Available online at www.sciencedirect.com

ScienceDirect

Procedia Social and Behavioral Sciences 2 (2010) 2360–2365

Procedia
Social and Behavioral Sciences

WCES-2010

First class high school students' behaviour in non-routine problems related to graphic representation

İlyas Yavuz[a] *, Mustafa Egin[a]

[a]*Ataturk Education Faculty, Marmara University, Istanbul, 34722, Turkey*

Received October 20, 2009; revised December 29, 2009; accepted January 11, 2010

Abstract

In general, the first concepts of calculus (such as functions) are taught by looking at examples, noticing their properties and generalizing from them in some implicit ways. Students have no means to discuss the general truth of a statement, or to examine the validity of a theorem, relative to the mathematical field. This knowledge is nonetheless required by teachers at the university level. The question is, therefore, if it is possible to organize activities for beginning calculus students, which would nevertheless lead them to working on statements and validity of theorems. This paper presents a teaching approach related to the concept of function, which aimed at leading students working within a graphic milieu to producing, discussing and testing the validity of mathematical statements and theorems. The intention of the approach was to use the procedural aspect of the graphs to provide a favorable milieu for linking the intuitive and the formal knowledge (such as required at the university for establishing proofs). The approach was experimented with a group of students. After the experiment, the students indeed became able to think of functions as objects and to engage with questions of validity of mathematical statements.
© 2010 Elsevier Ltd. All rights reserved.

Keywords: functions; graphic; proof; representatives; teaching situations.

1. Introduction

In general, the first concepts of calculus are taught apart from problems and without all the tools of formalization. Indeed the usual introduction at upper secondary school consists in looking at a few examples of functions and limits, noticing their properties and eventually reaching generalization in some implicit ways. This is supposed sufficient as a first approach to the concepts of analysis, assuming that, later (at the university level), students will learn to prove and justify the properties introduced at this point. But at the university, teachers often complain that students do not show the proper abilities to prove and that they use graphs and equations as if they were some kind of 'labels' for functions, rather than material means for expressing concepts and tools for proving. In this paper, I question the type of knowledge students build in the approach commonly used at upper secondary school. The main question posed here could be stated as follows: What type of knowledge is necessary for students to be able to

* İlyas Yavuz.
E-mail address: iyavuz@marmara.edu.tr

1877-0428 © 2010 Published by Elsevier Ltd.
doi:10.1016/j.sbspro.2010.03.337

produce and/or prove mathematical statements about functions and to test their validity? What kinds of situations could lead to this type of knowledge?

Related to this last question, this paper also describes and analyzes a teaching approach, in which 17–18-years old students are led to construct functions in a graphic milieu, as well as produce mathematical statements about these functions, discussing the validity of properties and the range of their application (For a discussion of the concept of milieu in the theory of situations, see Brousseau 1997).

In the seventies, calculus used to be taught with all the rules of formalization and proofs, such as the epsilon-delta definition for limits. Since this is no longer the case teachers have to try and make students perceive the objects, and the properties of these objects, by different means. A standard didactic sequence found in curricula and textbooks is, for instance, the following: students are given a graph of a function, and have to infer from it some properties of the function. The teacher can tell students that the function is bounded, and how it can be seen from the graph. The students are then invited to find the bounds of the function and to express the property in writing, in terms of an inequality. Research has shown that many students who were taught this way were not able to correctly grasp the nature of the mathematical objects (such as functions or limits) and to produce proofs in a calculus problem (see, for example, Slavit 1997, Dreyfus 1999).

Presently teaching is organized as follows. First, the teacher does a standard task in the classroom with his/her students, using a variety of representatives of the target concept (here – functions). Next, students are supposed to do a similar task, with other emblematic representations of the same concept. Students are expected to interpret the representatives used (graphs, tables of numbers, formulae . . .) in the same way as the teacher, that is, as representatives of functions and of their properties. This presentation is supposed to be more 'intuitive' than a formal one. But in fact, it does not bring out the fundamental mathematical knowledge. In doing this work, students indeed cannot learn or imagine what are the functions (or classes of functions) that are bounded; what is the use of this property in the mathematical organization, i.e., why it is useful to study functions; how it is possible to distinguish this property from other properties connected with order (extremes, increase, etc.); what are the functions (or classes of functions) that are not bounded; what is the opposite of being bounded, i.e. if the property 'p' is known, how can the property 'not p' be formulated? No further work is ever done about these questions. In other words, this ostensive way of teaching does not lead to real work on mathematical statements: it is a specificity of mathematical statements that they allow us to know what properties they determine, what mathematical objects satisfy these properties and what are those that do not satisfy the properties. But to do that we need tools to validate a property, otherwise we cannot do anything with it.

2. Settings and representatives

In the standard approach, graphs are used because they are seen as an easy way to show functions, that is, as 'good' representatives of functions: good for teaching, of course, for presenting the concept of function, saving time and avoiding the tediousness of calculation. It is expected that students can see functions and their properties through graphs. Yet, we have noticed that the usual handling of graphs does not seem appropriate for the pursued aims. But are there any other possibilities, in the same setting or in a different one? And how can each representative open the way to the concept? What is it possible to do to link various representatives in different settings? The settings at our disposal when working about functions are the following:

– numerical: tables of values;
– algebraic: formulae, equations;
– geometric: variable geometric magnitudes;
– graphic: straight lines, curves, axes of coordinates;
– formal: symbols such as f, f^{-1}, fog, f(x), . . .
– analytic: with symbols such as∞, or related to orders of magnitude; it is used for heuristic purposes but not for validation at upper secondary school.

These settings have different properties for mathematical work with functions. Two representatives in some setting do not allow the same validation for the same problems and do not exhibit the same properties of the function. It is important, first, to know the properties of each setting, in terms of how they are partial to the objects they represent, and then to make an inventory of the tasks they allow to organize for students. Teaching depends also on the previous knowledge built by students in each setting.

When a review of literature is done about the subject matter, several researches and developments are found especially related to the functionality of graphical representations in mathematical study and pupils' conceptions of these representations. Some researches have been especially concerned with the characteristics of various representation registers and treatment that is associated with *semiotics* which means the cognitive and didactic problems posed by the conversion between these registers, in particular, between graphical representations (Duval 1993, 1995, Bosch et Chevallard 1999, Lacasta 1995).

Other researches have targeted the development or use of specific software or specific programming languages (Gray et Tall 1994, Schwarz et Dreyfus 1995, Moschkovich, Schoenfeld et Arcavi,1993; Chauvat 1999). On the other hand, some other researches have worked on visualization which means the interpretation and manipulation of images in particular by focusing on the difficulty of interpreting graphs of functions (Eisenberg et Dreyfus 1994, Bloch 2000). According to these authors, rather than just being a source of errors for students, the visual aspects of a graph of function plays an important role in understanding the concepts of function and graph and in their learning. However the ability to interpret and give meaning to the properties of the graph is not innate but must be developed in an appropriate way.

3. Presentation of the questionnaire

In the planned situation, students have graphs at their disposal, and they are asked to build graphs of functions satisfying some constraints formulated in the instructions, like a fixed value or an inequality to be satisfied on an interval. I shall express graphs of functions using CGR (Cartesian Graph Representation). Students are also asked to justify that the CGR – either a given CGR or CGR they would draw – is consistent as a representative of the function, and that the properties of the graph are conform to those of the function.

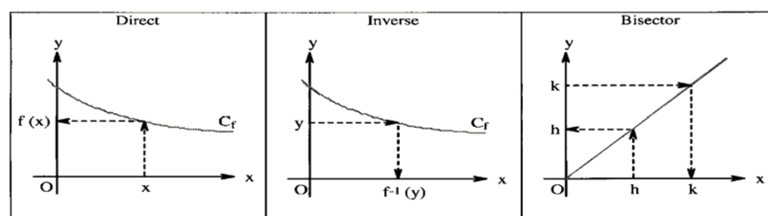

Figure 1. Three kinds of paths.

To do these tasks, students can use what is called a path (see Figure 1). A direct path starts at a point S (x, 0), follows the directions of the axis (first of the y-axis, and then of the x-axis) and goes through the corner C (x, f(x)) to the end point: E (0, f(x)). An inverse path starts from the yaxis. A bisector path starts from the x-axis and is relative to the identity function. This path is very useful in constructing inverse and compound functions. The paths can be used: 1) to justify that a curve represents a function, 2) to build new curves from others (for example sum or product or composite of two functions, or inverse of a function). Paths are defined by the teacher in the first phase of the situation, as tools for verifying if a graph is a representative of a function, and for finding an image or an antecedent of a value under the function.

Phase 2 of the situation

The aim of this phase is to let students produce graphs of functions, and interpret them in relation to well known properties of functions (like in creasing, or having given values on given points). This phase leads to the first family of situations: situations of graphs production.

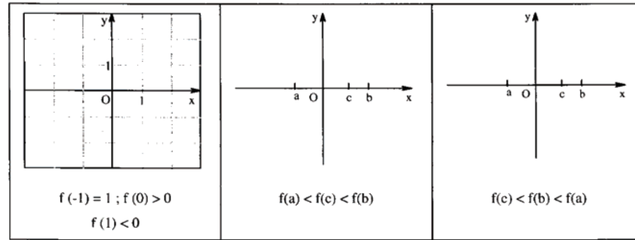

Figure 2. Examples of tasks in Phase 1(Draw a graph of a function f satisfying the given conditions)

The experiment showed that students may not be used to drawing graphs of 'arbitrary' functions, and that they may need some time to familiarize themselves with the idea. If paths are to be introduced, time is also necessary for the students to get used to them, and become convinced that they are convenient tools inmaking conjectures and prove statements about functions. Observations provide some evidence to the effect that exercises proposed in this first phase were quite useful in helping students to become familiar with all sorts of function graphs, to dare drawing a graph, and to make sure it corresponds to the required function.

Phase 3: Drawing graphs of sums and products of functions

The instructions are given for the product, but the same work can be done with the sum. (see Figure 3)

1. Draw the graph of the product function of f and g, using the values a,b, c, d, e.
2. What rules can you formulate about the graph of $f \times g$ relative to the graphs of f and g?
3. What could you say about the product of a constant function and linear
1. function ? of two linear functions ?

What kind of work is expected on these questions? It is first related to students' knowledge about numbers: sums and products of numbers, and their properties.

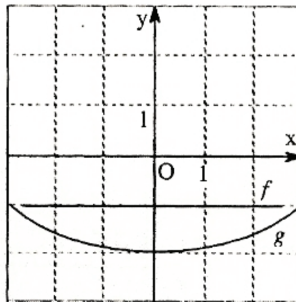

Figure 3. A graph drawn on squared paper to construct the product of the represented functions.

Phase 4. The inverse of a function and composition of functions

To deepen the formal control that students can gain by working within this approach, it is interesting to go on to work with inverse functions and composition of functions. Another aim is to make students work on functions as objects, as it is clear that both the questions and the notations imply a global view of function as an object. First students have to build the target function with a heuristic method, by drawing points, and this work can be done with paths, especially the bisector path. But very soon the questions focus on the existence of the inverse, or its properties, which both require looking at the function as an object in itself.

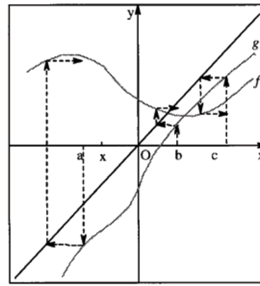

Figure 4. Composition of functions.

4. Method

The aim of this paper was to show how it is possible to construct an interesting situation with a theoretical control. But it is important, of course, that the situation achieves some success in practice. A whole class of 17-year-old science students followed the approach in 2009 (35 students). At the end of the year, a test was administered to four classes of students in this profile, including the experimental one (140 answers). Three questions of the test were taken from Schwarz and Dreyfus (1995), and aimed at probing students' ability to deal with the ambiguity of representatives of functions: given some aligned points of a graph, could it be the graph of a single linear function, for example, or of any arbitrary function whose CGR contains the given points? Or, given a CGR with asymptotes, and four different formulae, what could be the equation of the function? Yet another question was to build a new function (actually, a graph) with three 'pieces' of graphs; if it was possible to put them together to make a new graph, and express the function with one formula or more. Three other questions tested the common contract about functions, and these tasks were quite traditional at that level, such as read the roots (xintercepts) of a function from its CGR, or describe the variations of a given function.

5. Results and Discussion

We read the students' responses trying to analyze their procedures. The results of the test indicated that, in the experimental class, knowledge about arbitrary functions, links between settings and CGR had been rather well understood and mastered. Students were well able to change from a graph to an equation, to concatenate three bits of different representations to recognize a function, while students from other classes were easily destabilized by such questions, saying there were *three* functions and not *one*. They could also pass from a local to a global point of view without difficulty, speaking in the same item of a point of the graph and of the reciprocal function in graphic or algebraic terms, for instance. One advantage, in any case, of the experimental approach was that it allowed the students to encounter many more different representations of a larger variety of functions than the classic organization usually does. Students got used to the variety, which was one of the aims of the situation.

A more important observation was that students in the experimental class showed more irregular success than the others, but they tried to solve more questions. It seemed that knowledge in this class was more dispersed, which was also confirmed by the fact that in the non experimental classes we could distinguish between papers of students who usually succeeded well in mathematics from the papers of the other ones, while this was not pertinent for the experimental class because all students attempted heuristic work: they were trying to answer the questions with various means and made some mistakes, but none of them was helpless and unable to answer.

References

Dreyfus, T. (1999). Why Johnny can't prove, *Educational Studies in Mathematics* 38, 85–109.

Slavit, D.(1997). An alternate route to the reification of function, Educational Studies in Mathematics 33–3, 259–281.

Brousseau, G.(1997). *Theory of Didactical Situations in Mathematics*, Edited and translated by Nicolas Balacheff, Martin Cooper, Rosamund Sutherland and VirginiaWarfield, Mathematics Education Library, Kluwer Academic Publishers, The Netherlands.

Duval, R. (1993). Registres de représentations sémiotiques et fonctionnement cognitif de la pensée, *Annales de Didactique et de Sciences Cognitives*, Vol. 5, pp. 37-65.

Duval, R. (1995) *Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels*, Peter Lang, Bern.

Bosch, M. et Chevallard, Y. (1999) La sensibilité de l'activité mathématique aux ostensifs, *Recherches en Didactique des Mathématiques*, Vol. 19.1, pp. 77-123, La Pensée Sauvage, Grenoble.

Lacasta, E. (1995). Les graphiques cartésiens de fonctions dans l'enseignement secondaire des mathématiques : illusions et contrôles, Thèse, Université Bordeaux I.

Gray, E. M. et Tall, D. O. (1994). Duality, ambiguity and flexibility : a proceptual view of simple arithmetic, Journal for Research in Mathematics Education n. 26, pp. 115-141.

Schwarz, B. et Dreyfus, T. (1995). New action upon old objects: a new ontological perspective on functions, Educational Studies in Mathematics 29, pp. 259-291, Kluwer Academic Publishers.

Moschkovich, J., Schoenfeld, A. H. et Arcavi, A (1993). Aspects of understanding : on multiple perspectives and representations of linear relations and connections among them, in T. A. Romberg, E. Fennema et T. P. Carpenter (éds.), Integrating research on the graphical representation of function, pp.69-100, Lawrence Erlbaun Associated, Hillsdale-NJ.

Chauvat, G. (1999). Courbes et fonctions au collège*, Petit x* 51, pp. 23-44.

Eisenberg, T. et Dreyfus, T. (1994). On understanding how students learn to visualize function transformations, in E. Dubinsky, A. Schoenfeld et J. Kaput (éds.), Research in Collegiate Mathematics Education I, *CBMS Issues in Mathematics Education*, 4 , pp. 45-68.

Bloch, I. (2000) L'enseignement de l'analyse à la charnière lycée / université Savoirs, connaissances et conditions relatives à la validation, *Thèse université Bordeaux I.*