^\star x = 0$ or $s^\star y = 0$. Choose an ideal J of A containing P. Since A is an S-Principal ideal ring, we can choose $s_1, s_2 \in S$; $x, y \in A$ such that $s_1P \subseteq (x) \subseteq P$ and $s_2J \subseteq (y) \subseteq J$. This implies that $s_1s_2P \subseteq s_2(x) \subseteq s_2P \subseteq s_2J \subseteq (y) \subseteq J$. Then we have $s_2x = by \in P$ for some $b \in A$. Since P is S-prime ideal, we have either $sb \in P$ or $sy \in P$.

First Case: Let $sy \in P$. Then we have $s_2sJ \subseteq s(y) \subseteq P$, which implies that $J \subseteq (P : s_2s) \subseteq (P : s)$ and so $sJ \subseteq P$.

Second Case: Let $sy \notin P$ and $sb \in P$. Then we have $s_1sb \in s_1P \subseteq (x)$. This implies that $s_1sb = ax$ for some $a \in A$. Then we have $s_1s_2sx = s_1sby = axy$ and thus $x(s_1s_2s - ay) = 0$. Since A is an S-integral domain, we get $s^\star x = 0$ or $s^\star(s_1s_2s - ay) = 0$. If $s^\star x = 0$, then we conclude that $s^\star s_1P = (0)$, which is a contradiction. So we have $s^\star(s_1s_2s - ay) = 0$, which implies that $s^\star s_1s_2s = s^\star ay \in J \cap S$. Therefore, P is an S-maximal ideal of A.

(ii) : Follows from (i). □

In [6], Anderson and Fuller defined the submodule N of an A-module M as a pure submodule if $IN = N \cap IM$ for every ideal I of A. Afterwards, F. Farshadifar introduced [12] the concept of S-pure submodules as follows: A submodule N of an A-module M is S-pure if there exists an $s \in S$ such that $s(N \cap IM) \subseteq IN$ for every ideal I of A.

Theorem 7. *Let M be an S-PI-multiplication module over A and N a proper submodule of M. If N is an S-pure submodule, then N is an S-PI-multiplication module over A.*

Proof. Suppose that N is an S-pure submodule. Choose a proper submodule K of N. Then since M is S-PI-multiplication, there are $s \in S$ and $x \in A$ such that $sK \subseteq xM \subseteq K$. On the other hand, as N is S-pure, there is $s' \in S$ such that $s'(N \cap IM) \subseteq IN$ for every ideal I of A. Consider $I = xA$. Thus, we have $s'(K \cap xM) \subseteq s'(N \cap xM) \subseteq xN$. Then $s'(K \cap xM) = s'(xM)$ by $xM \subseteq K$. Moreover, as $sK \subseteq xM$, we get $s'(sK) \subseteq s'(xM) \subseteq s'(N \cap xM) \subseteq xN \subseteq xM \subseteq K$. This completes the proof. □

Theorem 8. *Let* $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ *be a short exact sequence, where* M_i*'s are* A*-modules. Suppose* $f(M_1)$ *is an* S*-pure submodule of* M_2*. If* M_2 *is an* S*-PI-multiplication module, then* M_1 *and* M_3 *are* S*-PI-multiplication modules.*

Proof. Since f is monic, we may assume that M_1 is an S-pure submodule of M_2. As M_2 is an S-PI-multiplication module, by Theorem 7, M_1 is an S-PI-multiplication module. Since g is epic and $\mathrm{Im} f = Ker g$, we conclude that $M_3 \cong M_2/M_1$. As M_2 is an S-PI-multiplication module, by Corollary 1, we have M_3 is an S-PI-multiplication module. □

Theorem 9. *Let* M_i *be an* A_i*-module and* S_i *be a multiplicatively closed set of* A_i *for each* $i = 1, 2, \ldots, n$*. Suppose that* $M = M_1 \times M_2 \times \cdots \times M_n$*,* $A = A_1 \times A_2 \times \cdots \times A_n$ *and* $S = S_1 \times S_2 \times \cdots \times S_n$*. Then the following statements are equivalent.*

(i) *M is an S-PI-multiplication A-module.*
(ii) *M_i is an S_i-PI-multiplication A_i-module for each $i = 1, 2, \ldots, n$.*

Proof. Suppose that M is an S-PI-multiplication A-module. Now, we will show that M_i is an S_i-PI-multiplication A_i-module for each $i = 1, 2, \ldots, n$. Let N_i be a submodule of M_i. Now, put $N = 0 \times 0 \times \cdots \times 0 \times N_i \times 0 \times \cdots \times 0$. Since M is an S-PI-multiplication A-module, there exist $s = (s_1, s_2, \ldots, s_n) \in S$ and $a = (a_1, a_2, \ldots, a_n) \in A$ such that $sN \subseteq aM \subseteq N$, which implies that $s_i N_i \subseteq a_i M_i \subseteq N_i$. Therefore, M_i is an S_i-PI-multiplication A_i-module.

For the converse, assume that M_i is an S_i-PI-multiplication A_i-module for each $i = 1, 2, \ldots, n$. Let N be a submodule of M. Then we can write $N = N_1 \times N_2 \times \cdots \times N_n$ for some submodules N_i of M_i. Since M_i is an S_i-PI-multiplication A_i-module, then there exists $s_i \in S_i$ and $x_i \in A_i$ such that $s_i N_i \subseteq x_i M_i \subseteq N_i$. Now, put $s = (s_1, s_2, \ldots, s_n) \in S$ and $x = (x_1, x_2, \ldots, x_n) \in A$. Then note that

$$\begin{aligned} sN &= \prod_{i=1}^{n}(s_i N_i) \subseteq \prod_{i=1}^{n}(x_i M_i) = xM \\ &\subseteq \prod_{i=1}^{n} N_i = N. \end{aligned}$$

Therefore, M is an S-PI-multiplication A-module. □

Let M be an A-module. *Trivial extension* $A \ltimes M = A \oplus M$ of M is a commutative ring with the componentwise addition and the following multiplication $(a, m)(a', m') = (aa', am' + a'm)$ for each $a, a' \in A$ and $m, m' \in M$. Now, we characterize S-PI-multiplication modules in terms of its trivial extension.

Recall that an ideal P of A is said to be an *S-PI-multiplication ideal* if it is an S-PI-multiplication A-module.

Theorem 10. *Let* N *be a submodule of an* A*-module* M *and* S *a multiplicatively closed set of* A*. The following statements are equivalent.*

(i) *N is an S-PI-multiplication module.*

(ii) $(0) \ltimes N$ *is an* $(S \ltimes 0)$*-PI-multiplication ideal of* $A \ltimes M$.
(iii) $(0) \ltimes N$ *is an* $(S \ltimes M)$*-PI-multiplication ideal of* $A \ltimes M$.

Proof. $(i) \Rightarrow (ii)$: Suppose that N is an S-PI-multiplication module. Take an ideal I of $A \ltimes M$ such that $I \subseteq (0) \ltimes N$. Then there exists a submodule N' of N such that $I = (0) \ltimes N'$. Since N is an S-PI-multiplication module, there exist $s \in S$ and $x \in A$ such that $sN' \subseteq xN \subseteq N'$, which implies that $(s,0)I = (0) \ltimes sN' \subseteq (0) \ltimes xN = (x,0)\,[(0) \ltimes N] \subseteq I$. Therefore, $(0) \ltimes N$ is an $(S \ltimes 0)$-PI-multiplication ideal of $A \ltimes M$.

$(ii) \Rightarrow (iii)$: Follows from Proposition 1.

$(iii) \Rightarrow (i)$: Suppose that $(0) \ltimes N$ is an $(S \ltimes M)$-PI-multiplication ideal of $A \ltimes M$. Let N' be a submodule of N. Put $I = (0) \ltimes N'$. Then note that I is an ideal of $A \ltimes M$ contained in $(0) \ltimes N$. Since $(0) \ltimes N$ is an $(S \ltimes M)$-PI-multiplication ideal of $A \ltimes M$, there exist $(s,m) \in (S \ltimes M)$ and $(x,m') \in A \ltimes M$ such that $(s,m)I \subseteq (x,m')(0) \ltimes N \subseteq I$. Let $n' \in N'$. Then we have

$$(0, sn') = (s,m)(0,n') \in (s,m)I \subseteq (x,m')(0) \ltimes N$$

which implies that $(0, sn') = (x,m')(0,n)$ for some $n \in N$. Then we conclude that $sn' = xn \in xN$, that is, $sN' \subseteq xN$. On the other hand, $(x,m')(0) \ltimes N \subseteq I$ gives that $xN \subseteq N'$ and so we have $sN' \subseteq xN \subseteq N'$. Therefore, N is an S-PI-multiplication module. □

Let M be an A-module and S a multiplicatively closed set of A. Recall from [1] that M is said to be an S-cyclic if there exist $s \in S$ and $m \in M$ such that $sM \subseteq Am$. Note that if $S \subseteq u(A)$, where $u(A)$ is the set of all units in A, then S-cyclic modules are in fact cyclic modules.

Theorem 11. *(i) Let* M *be an* S*-cyclic module and* N *a submodule of* M*. Then* N *is an* S*-cyclic module if and only if* N *is an* S*-multiple of* M*.*

(ii) Let M *be an* S*-cyclic module. Then* M *is an* S*-PI-multiplication module if and only if every submodule of* M *is* S*-cyclic.*

(iii) Let M *be a cylic module. Then* M *is a PI-multiplication module if and only if its every submodule is cylic.*

Proof. (i) : Suppose that M is an S-cyclic module. Then there exist $s \in S$ and $m \in M$ such that $sM \subseteq Am$. $(\Rightarrow)$: Now, let N be an S-cyclic module. We may assume that $sN \subseteq An$ for some $n \in N$. Then we can write $sn = xm$ for some $x \in A$. This gives $s^2N \subseteq Asn = Axm$ which implies that $s^3N \subseteq xAsm \subseteq xsM$. Let $m^* \in M$. Then $sm^* \in sM \subseteq Am$. Then there exists $a \in A$ such that $sm^* = am$ and so $xsm^* = axm = asn \in N$. Thus, we conclude that $s^3N \subseteq xsM \subseteq N$. Hence, N is an S-multiple of M. $(\Leftarrow)$: Now, let N be an S-multiple of M. Then we may assume that $sN \subseteq xM \subseteq N$ for some $x \in A$. Now, fix $n = xm$. It is clear that $Axm \subseteq N$. Choose $n^* \in N$. Then $sn^* = xm'$ for some $m' \in M$. This gives $s^2n^* = sxm' = x(sm') = a(xm) \in Axm$. Then we conclude that $s^2N \subseteq Axm \subseteq N$, that is, N is an S-cyclic module.

(ii) : Follows from (i).

(iii) : Take $S = \{1\}$ and apply (ii). □

For any ring A, we denote the set of all nilpotent elements of A by $Nil(A)$. It is known that $Nil(A)$ is the intersection of all prime ideals of A. Now, we investigate the conditions under which the trivial extension $A \ltimes M$ is an $(S \ltimes 0)$-PIR, where S is a multiplicatively closed set of A and M is an A-module.

Theorem 12. *Let* S *be a multiplicatively closed set of* A *and* M *an* A*-module. Consider the following statements.*

(i) $A \ltimes M$ *is an* $(S \ltimes 0)$*-PIR.*

(ii) $A \ltimes M$ *is an* $(S \ltimes M)$*-PIR.*
(iii) A *is an* S*-PIR,* M *is an* S*-cyclic module and* S*-PI-multiplication module.*

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$*. In addition, if* $tM \subseteq Nil(A)M$ *for some* $t \in S$*, we have* $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$*.*

Proof. $(i) \Rightarrow (ii)$: It is easy.

$(ii) \Rightarrow (iii)$: First we will show that M is an S-cyclic module. Put $J = 0 \ltimes M$. Since $A \ltimes M$ is an $(S \ltimes M)$-PIR, there exist $(s, m) \in S \ltimes M$ and $(0, m') \in J$ such that $(s, m)J \subseteq (0, m')A \ltimes M$. For any $m^* \in M$, we have $(s, m)(0, m^*) = (0, sm^*) \in (s, m)J \subseteq (0, m')A \ltimes M$. Then there exists $(y, m'') \in A \ltimes M$ such that $(0, sm^*) = (0, m')(y, m'') = (0, ym')$ which implies that $sm^* \in Am'$. Then we conclude that $sM \subseteq Am'$ that is M is an S-cyclic module. A similar argument shows that every submodule N of M is an S-cyclic module. Then by Theorem 11 (ii), M is an S-PI-multiplication module. Now, take an ideal I of A. Since $A \ltimes M$ is an $(S \ltimes M)$-PIR, there exist $(s, m) \in S \ltimes M$ and $(x, m') \in I \ltimes M$ such that $(s, m)(I \ltimes M) \subseteq (x, m')A \ltimes M$. This implies that $sI \subseteq xA \subseteq I$. Thus, A is an S-PIR.

For the rest, assume that $tM \subseteq Nil(A)M$ for some $t \in S$ and also A is an S-PIR, M is an S-cyclic module and S-PI-multiplication module. By [2, Proposition 16], it is sufficient to show that every prime ideal of $A \ltimes M$ which is disjoint from $(S \ltimes 0)$ is an $(S \ltimes 0)$-principal ideal. To see this, take a prime ideal J of $A \ltimes M$. Then by [3, Theorem 3.2], there exists a prime ideal P of A such that $J = P \ltimes M$. Now, let $J \cap S \ltimes 0 = \emptyset$. Then note that $P \cap S = \emptyset$. Since A is an S-PIR, we may write $s'P \subseteq Ap$ for some $p \in P$ and $s' \in S$. Since M is an S-cyclic module, we can write $s''M \subseteq Am$ for some $m \in M$ and $s'' \in S$. Now, put $s = s's''t \in S$. Then note that $sP \subseteq Ap$, $sM \subseteq Am$ and $sM \subseteq Nil(A)M$. Since $sM \subseteq Nil(A)M$, we have $sM \subseteq PM$ which implies that $s^2M \subseteq sPM \subseteq pM$. Now, we will show that $(s^2, 0)P \ltimes M \subseteq (p, 0)A \ltimes M \subseteq P \ltimes M$. It is clear that $(p, 0)A \ltimes M \subseteq P \ltimes M$. Now, let $(p^*, m^*) \in P \ltimes M$. Since $sP \subseteq Ap$ and $s^2M \subseteq pM$, we have that $(s^2, 0)(p^*, m^*) = (s^2p^*, s^2m^*) = (sap, pm') = (p, 0)(sa, m') \in (p, 0)A \ltimes M$. Thus we have $P \ltimes M$ is an $(S \ltimes 0)$-principal ideal. Hence, by [2, Proposition 16], $A \ltimes M$ is an $(S \ltimes 0)$-PIR. □

In Theorem 12 $(iii) \Rightarrow (i)$ is not true in general. See the following example.

Example 5. Consider the $A = \mathbb{Z}$-module $M = \mathbb{Z}_2$ and the trivial extension $A \ltimes M$. Take $S = \{1\}$. Then A is an S-PIR, M is an S-cyclic module and S-PI-multiplication module. However, $A \ltimes M$ is not an $(S \ltimes 0)$-PIR since $N = 2\mathbb{Z} \ltimes \mathbb{Z}_2$ is not an S-principal ideal of $\mathbb{Z} \ltimes \mathbb{Z}_2$.

Cohen type theorem for principal ideal rings states that a ring A is a principal ideal ring if and only if its every prime ideal is principal. Now, in the following, we prove a general result for S-PI-multiplication modules.

Theorem 13. ***(Cohen type Theorem for S-PI-multiplication modules)*** *Let* M *be an* S*-finite* A*-module for a multiplicatively closed set* S *of* A*. The following statements are equivalent.*

(i) M *is an* S*-PI-multiplication module.*
(ii) Every S*-prime submodule is an* S*-multiple of* M*.*

Proof. $(i) \Rightarrow (ii)$: It is clear.

$(ii) \Rightarrow (i)$: Suppose that M is an S-finite module in which every S-prime submodule of M is S-multiple. Now, we will show that M is an S-PI-multiplication module. Suppose to the contrary. Then $\Delta = \{N : N \text{ is a submodule of } M \text{ which is not } S\text{-multiple of } M\}$ is a nonempty (partially ordered) set. Take a chain $\{N_i\}_{i \in I}$ in Δ and let $N = \bigcup_{i \in I} N_i$. Note that N is not an S-multiple of M. Indeed, if N is an S-multiple of M, then there exist $s \in S$ and $x \in A$ such that $sN \subseteq xM \subseteq N$. On the other hand, there exist $l \in S$ and $m_1, m_2, \ldots, m_n \in M$ such that $lM \subseteq Am_1 + Am_2 + \cdots + Am_n$. Also note that there exists $k \in I$ such that $xm_i \in N_k$ for each $i = 1, 2, \ldots, n$. Let $m \in M$. Then $lm = a_1m_1 + a_2m_2 + \cdots + a_nm_n$ for some $a_1, a_2, \ldots, a_n \in A$. Thus, we conclude that $lxm = a_1(xm_1) + a_2(xm_2) + \cdots + a_n(xm_n) \in N_k$ which implies that $slN_k \subseteq slN \subseteq lxM \subseteq N_k$. Then N_k is an S-multiple of M, a contradiction. Thus, N is the supremum of the chain $\{N_i\}_{i\in I}$ in Δ. By Zorn's lemma, there exists a maximal element, say $P \in Max(\Delta)$. First note that $(P : M) \cap S = \emptyset$. Otherwise, we would have $sP \subseteq sM \subseteq P$ for some $s \in S$ which contradicts with $P \in \Delta$.

Step 1: First we will show that $(P : M)$ is a prime ideal of A. Suppose not. Then there exists $a, b \in A - (P : M)$ such that $ab \in (P : M)$. Then note that $P \subsetneq P + aM$ and $P \subsetneq (P :_M a)$. Thus, there exists $u \in S$ such that $u(P + aM) \subseteq xM \subseteq P + aM$ and $u(P :_M a) \subseteq yM \subseteq (P :_M a)$ for some $x, y \in A$. This gives $S^{-1}(P + aM) = S^{-1}(xM)$ and so $S^{-1}(xyM) = S^{-1}(yP + ayM) \subseteq S^{-1}P$. Since $lM \subseteq \sum_{i=1}^{n} Am_i$, note that $S^{-1}M = \sum_{i=1}^{n} S^{-1}(Am_i)$. This implies that $txyM \subseteq P$ for some $t \in S$. Thus, we have $xy \in (P : tM)$. Now, we will show that $(P :_M t) = P$. Otherwise, we would have $t'(P :_M t) \subseteq cM \subseteq (P :_M t)$ for some $t' \in S$ and $c \in A$. This gives $tt'P \subseteq tt'(P :_M t) \subseteq ctM \subseteq P$, that is P is an S-multiple of M, a contradiction. Thus, we have $(P :_M t) = P$ which yields that $(P : tM) = (P : M)$. Then we have $xy \in (P : M)$. We may assume that $y \notin (P : M)$. Indeed, if $y \in (P : M)$, then we have $uP \subseteq u(P :_M a) \subseteq yM \subseteq P$, a contradiction. Thus, we conclude that $P \subsetneq (P :_M x)$. Then there exists $w \in S$ such that $w(P :_M x) \subseteq zM \subseteq (P :_M x)$ for some $z \in A$. Let $m^\star \in P$. Then $um^\star = xm'$ for some $m' \in M$. Which implies that $m' \in (P :_M x)$. Then we get $wm' = zm''$ for some $m'' \in M$. Thus, we have $wum^\star = wxm' = xzm'' \in xzM$. This implies that $wuP \subseteq xzM \subseteq P$, that is, P is an S-multiple of M which is a contradiction. Hence, $(P : M)$ is a prime ideal of A.

Step 2: Now, we will show that P is a prime submodule of M. Deny. There exist $r \in A - (P : M)$ and $m \notin P$ such that $rm \in P$. This gives $P \subsetneq (P :_M r)$ and $P \subsetneq P + rM$. Then there exist $v, t \in S$ such that $v(P :_M r) \subseteq dM \subseteq (P :_M r)$ and $t(P + rM) \subseteq eM \subseteq P + rM$ for some $d, e \in A$. Since $(P : M)$ is a prime ideal and $rdM \subseteq P$, we have either $d \in (P : M)$ or $r \in (P : M)$. **Case 1:** Let $r \in (P : M)$. Then we have $tP \subseteq t(P + rM) \subseteq eM \subseteq P + rM \subseteq P$, that is, P is an S-multiple of M, a contradiction. **Case 2:** Let $d \in (P : M)$. Then we have $vP \subseteq v(P :_M r) \subseteq dM \subseteq P$ which is again a contradiction. Therefore, P is a prime submodule of M. Since $(P : M) \cap S = \emptyset$, by [16, Proposition 2.2], P is an S-prime submodule of M.

Then by assumption, P is an S-multiple of M which contradicts with $P \in \Delta$. Therefore, M is an S-PI-multiplication module. □

Theorem 14. *Let M be a finitely generated A-module. Then M is a PI-multiplication module if and only if every prime submodule P of M is a multiple of M.*

Proof. Suppose that M is a finitely generated module and $S = \{1\}$. Then M is an S-finite. By Theorem 13, M is an S-PI-multiplication module $\Leftrightarrow$ Every S-prime submodule is S-multiple of M. On the other hand, M is an S-PI-multiplication module $\Leftrightarrow$ M is a PI-multiplication module and also note that prime submodule and S-prime submodules are coincide which completes the proof. □

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