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Dynamic transitions and bifurcations of 1D reaction–diffusion equations: The self-adjoint case

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This paper deals with the classification of transition phenomena in the most basic dissipative system possible, namely, the 1D reaction–diffusion equation. The emphasis is on the relation between the linear and nonlinear terms and the effect of the boundaries which influence the first transitions. We consider the cases where the linear part is self-adjoint with second-order and fourth-order derivatives which is the case which most often arises in applications. We assume that the nonlinear term depends on the unknown function and its first derivative which is basically the semilinear case for the second-order reaction–diffusion system. As for the boundary conditions, we consider the typical Dirichlet, Neumann, and periodic boundary settings. In all the cases, the equations admit a trivial steady state which loses stability at a critical parameter. We aim to classify all possible transitions and bifurcations that take place. Our analysis shows that these systems display all three types of transitions: continuous, jump and mixed. Moreover they exhibit transcritical, supercritical bifurcations with bifurcated states such as finitely many equilibria, circle of equilibria, and slowly rotating limit cycle. Many applications found in the literature are basically corollaries of our main results. We apply our results to classify the first transitions of the Chaffee–Infante equation, the Fisher-KPP equation, the Kuramoto–Sivashinsky equation, and the Swift–Hohenberg equation.

KEYWORDS
bifurcations, center manifold reduction, dynamic transitions, reaction–diffusion

MSC CLASSIFICATION
35B32; 37L10

1 | INTRODUCTION

The reaction–diffusion equations play an important role as models of diverse physical, chemical, and biological phenomena. One basic aspect of these equations is the transition behavior which is observed when the stability of a stable state changes as a system parameter crosses a threshold.[1] A recent and fruitful paradigm to understand the transition behavior is the dynamical transition theory introduced in Ma and Wang.[2] The philosophical basis of dynamic transition theory is to search for a full set of transition states thus yielding a complete characterization of stability and transition. The application of this methodology has given rise to a lot of results regarding the transitions of reaction–diffusion equations.[3–18] Our main goal is to generalize these results which can cover most of the results obtained in these works. Namely, we want to obtain the relation between the linear and nonlinear terms and the effect of the boundaries which influence the transition phenomena in the most basic dissipative system possible, the 1D reaction–diffusion equation. To keep things simple, we limit ourselves only with the loss of stability of a single constant steady state solution.

We consider 1D reaction–diffusion type equations with second- and fourth-order linear self-adjoint operator and a general nonlinear term depending only on u and u_x

$$\frac{\partial u}{\partial t} = Lu + g(u, u_x). \tag{1}$$

Here, $u = u(x, t)$ is the unknown scalar function, $x \in I$ is the spatial variable, I is an open interval in $\mathbb{R}$, $t \geq 0$ is the time. We consider two cases which arise most often in practice: the case where L is a self-adjoint operator of order two

$$Lu = \frac{\partial^2 u}{\partial x^2} + \lambda u \tag{2}$$

and of order four

$$Lu = -\frac{\partial^4 u}{\partial x^4} - \delta \frac{\partial^2 u}{\partial x^2} + \lambda u, \tag{3}$$

respectively, where $\lambda, \delta \in \mathbb{R}$ are parameters. The nonlinear operator is at least of quadratic order, and we consider its following expansion:

$$g(u, u_x) = a_1 u^2 + a_2 u u_x + a_3 u_x^2 + a_4 u^3 + a_5 u^2 u_x + a_6 u u_x^2 + a_7 u_x^3 + o(|(u, u_x)|^3), \tag{4}$$

with $a_j \in \mathbb{R}$ being constants.

We consider three types of boundary conditions supplementing Equation (1). Namely, the homogeneous Dirichlet, homogeneous Neumann, and the periodic boundary conditions.

The equations in each case possess the trivial steady-state $u \equiv 0$. This steady state is stable for $\lambda < \lambda_c$ and loses its stability on $\lambda > \lambda_c$ for some critical value λ_c which is determined by the linear operator together with the boundary conditions. Since the linear operators in (2) and (3) are self-adjoint, the eigenvalues are real, and Hopf bifurcations cannot occur.

The Equation (1) can be put into an abstract ordinary differential equation (ODE)

$$\frac{du}{dt} = L_\lambda u + G(u) \tag{5}$$

where L is the linear operator, G is the nonlinear operator, $u(\cdot) \in X = L^2(I)$ (or $L^2_{\text{per}}(I)$ in the periodic setting) where $I \subset \mathbb{R}$ is the spatial domain.

The dynamic transition theory comes with the following definitions.[2] We say that the system undergoes a **dynamic transition** at $\lambda = \lambda_c$ if $u \equiv 0$ is asymptotically stable for $\lambda < \lambda_c$ and for $\lambda > \lambda_c$ and $u_0 \in U \setminus \Gamma_\lambda$,

$$\lim_{t \to \infty} \|u_\lambda(t; u_0)\|_X \geq \delta(\lambda), \qquad \lim_{\lambda \to \lambda_c^+} \delta(\lambda) \geq 0,$$

where $U \subset X$ is some neighborhood of 0, Γ_λ is the stable manifold of $u = 0$ and $u_\lambda(t; u_0)$ is the solution of (5) with initial condition u_0.

1. If $\lim_{\lambda \to \lambda_c^+} \delta(\lambda) = 0$ then the transition is called **continuous**.
2. If for some $\epsilon > 0$, and some $\delta_0 > 0$, and $\lambda_c < \lambda < \lambda_c + \epsilon$, $\delta(\lambda) > \delta_0$ then the transition is called **jump** (also called catastrophic).
3. Finally if the neighborhood U can be decomposed into open sets U_1, U_2 with $\overline{cU} = \bar{U}_1 \cup \bar{U}_2$ and $U_1 \cap U_2 = \emptyset$ such that in U_1, the transition is continuous and on U_2 the transition is jump then we call the overall transition as **mixed** (also called random).

Our main goal is to classify the first transitions of the trivial-steady state $u \equiv 0$ which occur as the first critical eigenvalues of the linear operator become unstable. The classification of possible transitions of the system is accompanied by a bifurcation of new states and the stability of these new states.

We present the principle of exchange of stabilities (PES) condition which characterizes the stability of the trivial steady state in each case considered. PES condition is in principle determines the critical parameter λ_c, the multiplicity of the first critical eigenvalues that become unstable as well as whether these eigenvalues are real or complex.

Next step is to obtain the reduced equation(s) which gives a full picture of the local transition phenomena near $\lambda = \lambda_c$ in a small neighborhood of $u \equiv 0$ in the phase space.

Once the reduced system of equation(s) is obtained, the stability analysis of these equations describes the type of transition and accompanied bifurcation and the bifurcating stable/unstable states which can be steady states, limit cycles, or circle of equilibria.

We display the generality of our main results by considering some well-known examples such as the Chaffee–Infante equation, the Kolmogorov-Fisher equation, the Swift–Hohenberg equation, and the Kuramoto–Sivashinsky equation. All these examples which can be considered as corollaries of our main results have been recently tackled as stand-alone problems in the literature.

The paper is organized as follows: Section 1 is the introduction. Section 2 states the main results. In Section 3, we apply our theoretical results to several well-known reaction–diffusion models. In Section 4, we discuss the conclusions. Finally, we give the proofs of main results in Section 5.

2 | MAIN RESULTS

2.1 | Second-order linear self-adjoint operator

We first consider the second-order reaction–diffusion equation given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda u + g(u, u_x) \tag{6}$$

with g as in (4) and subject to following three different types of boundary conditions:

$$u(0,t) = u(\pi,t) = 0, \tag{Dirichlet}$$

$$\frac{\partial}{\partial x} u(0,t) = \frac{\partial}{\partial x} u(\pi,t) = 0, \tag{Neumann}$$

$$u(x+2\pi) = u(x), \qquad \forall x \in \mathbb{R} \tag{periodic}$$

The eigenvalues of the linear operator $\partial_{xx} + \lambda$ in all the above boundary settings are

$$\beta_k = \lambda - k^2,$$

with corresponding eigenvectors

$$\begin{aligned} &\text{Dirichlet:} && e_k = \sin kx, && k \in \mathbb{Z}_{>0} = \{1, 2, \dots\} \\ &\text{Neumann:} && e_k = \cos kx, && \beta_k = \lambda - k^2, \qquad k \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\} \\ &\text{periodic:} && e_k = e^{ikx}, && \beta_k = \lambda - k^2, \qquad k \in \mathbb{Z} \end{aligned} \tag{7}$$

For the Dirichlet and Neumann boundary conditions, the eigenvalues satisfy the PES condition

$$\begin{aligned} &\beta_1 \begin{cases} < 0, & \lambda < 1 \\ = 0, & \lambda = 1 \\ > 0, & \lambda > 1 \end{cases} \\ &\beta_n|_{\lambda=1} < 0 \qquad n \geq 2 \end{aligned} \tag{8}$$

while for the periodic setting, the PES condition reads

$$\beta_1 = \beta_{-1} \begin{cases} < 0, & \lambda < 1 \\ = 0, & \lambda = 1 \\ > 0, & \lambda > 1 \end{cases} \qquad (9)$$
$$\beta_n|_{\lambda=1} < 0 \qquad |n| \geq 2$$

Under these settings, by the linear stability theory, the steady-state $u \equiv 0$ is stable for $\lambda < 1$ and loses its stability on $\lambda > 1$. For the next theorem which deals with the Dirichlet/Neumann setting, we define the following transition numbers.

$$\begin{aligned}
A &= \begin{cases} (8a_1 + 4a_3)/(3\pi), & \text{Dirichlet BC} \\ -4a_2/(3\pi), & \text{Neumann BC} \end{cases} \\
B &= C + \sum_{k=0}^{\infty} B_k \\
C &= \frac{3a_4 + a_6}{4} \\
B_0 &= \begin{cases} 0, & \text{Dirichlet BC} \\ -2a_1(a_1 + a_3), & \text{Neumann BC} \end{cases} \\
B_1 &= 0 \\
B_2 &= \begin{cases} -\frac{a_2^2}{12}, & \text{Dirichlet BC} \\ \frac{(a_1+2a_3)(a_1-a_3)}{6}, & \text{Neumann BC} \end{cases} \\
B_k &= \begin{cases} \frac{8}{\pi^2}\frac{((-1)^k-1)^2}{k^2(k^2-4)^2}\frac{(2a_1+k^2a_3)[2a_1-(k^2-2)a_3]}{(k^2-1)}, & \text{Dirichlet BC} \\ -\frac{4a_2^2}{\pi^2}\frac{(k^2-3)((-1)^k-1)^2}{(k^2-1)(k^2-4)^2}, & \text{Neumann BC} \end{cases}, \qquad k \geq 3
\end{aligned} \qquad (10)$$

Here A and C are due to the self-interaction of the critical mode through quadratic and cubic nonlinearities respectively. On the other hand, B_k is due to the interaction of the critical mode with the kth stable mode through quadratic nonlinearity.

Theorem 1 (Second-order linear operator with Dirichlet or Neumann boundary conditions)**.** *Equation (6) with either Dirichlet or Neumann boundary conditions has locally stable equilibrium $u \equiv 0$ for $\lambda < 1$. For $\lambda > 1$, the system exhibits a first transition described by the following ODE:*

$$\frac{du_1}{dt} = (\lambda - 1)u_1 + Au_1^2 + Bu_1^3 + o(3) \qquad (11)$$

where $u_1 \in \mathbb{R}$ denotes the amplitude of the first critical eigenvector $e_1 = \sin x$ for the Dirichlet case and $e_1 = \cos x$ for the Neumann case.

Moreover, the following statements hold true.

1. *If $A \neq 0$ then the first transition is of mixed type accompanied by a transcritical bifurcation. There is one bifurcated steady-state solution given by*
$$u = \frac{-(\lambda - 1)}{A}e_1 + o(|(\lambda - 1)|).$$
2. *If $A = 0$ and $B \neq 0$ then the first transition is continuous if $B < 0$ and jump if $B > 0$ accompanied by a pitchfork bifurcation. There are two bifurcated steady-state solutions given by*
$$u = \pm\sqrt{\frac{-(\lambda - 1)}{B}}e_1 + o(|(\lambda - 1)|^{1/2}).$$

Remark 1. For the Neumann case, although we present the coefficients B_k for $k \geq 3$, those coefficients are never necessary to be computed. This is because if $a_2 \neq 0$, then the reduced equation is determined by A and B_k need not be computed. On the other hand, if $a_2 = 0$ then $B_k = 0$ for $k \geq 3$. Similarly for the Dirichlet case, the coefficient $B_k,\ k \geq 3$ needs to be computed only in the case $8a_1 + 4a_3 = 0,\ a_1 \neq 0,\ a_3 \neq 0$.

For the periodic setting, we define the transition number as

$$\begin{aligned} P &= C + P_0 + P_2 \\ C &= 3a_4 + ia_5 + a_6 + 3ia_7 \\ P_0 &= -2\,(2a_1 + ia_2)\,(a_1 + a_3) \\ P_2 &= \frac{1}{3}\,[2a_1 - (i+2)a_2 + 4ia_3]\,[a_1 + ia_2 - a_3] \end{aligned} \tag{12}$$

Theorem 2 (Second-order linear operator with periodicity condition)**.** *The Equation (6) with periodic boundary condition has locally stable equilibrium $u \equiv 0$ for $\lambda < 1$. For $\lambda > 1$, the system exhibits a first transition described by the following ODE:*

$$\frac{dz}{dt} = (\lambda - 1)\,z + Pz|z|^2 + o(3), \tag{13}$$

where $z(\cdot) \in \mathbb{C}$ denotes the amplitude of the first critical eigenvector $e_1 = e^{ix}$. Moreover, we have the following.

1. *If $\Re P < 0$ then the first transition is continuous. Furthermore, the following statements hold true.*
   (a) *If $\Im P = 0$, the system has a circle C of neutrally stable equilibria given by*

$$C = \left\{ u_\theta(x) = 2\sqrt{\frac{-(\lambda-1)}{\Re P}}\cos(\theta + x) + o(|\lambda - 1|^{1/2}) | \theta \in [0, 2\pi) \right\}$$

   That is for each $\theta \in [0, 2\pi)$, $u_\theta \in C$ is an equilibrium solution.
   (b) *If $\Im P \neq 0$, then a unique stable limit cycle bifurcates given by*

$$u(x,t) = 2\sqrt{\frac{-(\lambda-1)}{\Re P}}\cos\left(x - \frac{(\lambda-1)\Im P}{\Re P}t\right). \tag{14}$$

2. *If $\Re P > 0$, then the first transition is jump and similar assertions hold, that is in the case $\Im P = 0$, a circle of unstable equilibria bifurcates and in the case $\Im P \neq 0$, a unique unstable limit cycle bifurcates.*

The following corollary of Theorem 1 and Theorem 2 describes some of the transition possibilities in terms of the coefficients of the nonlinear operator.

Corollary 1. *Consider the Equation (6).*

1. *For the Dirichlet boundary conditions, we have*
   (a) *If $8a_1 + 4a_3 \neq 0$, then the transition is of mixed type.*
   (b) *If $a_2 \neq 0$ and $a_1 = a_3 = a_4 = a_6 = 0$, the transition is continuous.*
2. *For the Neumann boundary conditions, we have*
   (a) *If $a_2 \neq 0$ the transition is of mixed type.*
   (b) *If $a_2 = a_4 = a_6 = 0$ then the transition can be of jump or continuous depending on a_1 and a_3 as in Figure 1.*
3. *For the periodic case when $a_4 = a_5 = a_6 = a_7 = 0$, the transition diagram in $a_1 - a_3$ plane is qualitatively as in Figure 2.*

An important aspect of the transition number is it is composed of the effects of different modes. In Figure 3, we demonstrate which stable mode is more dominant on the first transition number for the 2D Neumann case with $a_2 = 0$: the stable mode $e_0 = \cos 0x = 1$ or $e_2 = \cos 2x$.

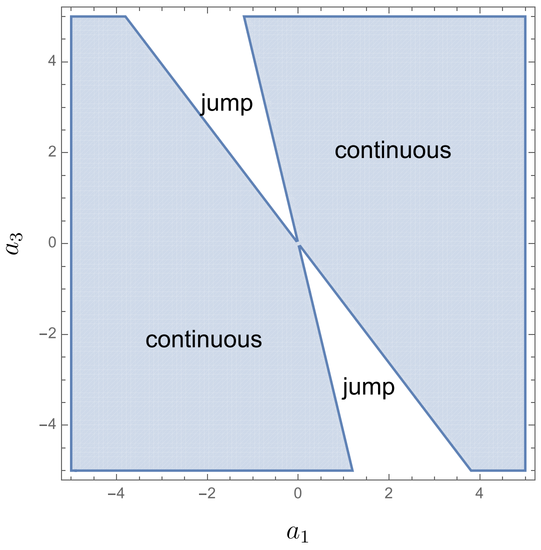

FIGURE 1 The transition diagram for the Neumann boundary conditions with second-order linear operator in the case $a_2 = a_4 = a_6 = 0$ [Colour figure can be viewed at wileyonlinelibrary.com]

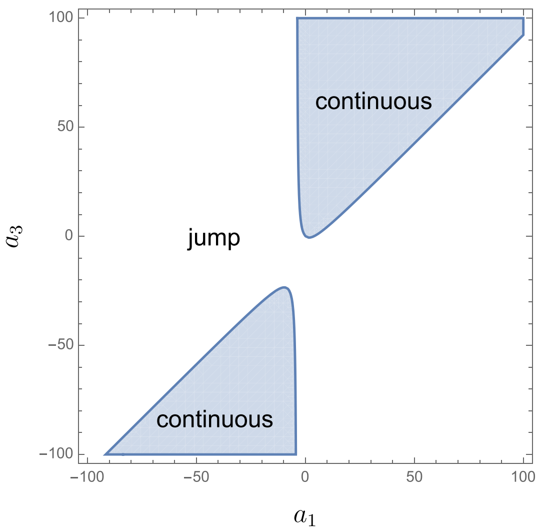

FIGURE 2 The transition diagram for the periodic boundary conditions with second-order linear operator in the bilinear nonlinearity case $a_4 = a_5 = a_6 = a_7 = 0$. Here, $a_2 = 1$, but the figure is qualitatively unchanged for any a_2 [Colour figure can be viewed at wileyonlinelibrary.com]

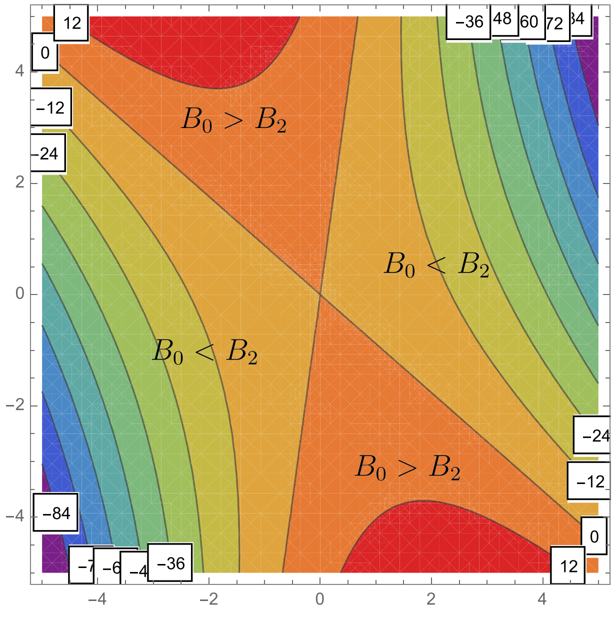

FIGURE 3 Contour plot of $B_0 - B_2$ for the second-order Neumann case which gives an indication on which stable mode is more dominant on the transition number: B_0 due to the zeroth mode $\cos 0x = 1$ or B_2 due to the second mode $\cos 2x$ [Colour figure can be viewed at wileyonlinelibrary.com]

2.2 | Fourth-order linear self-adjoint operator

Here, we consider a 1D reaction–diffusion equation with a fourth-order linear self-adjoint operator as follows:

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial^4 u(x,t)}{\partial x^4} - \delta \frac{\partial^2 u(x,t)}{\partial x^2} + \lambda u(x,t) + g(u, u_x), \tag{15}$$

where $\delta \in \mathbb{R}$ is a parameter and $g(u, u_x)$ is as defined in (4). The above equation is supplemented with one of the Dirichlet, Neumann, or periodic boundary conditions applied, respectively,

$$u(0,t) = u(\pi,t) = \frac{\partial^2}{\partial x^2} u(0,t) = \frac{\partial^2}{\partial x^2} u(\pi,t) = 0 \qquad \text{(Dirichlet)}$$

$$\frac{\partial}{\partial x} u(0,t) = \frac{\partial}{\partial x} u(\pi,t) = \frac{\partial^3}{\partial x^3} u(0,t) = \frac{\partial^3}{\partial x^3} u(\pi,t) = 0 \qquad \text{(Neumann)}$$

$$u(x+2\pi) = u(x), \qquad \forall x \in \mathbb{R}. \qquad \text{(periodic)}$$

Dirichlet and Neumann boundary conditions may not be the typical naming choice for the fourth-order problems. However, if we set $u_{xx} = v$, then the eigenvalue problem for the linear operator can be expressed as an eigenvalue problem for (u, v) where u and v separately satisfy the chosen Dirichlet or Neumann boundary conditions.

The linear eigenvalue problem reads

$$\beta e(x) = -e''''(x) - \delta e''(x) + \lambda e(x)$$

The eigenvalues of the linear operator are

$$\beta_k = \lambda - \lambda_k,$$

where

$$\lambda_k = k^4 - \delta k^2,$$

with corresponding eigenvectors as in (7).

To state the PES conditions, we define

$$\delta_k = 2k^2 + 2k + 1, \tag{16}$$

and

$$k_c = \begin{cases} 1, & \text{if } \delta < \delta_1 = 5 \\ k, & \text{if } \delta_{k-1} < \delta < \delta_k, \text{ and } k \geq 2 \end{cases} \tag{17}$$

For example, $k_c = 2$ if $5 < \delta < 13$ and $k_c = 3$ if $13 < \delta < 25$ and so on. Let us define

$$\Delta = \{\delta_k : k \in \mathbb{N}\} = \{5, 13, 25, \dots\}$$

2.2.1 | PES conditions for the fourth-order case with Neumann/Dirichlet boundary conditions

We have two different considerations for the PES condition. For $\delta \notin \Delta$, the multiplicity of the first critical eigenvalue is one, and the PES condition reads as

$$\beta_{k_c} \begin{cases} < 0, & \lambda < \lambda_{k_c} \\ = 0, & \lambda = \lambda_{k_c} \\ > 0, & \lambda > \lambda_{k_c} \end{cases}, \qquad \beta_k|_{\lambda=\lambda_{k_c}} < 0 \quad k \neq k_c \tag{18}$$

However, for $\delta \in \Delta$, the multiplicity of the first critical eigenvalue is two, and the PES condition reads as

$$\beta_{k_c} = \beta_{k_c+1} \begin{cases} < 0, & \lambda < \lambda_{k_c} \\ = 0, & \lambda = \lambda_{k_c} \\ > 0, & \lambda > \lambda_{k_c} \end{cases} \qquad \beta_k|_{\lambda=\lambda_{k_c}} < 0 \quad k \notin \{k_c, k_c+1\} \tag{19}$$

2.2.2 | PES conditions for the fourth-order case with periodicity condition

For the periodic setting, the PES condition is always higher multiplicity and can be stated as follows. For the generic case, $\delta \notin \Delta$, the multiplicity of the critical eigenvalue is two, and we have

$$\begin{aligned} &\beta_{k_c} = \beta_{-k_c} \begin{cases} < 0, & \lambda < \lambda_{k_c} \\ = 0, & \lambda = \lambda_{k_c} \\ > 0, & \lambda > \lambda_{k_c} \end{cases} \\ &\beta_k|_{\lambda=\lambda_{k_c}} < 0 \qquad k \notin \{k_c, -k_c\} \end{aligned} \tag{20}$$

while for the non-generic case, $\delta \in \Delta$, the critical multiplicity is four, and we have

$$\begin{aligned} &\beta_{k_c} = \beta_{-k_c} = \beta_{k_c+1} = \beta_{-k_c-1} \begin{cases} < 0, & \lambda < \lambda_{k_c} \\ = 0, & \lambda = \lambda_{k_c} \\ > 0, & \lambda > \lambda_{k_c} \end{cases} \\ &\beta_k|_{\lambda=\lambda_{k_c}} < 0 \qquad k \notin \{k_c, -k_c, k_c+1, -k_c-1\}. \end{aligned} \tag{21}$$

We first deal with the generic PES conditions (18) in Theorem 3 and (20) in Theorem 4 below. In the non-generic cases, the number of possible transition scenarios is much higher, and it is much more difficult to write a general transition theorem in that case. For this reason, for the non-generic PES condition, we only give the reduced ODE system for the Dirichlet boundary condition in Theorem 5.

For Theorem 3, we define the following transition numbers. The coefficients in the above equation are

$$A = \begin{cases} \left(4a_1 + 2k_c^2 a_3\right)\left(1 - (-1)^{k_c}\right)/(3k_c\pi), & \text{Dirichlet BC} \\ -2a_2\left(1 - (-1)^{k_c}\right)/(3\pi), & \text{Neumann BC} \end{cases} \tag{22}$$

$$\begin{aligned} &B = C + \sum_{m=0}^{\infty} B_m \\ &C = \frac{3a_4 + k_c^2 a_6}{4} \\ &B_0 = \begin{cases} 0, & \text{Dirichlet BC} \\ \frac{2a_1(a_1+k_c^2 a_3)}{k_c^2(\delta-k_c^2)}, & \text{Neumann} \end{cases} \\ &B_{k_c} = 0 \\ &B_{2k_c} = \begin{cases} \frac{a_2^2}{12(\delta-5k_c^2)}, & \text{Dirichlet} \\ -\frac{(a_1+2k_c a_3)(a_1-k_c^2 a_3)}{6k_c^2(\delta-5k_c^2)}, & \text{Neumann} \end{cases} \\ &B_m = \begin{cases} \frac{8k_c^4}{\pi^2}\frac{\left(1-(-1)^m\right)^2}{m^2\left(4k_c^2-m^2\right)^2}\frac{(2a_1+m^2a_3)\left[2a_1+\left(2k_c^2-m^2\right)a_3\right]}{\delta\left(m^2-k_c^2\right)+k_c^4-m^4}, & \text{Dirichlet} \\ -\frac{4k_c^2a_2^2}{\pi^2}\frac{\left(1-(-1)^m\right)^2}{\left(4k_c^2-m^2\right)^2}\frac{\left(3k_c^2-m^2\right)}{\delta\left(m^2-k_c^2\right)+k_c^4-m^4}, & \text{Neumann} \end{cases}, \qquad m \neq 0, k_c, 2k_c. \end{aligned} \tag{23}$$

The remark for A, B_k and C after the definition (10) is valid here as well.

Theorem 3 (Fourth-order linear operator with Dirichlet or Neumann boundary conditions). *Assume that $\delta \notin \Delta$ so that the PES condition (18) is valid. Then the Equation (15) with either Dirichlet or Neumann boundary conditions has locally stable equilibrium $u \equiv 0$ for $\lambda < \lambda_{k_c}$. For $\lambda > \lambda_{k_c}$, the system exhibits a first transition described by the following ODE:*

$$\frac{du}{dt} = (\lambda - \lambda_{k_c})u + Au^2 + Bu^3 + o(3), \tag{24}$$

where $u \in \mathbb{R}$ denotes the amplitude of the first critical eigenvector $e_{k_c} = \sin k_c x$ for the Dirichlet case and $e_1 = \cos k_c x$ for the Neumann case. For A and B as defined in (22) and (23), we have the following.

1. *If $A \neq 0$, then the first transition is of mixed type accompanied by a transcritical bifurcation. The bifurcated steady-state solution is given by*

$$u = \frac{-(\lambda - \lambda_{k_c})}{A} e_{k_c} + o(|(\lambda - \lambda_{k_c})|).$$

2. *If $A = 0$ and $B \neq 0$ then the first transition is continuous if $B < 0$ and jump if $B > 0$ accompanied by a pitchfork bifurcation. There are two bifurcated steady-state solutions given by*

$$u = \pm\sqrt{\frac{-(\lambda - \lambda_{k_c})}{B}} e_{k_c} + o(|(\lambda - \lambda_{k_c})|^{1/2}).$$

Remark 2. An observation which is useful in determining the sign of B in the above theorem is that

$$\delta - 5k_c^2 < 0. \tag{25}$$

This can be proved using (17) by noticing that $\delta < \delta_{k_c}$ and as a result,

$$\delta - 5k_c^2 < \delta_{k_c} - 5k_c^2 = 2k_c^2 + 2k_c + 1 - 5k_c^2 < 0$$

since $k_c \geq 1$.

Theorem 4 (Fourth-order linear operator with periodicity condition). *Assume that $\delta \notin \Delta$ so that the PES condition (21) is valid. Then the Equation (15) with periodic boundary condition has locally stable equilibrium $u \equiv 0$ for $\lambda < \lambda_{k_c}$. For $\lambda > \lambda_{k_c}$, the system exhibits a first transition described by the following ODE:*

$$\frac{dz}{dt} = \left(\lambda - \lambda_{k_c}\right) z + Pz|z|^2 + o(3), \tag{26}$$

where $z \in \mathbb{C}$ denotes the amplitude of the first critical eigenvector $e_{k_c} = e^{ik_c x}$ and

$$\begin{aligned}
P &= P_0 + P_2 + C \\
C &= 3a_4 + ik_c a_5 + k_c^2 a_6 + 3ik_c^3 a_7 \\
P_0 &= \frac{2\left(2a_1 + ik_c a_2\right)\left(a_1 + k_c^2 a_3\right)}{k_c^2\left(\delta - k_c^2\right)} \\
P_2 &= -\frac{\left[2a_1 - k_c\left(i + 2k_c\right)a_2 + 4ik_c^3 a_3\right]\left[a_1 + ik_c a_2 - k_c^2 a_3\right]}{3k_c^2\left(\delta - 5k_c^2\right)}.
\end{aligned} \tag{27}$$

Moreover, we have the following.

1. *If $\Re P < 0$, then the first transition is continuous. Furthermore, the following statements hold true.*
   (a) *If $\Im P = 0$, the system has a circle C of neutrally stable equilibria given by*

$$C = \left\{ u_\theta(x) = 2\sqrt{\frac{-(\lambda - \lambda_{k_c})}{\Re P}} \cos(\theta + k_c x) + o(|\lambda - \lambda_{k_c}|^{1/2}) | \theta \in [0, 2\pi) \right\} \tag{28}$$

That is for each $\theta \in [0, 2\pi)$, $u_\theta \in C$ is an equilibrium solution.

(b) *If $\Im P \neq 0$, then a unique stable limit cycle bifurcates given by*

$$u(x,t) = 2\sqrt{\frac{-(\lambda - \lambda_{k_c})}{\Re P}} \cos\left(k_c x - \frac{(\lambda - \lambda_{k_c})\Im P}{\Re P} t\right) \tag{29}$$

2. *If $\Re P > 0$, then the first transition is jump, and similar assertions hold. That is in the case $\Im P = 0$, a circle of unstable equilibria bifurcates and in the case $\Im P \neq 0$, a unique unstable limit cycle bifurcates.*

Unlike the second-order case, since the transition number for the fourth-order case depends on the parameters δ and k_c, for the fourth-order case, it is much more difficult to give a corollary as in Corollary 1 which establishes the effect of different coefficients of the nonlinear operator on the first transition.

When the linear operator is of fourth-order, a non-generic higher multiplicity transition is also possible as described by the PES conditions (19) or (21). In these cases, the reduced equations are two dimensional for (19) (the Dirichlet/Neumann case) or four dimensional (21) (periodic case). Hence as a result, the bifurcated attractors/repellors will be of the corresponding dimension, two and four, respectively.

In the higher multiplicity case, we derive the reduced equations only for the Dirichlet case which we give below. These equations display much richer dynamical behavior depending on the parameters, and a general transition theorem covering all possible cases is not possible. Thus, we only give the reduced equations without the accompanying transition analysis.

Theorem 5. *Assume that $\delta \in \Delta$ so that the PES condition (19) is valid. Then the Equation (15) with the Dirichlet boundary conditions has locally stable equilibrium $u \equiv 0$ for $\lambda < \lambda_{k_c}$. For $\lambda > \lambda_{k_c}$, the system exhibits a first transition described by the following ODE:*

$$\begin{aligned}
\frac{du_{k_c}}{dt} &= \beta_{k_c} u_{k_c} + A\left(u_{k_c}^2, u_{k_c}u_{k_c+1}, u_{k_c+1}^2\right) + B\left(u_{k_c}^3, u_{k_c}^2 u_{k_c+1}, u_{k_c}u_{k_c+1}^2, u_{k_c+1}^3\right) + o(3) \\
\frac{du_{k_c+1}}{dt} &= \beta_{k_c+1} u_{k_c+1} + C\left(u_{k_c}^2, u_{k_c}u_{k_c+1}, u_{k_c+1}^2\right) + D\left(u_{k_c}^3, u_{k_c}^2 u_{k_c+1}, u_{k_c}u_{k_c+1}^2, u_{k_c+1}^3\right) + o(3)
\end{aligned} \tag{30}$$

where $u_k, u_{k+1} \in \mathbb{R}$ denotes the amplitude of the first critical eigenvectors $e_{k_c} = \sin k_c x$ and $e_{k_c+1} = \sin(k_c+1)x$. The coefficients in the above equation are

$$\begin{aligned}
A = {} & u_k^2 \left[\frac{2\left(1-(-1)^k\right)}{3k\pi}\left(2a_1 + k^2 a_3\right)\right] + u_k u_{k+1}\left[\frac{4k^2\left(1+(-1)^k\right)}{\pi(k-1)(3k+1)}\left(\frac{2a_1}{k+1} + a_3(k+1)\right)\right] \\
& + u_{k+1}^2\left[\frac{2\left(1-(-1)^k\right)(k+1)^2}{\pi k(k+2)(3k+2)}\left(2a_1 + a_3\left(k^2+4k+2\right)\right)\right] \\
B = {} & u_k^3\left[\frac{k a_2^2}{12k\left(\delta - 5k^2\right)} + \frac{3a_4 + k^2 a_6}{4}\right] \\
& + u_k^2 u_{k+1}\left[-\frac{8k^3(k+1)a_5}{\pi(2k-1)(2k+1)(4k+1)} + \frac{12k^3(k+1)\left(-2k^2+2k+1\right)a_7}{\pi(2k-1)(2k+1)(4k+1)}\right] \\
& + u_k u_{k+1}^2\left[\frac{(2k+1)k a_2^2}{4\left(\delta(2k+1)^2 - \delta k^2 + k^4 - (2k+1)^4\right)} + \frac{3a_4 + (k+1)^2 a_6}{2}\right] \\
& + u_{k+1}^3\left[-\frac{8k(k+1)^3 a_5}{\pi(2k+1)(2k+3)(4k+3)} - \frac{4k(k+1)^3\left(6k^2+14k+7\right)a_7}{\pi(2k+1)(2k+3)(4k+3)}\right]
\end{aligned}$$

$$C = u_k^2 \left[\frac{2k^2\left(1+(-1)^k\right)}{\pi\,(k-1)\,(k+1)(3k+1)} \left(2a_1 + \left(k^2-2k-1\right) a_3\right) \right]$$
$$+ u_k u_{k+1} \left[\frac{4(k+1)^2\left(1-(-1)^k\right)}{\pi(k+2)(3k+2)} \left(\frac{2a_1}{k} + k a_3 \right) \right] + u_{k+1}^2 \left[\frac{2\left(1+(-1)^k\right)}{3\pi} \left(\frac{2a_1}{k+1} + (k+1)\,a_3 \right) \right]$$
$$D = u_k^3 \left[\frac{8k^3(k+1)a_5}{\pi\,(2k-1)\,(2k+1)(4k+1)} + \frac{2k^3\left(12k^3+8k^2-6k-2\right)a_7}{\pi\,(2k-1)\,(2k+1)\,(4k+1)} \right]$$
$$+ u_k^2 u_{k+1} \left[\frac{(k+1)(2k+1)a_2^2}{4\left(\delta(2k+1)^2 - \delta k^2 + k^4 - (2k+1)^4\right)} + \frac{3a_4 + k^2 a_6}{2} \right]$$
$$+ u_k u_{k+1}^2 \left[\frac{8k(k+1)^3 a_5}{\pi\,(2k+1)\,(2k+3)(4k+3)} + \frac{6k(k+1)^2\left(4k^3+16k^2+18k+6\right)a_7}{\pi\,(2k+1)\,(2k+3)(4k+3)} \right]$$
$$+ u_{k+1}^3 \left[\frac{(k+1)^2 a_2^2}{4\left(\delta(2k+2)^2 - \delta k^2 + k^4 - (2k+2)^4\right)} + \frac{3a_4 + (k+1)^2 a_6}{4} \right]$$

3 | APPLICATIONS OF THE THEORETICAL RESULTS

In this section, we show that our general results can be used to describe the first transitions of some important reaction–diffusion equations studied very often.

3.1 | The Chaffee–Infante equation

The Chaffee–Infante equation was first studied in Chafee and Infante[19] and has been described as a jewel of dynamical systems theory in Henry.[20] The equation is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda u - b u^3$$

where λ, b are constants. It is known that under Dirichlet boundary conditions, the equation has only the zero equilibrium if $\lambda < 1$ which is globally asymptotically stable. The equations have $2n+1$ equilibria if $n^2 \leq \lambda \leq (n+1)^2$ which are contained in a global attractor of dimension n.[20,21]

The coefficient of $a_4 = -b$ while the rest of the coefficients in (4) are zero. By Theorem 1 and Theorem 2, the system has a first transition at $\lambda = 1$ and we have the following assertions.

1. Both for the Dirichlet and Neumann cases, $A = 0$, $B = -\frac{3b}{4}$. Hence, the transition is continuous if $b > 0$ and jump if $b < 0$. The corresponding bifurcation is pitchfork, and the bifurcated steady states are

$$u = \pm\sqrt{\frac{4(\lambda-1)}{3b}}\, e_1(x) + o(|(\lambda-1)|^{1/2}),$$

where $e_1(x) = \sin x$ for the Dirichlet case and $e_1(x) = \cos x$ for the Neumann case.
2. For the periodic boundary setting, the transition number is $P = -3b$ which implies that the first transition is continuous if $b > 0$ and jump if $b < 0$ and we have a circle of equilibria given by

$$C = \left\{ u_\theta(x) = 2\sqrt{\frac{(\lambda-1)}{3b}} \cos(\theta + x) + o(|\lambda-1|^{1/2}) | \theta \in [0, 2\pi) \right\}.$$

3.2 | The Kolmogorov–Fisher equation

The Kolmogorov–Fisher model is often used to model population growth and wave propagation.[22] The equation is given by

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = \lambda u(1-u),$$

where $\lambda,\ D > 0$ are constants. By the change of variables

$$x = \sqrt{D}X, \qquad u = \frac{U}{\lambda},$$

the equation becomes

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial X^2} + \lambda U - U^2$$

which is now in the standard form (6).

The coefficient of $a_1 = -1$ while the rest of the coefficients in (4) are zero. According to our analysis, near the critical transition number $\lambda = 1$, we have the following:

1. For the Dirichlet case, $A = -\frac{8}{3\pi}$. Hence, the transition is of mixed type accompanied by a transcritical bifurcation. The bifurcated steady state is given by

$$u = \frac{U}{\lambda} = \frac{3\pi(\lambda-1)}{8\lambda}\sin x + o(|(\lambda-1)|).$$

2. For the Neumann case, $A = 0$ and $B = -\frac{11}{6}$ and the transition is continuous accompanied by a supercritical pitchfork bifurcation with two bifurcated steady states on $r < 1$ given by

$$u = \pm\frac{1}{\lambda}\sqrt{\frac{6(\lambda-1)}{11}}\cos x + o(|(\lambda-1)|^{1/2}).$$

3. For the periodic boundary setting, the transition number is $P = -\frac{10}{3}$ which implies that the first transition is continuous, and we have a circle of equilibria given by

$$C = \left\{ u_\theta(x) = \frac{2}{\lambda}\sqrt{\frac{3(\lambda-1)}{10}}\cos(\theta+x) + o(|\lambda-1|^{1/2}) | \theta \in [0, 2\pi) \right\}$$

3.3 | The Swift–Hohenberg equation

The Swift–Hohenberg equation has been introduced as a model equation to describe the transition behavior of the Rayleigh–Bénard equations near the criticality.[23] In recent years, the dynamical transitions of the system under various settings have been investigated[8,10,12,24]; see also for the global attractor of the equations.[25] The equation reads as

$$\frac{\partial u}{\partial t} = \beta u - (\partial_{xx}+1)^2 u - u^3.$$

We can put the equation in the standard form (15) if we let

$$\lambda = \beta - 1, \qquad \delta = 2.$$

The coefficients of the nonlinear operator are $a_4 = -1$ and the rest are zero. By (17), the critical wavenumber is $k_c = 1$. By Theorem 3 and Theorem 4, the system has a first transition at $\beta = 2$, and we have the following assertions.

1. Both for the Dirichlet and Neumann cases, $A = 0$ and $B = -\frac{3}{4}$. Hence, the transition is of continuous type accompanied by a supercritical pitchfork bifurcation, and the two bifurcated steady states are

$$u = \pm\sqrt{\frac{3(\beta-2)}{4}}e_1(x) + o(|(\beta-2)|^{1/2}),$$

where $e_1(x) = \sin x$ for the Dirichlet case and $e_1(x) = \cos x$ for the Neumann case.
2. For the periodic boundary setting, the transition number is $P = -3$, which implies that the first transition is continuous, and we have a circle of equilibria given by

$$C = \left\{ u_\theta(x) = 2\sqrt{\frac{(\beta-2)}{3}}\cos(\theta + x) + o(|\beta - 2|^{1/2}) | \theta \in [0, 2\pi) \right\}$$

3. If the operator $(\partial_{xx} + 1)^2 \to (\partial_{xx} + \delta/2)^2$ with $\delta = 5$ then the Swift–Hohenberg equation can be written in the form (15) with $\lambda = \beta - 25/4$. The first transition is of multiplicity two with two critical wavenumbers $k_c = 1, 2$ which occurs at $\lambda = \lambda_c = \lambda_1 = 1 - \delta = -4$ and transition scenario is governed by Theorem 5 with the reduced equations given by

$$\frac{du_1}{dt} = (\lambda + 4)u_1 - \frac{3}{4}u_1(u_1^2 + 2u_2^2),$$
$$\frac{du_2}{dt} = (\lambda + 4)u_2 - \frac{3}{4}u_2(2u_1^2 + u_2^2).$$

The above equations has a continuous transition on $\lambda > -4$. It can be shown that the system has an S^1 attractor bifurcation[26] and the bifurcated attractor consists of eight steady states. Four of these steady states are saddles which are mixed modes

$$u_{1,2,3,4} = \frac{2\sqrt{\lambda+4}}{3}(\pm \sin x \pm \sin 2x) + o\left(\sqrt{\lambda+4}\right),$$

and the remaining four are stable steady states which are single mode

$$u_{5,6} = \sqrt{\frac{4(\lambda+4)}{3}}\sin x + o\left(\sqrt{\lambda+4}\right), \qquad u_{7,8} = \sqrt{\frac{4(\lambda+4)}{3}}\sin 2x + o\left(\sqrt{\lambda+4}\right).$$

3.4 | The Kuramoto–Sivashinsky equation

The Kuramoto–Sivashinsky equation arises in many important physical contexts and has been widely used as a model for instabilities of flame fronts and ion plasma.[21,27,28] The dynamic transitions of this system has been studied for example in Ong[14] and Ma and Wang.[26]

$$\frac{\partial u}{\partial \tau} + \mu\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} - \beta u + u\frac{\partial u}{\partial x} = 0. \tag{31}$$

We can put the equation in the standard form (15) if we let

$$t = \mu\tau, \qquad \delta = \frac{1}{\mu}, \qquad \lambda = \frac{\beta}{\mu}.$$

The coefficients of the nonlinear operator are $a_2 = -1/\mu$, and the rest are zero. We define the critical wavenumber k_c by (17). By Theorem 3 and Theorem 4, the system has a first transition at $\frac{\beta}{\mu} = \lambda_{k_c} = k_c^4 - \delta k_c^2$, and we have the following assertions.

1. For the Dirichlet case, under the generic PES condition (simple critical eigenvalue) $A = 0$ and $B = \frac{1}{12\mu^2(\delta - 5k_c^2)} < 0$ by (25). Hence, the transition is of continuous type accompanied by a supercritical pitchfork bifurcation, and the two bifurcated steady states are

$$u = \pm\sqrt{\frac{-(\lambda - \lambda_{k_c})}{B}}\sin k_c x + o(|(\lambda - \lambda_{k_c})|^{1/2}).$$

2. For the Neumann case, under the generic PES condition, the parity of k_c (the wave index of first critical mode) plays a crucial role. When k_c is odd, $A = 4/(3\mu\pi)$ and the first transition is of mixed type accompanied by a transcritical bifurcation and there is one bifurcated steady state given by

$$u = \frac{-3\mu\pi}{4}(\lambda - \lambda_{k_c}) \cos k_c x + o(|\lambda - \lambda_{k_c}|).$$

When k_c is even, $A = 0$ and B is expressed as an infinite series as in (23), that is $B = \sum_{m\neq 0,k_c,2k_c}^{\infty} B_m$. In this case, we numerically experiment with B. To do so, we pick an even k_c, and limit δ to the interval $(\delta_{k_c-1}, \delta_{k_c})$ by (25) and (17) and finally truncating the infinite sum when a satisfactory convergence is obtained. This numerical experimentation shows that B is positive. This means that the system exhibits a jump transition accompanied by a subcritical pitchfork bifurcation with two steady states given by

$$u = \pm\sqrt{\frac{-(\lambda - \lambda_{k_c})}{B}} \cos k_c x + o(|(\lambda - \lambda_{k_c})|^{1/2}).$$

3. For the periodic boundary setting, under the generic PES condition, (double eigenvalue) the transition number P has $\Re P = \frac{-1}{3\mu^2(\delta - 5k_c^2)} > 0$ by (25) and $\Im P = \frac{2k_c}{3\mu^2(\delta-5k_c^2)}$ which implies that the first transition is jump and a an unstable limit cycle given by

$$u(x,t) = 2\sqrt{\frac{-(\lambda - \lambda_{k_c})}{\Re P}} \cos\left(k_c x - \frac{(\lambda - \lambda_{k_c})\Im P}{\Re P} t\right)$$

is bifurcated from the steady state.

4 | CONCLUSIONS, DISCUSSIONS, AND PHYSICAL REMARKS

We consider reaction–diffusion equations with self-adjoint second- and fourth-order linear operators, a general nonlinear term with quadratic/cubic nonlinearities and three different boundary conditions, namely, Dirichlet, Neumann, and the periodic boundary conditions. We denote these by (2DN), (4DN), (2P), and (4P). For example, (2DN) is the setting of the second-order equation with Dirichlet or Neumann boundaries. We have the following conclusions.

1. **Classification of transition types**. For (2DN) and (4DN), all three types of transitions are possible, namely, continuous, jump, and transition. For (2P) and (4P), only continuous and jump transition types are possible and the mixed type transition is not possible.
2. **Multiplicity of first critical eigenvalue and the bifurcated states**. The bifurcated states of the system depend on the multiplicity of the first eigenvalue. Depending on the multiplicity of the first eigenvalue, isolated steady states, circle of equilibria, limit cycles, and invariant torus can bifurcate. As we consider only self-adjoint linear operators, in all cases studied the spectrum consists of real eigenvalues. For (2DN) always and for (4DN) generically, the transition is from a single multiplicity eigenvalue which give rise to bifurcated steady states. For (2P) always and for (4P) generically, transition is from a double multiplicity eigenvalue which give rise to either a slowly rotating bifurcated limit cycle or a bifurcated circle of equilibria.

   For (4DN) a non-generic double real eigenvalue transition is possible. For (4P) a non-generic quadruple real eigenvalue transition is possible. We find the general reduced system for the (4D) and apply the results for the Swift–Hohenberg equation which display an S^1 attractor bifurcation with four bifurcated stable pure modes and four bifurcated saddle mixed modes.
3. **The effect of modes on the transition numbers**.

   (a) For (2DN) and (4DN) cases the transition numbers A and B in Theorem 1 and Theorem 3 completely describe the transition behavior. A is determined by the self-interaction of the critical mode through the quadratic nonlinearities while B is determined by two different effects: (i) sum of coefficients B_k which are due to the interaction of the critical mode with kth stable through quadratic nonlinearity, (ii) C, the cubic self-interaction of the critical mode.

   If both A and B are zero, our theorem does not give a conclusion and higher order coefficients of (11) need to be computed.

(b) For (2P) and (4P) cases the transition number P in Theorem 2 and Theorem 4 completely describe the transition behavior. In the periodic setting, the only contributions to P come from P_0, the interaction of the critical mode with zero wavenumber mode and P_2, the interaction of the critical mode with twice the critical wavenumber mode through quadratic nonlinearities and C the self-interaction of the critical mode through cubic nonlinearities.

(c) For (2N) and (2P), the modes with wavenumber other than 0 and 2 never and for (2D) generically (unless the quadratic nonlinearity satisfies a very specific condition) do not contribute to first transition. Similarly for (4P), the only modes that play a role are those with 0 wavenumber and twice the critical wavenumber.

However for (4DN), the situation is more subtle and the parity of the wavenumber of the first critical mode plays a crucial role. For odd parities of the critical wavenumber, the situation is as in (4P). However, for even parities, there are infinitely many modes which contribute to the first transition.

4. **The critical wavenumber**. The main difference between (2DN)/(2P) and (4DN)/(4P) cases are, for the (2DN)/(2P) cases, the critical wavenumber is always one whereas for the (4DN)/(4P) cases, the critical wavenumber depends on the strength of the second-order diffusion and can be higher than one.
5. **Nature of the bifurcated slow limit cycle**. We note that there is possibility of bifurcated limit cycle in the periodic settings (2P)/(4P) which is unlike the Hopf bifurcation. In our case, thanks to the self-adjointness of the linear operator, the eigenvalues are real and Hopf bifurcation is not possible. For the bifurcated limit cycle we have, the period of the limit cycle (14) goes to infinity as $\lambda \downarrow \lambda_c = 1$. Thus, near the transition point $\lambda_c = 1$, the limit cycle behaves almost as a circle of equilibria.
6. **The mean zero condition for (2P) and (4P)**. Under the periodicity condition if $\int_{-\pi}^{\pi} g(u, u_x)dx = 0$ (e.g., if $g(u, u_x) = a_2 u u_x$) then it is natural to impose the zero-mean condition on the solution. This is because, if the initial mean is zero, $\int_{-\pi}^{\pi} u(x, 0)dx = 0$, then it stays zero for all t, $\int_{-\pi}^{\pi} u(x, t)dx = 0$. Thus for such equations (e.g., see Kuramoto–Sivashinsky Equation 31) an additional condition, namely, the zero-mean condition is usually imposed. In such a case, the constant stable mode $e_0 \equiv 1$ is not present in the spectrum. Theorem 2 and Theorem 4 still hold true for the zero-mean case with the modification $P = C + P_2$ in (12), that is with P_0 removed from the definition of P.
7. **Possible future research problems**. There are several problems left open in this work which will be addressed in the later studies.

(a) One can consider the case where the parameters of the equation are x-dependent. This case can be tackled using numerical treatment of the eigenvalue problem, see Sengul et al[32] and Lu et al.[30]

(b) Another direction is to consider the case of semilinear fourth-order equations $g = g(u, u_x, u_{xx}, u_{xxx})$.

(c) As we consider only the self-adjoint linear operators in this study, a natural next step is to obtain a classification of non-self-adjoint linear operators which naturally give rise to Hopf bifurcations.

(d) The classification of reaction–diffusion equations in two and more spatial dimensions is open. These systems include applications such as the Fitzhugh–Nagumo, the Belusov–Zhabotinsky, and the Brusselator equations as well as the higher spatial dimensional counterparts of the equations considered in this paper.

(e) The addition of white noise in to the system can be tackled using the tools introduced in Chekroun et al.[31,32] In the stochastic context, due to large excursions of the solutions caused by the white noise, the condition that the amplitudes of the critical modes remain sufficiently small is violated even when the magnitude of the noise is small. The way to deal with this difficulty (the stochastic parameterizing manifold approach) is introduced in Chekroun et al.[31,22]

(f) Our classification deals only with standard boundary conditions. There are other homogeneous boundary conditions which arise often in applications such as Dirichlet–Neumann boundaries $u(0) = u'(\pi)$ and Robin boundary conditions for the second-order case and clamped boundary condition $u(0) = u(\pi) = u'(0) = u'(\pi)$ for the fourth-order case. All these different settings can be dealt similar to the analysis of this paper.

5 | PROOFS

In this section, we give the proofs of our main results. The proof is carried out by first reducing the system in the direction of first critical eigenvectors which yield a reduced ODE system. Then the transition and bifurcations are determined by the analysis of this reduced system.

5.1 | Second-order linear self-adjoint operator with Dirichlet boundary conditions

In the Dirichlet boundary case, the basis is given by $e_k(x) = \sin kx$, $k \geq 1$. The unknown function u can be written as

$$u = \sum_{k=1}^{\infty} u_k(t) e_k(x)$$

According to the center manifold theory, to understand the behavior of the system in the regime where both nonlinear effects as well as $\lambda > 1$ is small, it is sufficient to consider the dynamics on the center manifold. This is achieved by setting

$$u_k = \Phi_k(u_1) = b_k u_1^2 + o\left(u_1^2\right),\ u_1 \to 0,\ \forall k \geq 2,$$

where Φ_k are the components of the center manifold function Φ. In other words, we can write

$$\begin{aligned} u &= u_1 \sin x + \Phi, \\ \Phi &= \sum_{k=2}^{\infty} \Phi_k(u_1) \sin(kx) = \sum_{k=2}^{\infty} b_k u_1^2 \sin(kx) + o\left(u_1^2\right). \end{aligned} \tag{32}$$

The dynamics of the amplitude u_1 of the first mode can be obtained by plugging (32) in the main equation and taking inner product with the first critical mode e_1 and reads as

$$\frac{du_1}{dt} = (\lambda - 1) u_1 + g_1, \tag{33}$$

where for u as given in (32), we have

$$\begin{aligned} g_1 &= \frac{\int_0^{\pi} g(u, u_x) e_1 dx}{\int_0^{\pi} e_1^2 dx} \\ &= \frac{2}{\pi} \int_0^{\pi} \left[a_1 u^2 + a_2 u u_x + a_3 u_x^2 + a_4 u^3 + a_5 u^2 u_x + a_6 u u_x^2 + a_7 u_x^3 + o(|(u, u_x)|^3\right] \sin x dx \\ &= \frac{2a_1}{\pi} \int_0^{\pi} \left(u_1^2 \sin^3 x + 2u_1 \Phi \sin^2 x\right) dx \\ &+ \frac{2a_2}{\pi} \int_0^{\pi} \left(u_1^2 \sin^2 x \cos x + u_1 \Phi_x \sin^2 x + u_1 \Phi \sin x \cos x\right) dx \\ &+ \frac{2a_3}{\pi} \int_0^{\pi} \left(u_1^2 \sin x \cos^2 x + 2u_1 \Phi_x \sin x \cos x\right) dx \\ &+ \frac{2a_4}{\pi} \int_0^{\pi} u_1^3 \sin^4 x dx + \frac{2a_5}{\pi} \int_0^{\pi} u_1^3 \sin^3 x \cos x dx \\ &+ \frac{2a_6}{\pi} \int_0^{\pi} u_1^3 \sin^2 x \cos^2 x dx + \frac{2a_7}{\pi} \int_0^{\pi} u_1^3 \sin x \cos^3 x dx + o(3) \\ &= A u_1^2 + B u_1^3 + o(3). \end{aligned}$$

Here A and B are

$$A = \frac{2a_1}{\pi}\int_0^{\pi} \sin^3 x dx + \frac{2a_2}{\pi}\int_0^{\pi} \sin^2 x \cos x dx + \frac{2a_3}{\pi}\int_0^{\pi} \sin x \cos^2 x dx = \frac{8a_1 + 4a_3}{3\pi}$$

$$\begin{aligned} B = {} & \frac{4a_1}{\pi}\sum_{k\neq 1,2}^{\infty} b_k \int_0^{\pi} \sin(kx)\sin^2 x dx \\ & + \frac{2a_2}{\pi}\sum_{k=2}^{\infty} b_k \int_0^{\pi} \left[k\cos(kx)\sin^2 x + \sin(kx)\sin x\cos x\right] dx \\ & + \frac{4ka_3}{\pi}\sum_{k\neq 1,2}^{\infty} b_k \int_0^{\pi} \cos(kx)\sin x\cos x dx + \frac{2a_4}{\pi}\int_0^{\pi} \sin^4 x dx \\ & + \frac{2a_5}{\pi}\int_0^{\pi} \sin^3 x\cos x dx + \frac{2a_6}{\pi}\int_0^{\pi} \sin^2 x\cos^2 x dx + \frac{2a_7}{\pi}\int_0^{\pi} \sin x\cos^3 x dx \end{aligned}$$

The coefficient B can be written as follows.

$$\begin{aligned} B &= C + \sum_{k=2}^{\infty} B_k \\ C &= \frac{3a_4 + a_6}{4} \\ B_2 &= -\frac{a_2}{2} b_2 \\ B_k &= a_n \frac{4}{\pi}\frac{\left(2a_1 + k^2 a_3\right)}{k\left(k^2 - 4\right)}\left((-1)^k - 1\right) b_k, \qquad k \geq 3. \end{aligned} \tag{34}$$

The reduced equation is determined by the quadratic term if $A \neq 0$. However when $A = 0$, the computation of B is necessary. As seen from the formula for B, this requires the computation of quadratic coefficients b_k of the center manifold in above equation. For this purpose, we take the time derivative of

$$u_k(t) = \Phi_k = b_k u_1^2(t) + o\left(u_1^2\right)$$

to find

$$\dot{u}_k(t) = \dot{u}_1\left(2b_k u_1 + o\left(u_1\right)\right)$$

which gives

$$\beta_k\left(\lambda_c\right) u_k + g_k(t) = \dot{u}_k(t) = \left(\beta_1\left(\lambda_c\right) u_1 + g_1(t)\right)\left(2b_k u_1 + o\left(u_1\right)\right)$$

Since $g_1 = O\left(u_1^3\right)$ when $A = 0$ and

$$\beta_k\left(\lambda_c\right) = \lambda_c - k^2 = 1 - k^2 + O\left(|\beta_1|\right)$$

the above equation becomes

$$\beta_k\left(\lambda_c\right)\left(b_k u_1^2 + o\left(u_1^2\right)\right) + g_k = O\left(u_1^2 |\beta_1|\right) + O\left(u_1^3\right).$$

To find b_k, we need to approximate g_k to the lowest order in u_1. Since $u_k = \Phi_k(u_1) = O\left(u_1^2\right)$,

$$
\begin{aligned}
g_k &= \frac{2}{\pi} a_1 u_1^2 \int_0^\pi \sin^2 x \sin(kx) dx + \frac{2a_2}{\pi} u_1^2 \int_0^\pi \sin x \cos x \sin(kx) + \frac{2}{\pi} a_3 u_1^2 \int_0^\pi \cos^2 x \sin(kx) dx \\
&= u_1^2 \left[\frac{2}{\pi} \frac{\left[2a_1 - \left(k^2-2\right) a_3\right]}{k\left(k^2-4\right)} \left((-1)^k - 1\right) + \frac{a_2}{2} \right],
\end{aligned}
$$

then

$$
\begin{aligned}
b_2 &= \frac{a_2}{6} \\
b_k &= \frac{2}{\pi} \frac{\left[2a_1 - \left(k^2-2\right) a_3\right]}{k\left(k^2-1\right)\left(k^2-4\right)} \left((-1)^k - 1\right).
\end{aligned}
$$

Thus, B_k in (34) is found as

$$
\begin{aligned}
B_2 &= -\frac{a_2^2}{12} \\
B_k &= \frac{8}{\pi^2} \sum_{k \neq 1,2}^{\infty} \frac{\left((-1)^k - 1\right)^2}{k^2\left(k^2-4\right)^2} \frac{\left(2a_1 + k^2 a_3\right)\left[2a_1 - \left(k^2-2\right) a_3\right]}{\left(k^2-1\right)}, \qquad k \geq 3.
\end{aligned}
$$

Thus, we obtain the reduced Equation (11), and it can be easily seen that the assertions of Theorem 1 hold true in the Dirichlet boundary case.

5.2 | Second-order linear self-adjoint operator with Neumann boundary conditions

For the Neumann boundary conditions, the proof is quite similar. The only difference is that the eigenfunctions are now $e_k(x) = \cos kx$, $k \in \mathbb{Z}_{\geq 0}$. With these eigenfunctions, the coefficient A of the quadratic term u^2 of the reduced equation can be computed similarly. For the coefficient B of the cubic term u^3 however, in addition to the contributions from higher frequency stable modes e_k, $k \geq 2$, the contribution coming from the zeroth wavenumber mode must also be included. As will be discussed in the next section, the zeroth wavenumber stable mode has a contribution on the transition number for the periodic boundary condition as well. We omit the details of the proof for this case.

5.3 | Second-order linear self-adjoint operator with periodic boundary conditions

The reduction procedure for the periodic setting is somewhat different as it is more convenient to write the basis in terms of complex exponentials. For studying the dynamics on the center manifold, we write

$$
\begin{aligned}
u &= u_c + \Phi = z e^{ix} + \bar{z} e^{-ix} + \Phi \\
\Phi &= \sum_{|k| \neq 1}^{\infty} \Phi_k(z, \bar{z}) e^{ikx} = \sum_{|k| \neq 1}^{\infty} \left(b_{k,1} z^2 + b_{k,2} z\bar{z} + b_{k,3} \bar{z}^2\right) e^{ikx} + o(3),
\end{aligned}
$$

where Φ is the center manifold function. By the center manifold theorem, the center manifold is tangent to the center space at the origin; hence, we have

$$
\Phi_k(z, \bar{z}) = \left(b_{k,1} z^2 + b_{k,2} z\bar{z} + b_{k,3} \bar{z}^2\right) + o(3).
$$

Now plugging u into the main equation and taking inner product with $e_1 = e^{ix}$, we obtain

$$
\frac{dz}{dt} = (\lambda - 1) z + g_1.
$$

The nonlinear term g_1 of the above can be obtained easily by

$$g_1 = \frac{1}{2\pi}\int_0^{2\pi} \left[a_1u^2 + a_2uu_x + a_3u_x^2 + a_4u^3 + a_5u^2u_x + a_6uu_x^2 + a_7u_x^3 + o(|(u,u_x)|^3\right] e^{-ix}dx,$$

$$\begin{aligned} g_1 = &(2a_1 + ia_2)\left(b_{0,1}z^2 + b_{0,2}z\bar{z} + b_{0,3}\bar{z}^2\right) z + [2a_1 - (i+2)a_2 + 4ia_3]\left(b_{2,1}z^2 + b_{2,2}z\bar{z} + b_{2,3}\bar{z}^2\right)\bar{z} \\ &+ (3a_4 + ia_5 + a_6 + 3ia_7) z^2\bar{z} + o(4). \end{aligned}$$

The quadratic interactions vanish and only cubic interactions with the center manifold come from the zeroth and second wavenumber modes. Thus, we need to find $b_{0,1}, b_{2,1}, b_{0,2}, b_{2,2}, b_{0,3}$ and $b_{2,3}$. These coefficients are obtained as in the Dirichlet type boundary condition.

$$b_{0,1}z^2 + b_{0,2}z\bar{z} + b_{0,3}\bar{z}^2 = -2(a_1 + a_3)z\bar{z}$$

and

$$b_{2,1}z^2 + b_{2,2}z\bar{z} + b_{2,3}\bar{z}^2 = \frac{1}{3}(a_1 + ia_2 - a_3)z^2.$$

Hence, g_1 becomes

$$\begin{aligned} g_1 = &-2(2a_1 + ia_2)(a_1 + a_3)z^2\bar{z} + \frac{1}{3}[2a_1 - (i+2)a_2 + 4ia_3](a_1 + ia_2 - a_3)z^2\bar{z} \\ &+ (3a_4 + ia_5 + a_6 + 3ia_7)z^2\bar{z} + o(3) \\ = &Pz|z|^2 + o(3) \end{aligned}$$

where P is obtained as in Theorem 2.

Now we analyze the reduced equation

$$\dot{z} = \beta_1 z + Pz|z|^2 \tag{35}$$

with $z \in \mathbb{C}$. Writing $P = P_r + iP_i$ and $z = x_1 + ix_2$, the reduced equation becomes

$$\begin{aligned} \dot{x} &= \beta x + (P_r x - P_i y)(x^2 + y^2), \\ \dot{y} &= \beta y + (P_r y + P_i x)(x^2 + y^2). \end{aligned} \tag{36}$$

If $P_i = 0$ and $\beta P_r > 0$ the only equilibrium is the origin. However, if $P_i = 0$ and $\beta P_r < 0$ then there is a circle C of equilibria given by $x^2 + y^2 = -\frac{\beta}{P_r}$, that is

$$C = \left\{(x,y) = \sqrt{\frac{-\beta}{P_r}}(\cos\theta, \sin\theta) | \theta \in [0, 2\pi)\right\}$$

If $P_i \neq 0$, then there are no equilibrium solutions of (36) other than $(0,0)$ since in that case,

$$\frac{x}{P_r x - P_i y} = \frac{y}{P_r y + P_i x} + \ \Rightarrow\ P_i(x^2 + y^2) = 0.$$

To analyze $P_i \neq 0$ case, we consider (35) in polar coordinates $z = re^{i\theta}$.

$$\begin{aligned} \dot{r} &= \beta r + P_r r^3 \\ \dot{\theta} &= P_i r^2 \end{aligned}$$

If $P_i \neq 0$ and $\beta P_r < 0$, then there is a limit cycle $r^2 = -\beta/P_r$ with period $O(r^2)$ which is stable if $P_r < 0$ and unstable otherwise. The limit cycle is given by

$$z(t) = \sqrt{\frac{-\beta}{P_r}} e^{-i\frac{\beta P_i}{P_r}t}$$

The assertions of Theorem 2 follow from the above analysis.

5.4 | Fourth-order linear operator with Dirichlet/Neumann boundary conditions

For the fourth-order case, the main difference with the second-order case is that the first critical mode is e_{k_c} instead of e_1 where k_c is defined as in (17). For ease of notation, we will write k instead of k_c.

The proof for both Dirichlet and the Neumann cases are similar. Thus, we only proceed with the Dirichlet case.

The dynamics of the first mode u_k becomes

$$\frac{du_k}{dt} = (\lambda - \lambda_k)\, u_k + g_k, \qquad \lambda_k = k^4 - \delta k^2$$

By the center manifold theorem,

$$u_m = \Phi_m (u_k) = b_m u_k^2 + o\left(u_k^2\right),$$

where Φ_m are the components of the center manifold function Φ

$$\begin{aligned} u &= u_k \sin(kx) + \Phi \\ \Phi &= \sum_{m\neq k}^{\infty} \Phi_m (u_k) \sin(mx) = \sum_{m\neq k}^{\infty} b_m u_k^2 \sin(mx) + o\left(u_k^2\right). \end{aligned}$$

By performing similar operations as in the second-order linear self-adjoint operator, the following results are easily obtained.

$$\begin{aligned} g_k = {} & \frac{2a_1}{\pi}\int_0^{\pi} \left[u_k^2 \sin^3(kx) + 2u_k \Phi \sin^2(kx)\right] dx \\ & + \frac{2a_2}{\pi}\int_0^{\pi} \left[k u_k^2 \sin^2(kx)\cos(kx) + u_k \Phi_x \sin^2(kx) + k u_k \Phi \sin(kx)\cos(kx)\right] dx \\ & + \frac{2a_3}{\pi}\int_0^{\pi} \left[k^2 u_k^2 \sin(kx)\cos^2(kx) + 2k u_k \Phi_x \sin(kx)\cos(kx)\right] dx \\ & + \frac{2a_4}{\pi}\int_0^{\pi} u_k^3 \sin^4(kx) dx + \frac{2a_5}{\pi}\int_0^{\pi} k u_k^3 \sin^3(kx)\cos(kx) dx \\ & + \frac{2a_6}{\pi}\int_0^{\pi} k^2 u_k^3 \sin^2(kx)\cos^2(kx) dx \\ & + \frac{2a_7}{\pi}\int_0^{\pi} k^3 u_k^3 \sin(kx)\cos^3(kx) dx + \mathcal{O}\left(u_1^4\right) \\ = {} & A u_k^2 + B u_k^3 + o(3), \end{aligned}$$

where

$$\begin{aligned} A &= \frac{2a_1}{\pi}\int_0^{\pi} \sin^3(kx) dx + \frac{2ka_2}{\pi}\int_0^{\pi} \sin^2(kx)\cos(kx) dx + \frac{2k^2a_3}{\pi}\int_0^{\pi} \sin(kx)\cos^2(kx) dx \\ &= \frac{4a_1 + 2k^2 a_3}{3k\pi}\left(1 - (-1)^k\right) \end{aligned}$$

and

$$B = \frac{4k^2}{\pi}\sum_{m\neq k,2k}^{\infty} \frac{\left(2a_1 + m^2 a_3\right)}{m\left(4k^2 - m^2\right)}\left(1 - (-1)^m\right) b_m - \frac{ka_2}{2} b_{2k} + \frac{3a_4 + k^2 a_6}{4} \tag{37}$$

We need to find b_m. By taking the time derivative of u_m and using g_k, one obtains

$$\begin{aligned} b_{2k} &= -\frac{a_2}{6k\left(\delta - 5k^2\right)} \\ b_m &= -\frac{2k^2}{\pi}\frac{2a_1 + a_3\left(2k^2 - m^2\right)}{m\left(4k^2 - m^2\right)}\frac{\left(1 - (-1)^m\right)}{\delta\left(m^2 - k^2\right) + k^4 - m^4} \end{aligned} \tag{38}$$

By plugging (38) in (37), B is obtained as in Theorem 3. The conclusions of the theorem now can be obtained as in the second-order case.

5.5 | Fourth-order linear operator with periodic boundary conditions

In this case, the system can be reduced as

$$\frac{dz}{dt} = \left(\lambda - \lambda_{k_c}\right) z + g_{k_c},$$

where $\lambda_{k_c} = k_c^4 - \delta k_c^2$. The only difference with the second-order periodic setting is that the solution this time should be written as

$$\begin{aligned} u &= u_c + \Phi = z e^{ik_c x} + \bar{z} e^{-ik_c x} + \Phi, \\ \Phi &= \sum_{|m| \neq k_c}^{\infty} \Phi_m(z, \bar{z}) e^{imx} = \sum_{|m| \neq k_c}^{\infty} \left(b_{m,1} z^2 + b_{m,2} z\bar{z} + b_{m,3}\bar{z}^2\right) e^{imx} + o(3). \end{aligned}$$

With similar analysis, g_k and hence the conclusions of Theorem 4 are obtained.

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