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ON QUASI n-IDEALS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with a nonzero identity. In this study, we present a new class of ideals lying properly between the class of n-ideals and the class of $(2,n)$-ideals. A proper ideal I of R is said to be a quasi n-ideal if $\sqrt{I}$ is an n-ideal of R. Many examples and results are given to disclose the relations between this new concept and others that already exist, namely, the n-ideals, the quasi primary ideals, the $(2,n)$-ideals and the pr-ideals. Moreover, we use the quasi n-ideals to characterize some kind of rings. Finally, we investigate quasi n-ideals under various contexts of constructions such as direct product, power series, idealization, and amalgamation of a ring along an ideal.

Keywords: n-ideal; quasi n-ideal; $(2,n)$-ideal

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1. Introduction

In this article, we focus only on commutative rings with a nonzero identity and nonzero unital modules. Let R always denote such a ring and M denote such an R-module. The principal ideal generated by $a \in R$ is denoted by (a). Also the radical of I is defined as $\sqrt{I} := \{r \in R\colon r^k \in I \text{ for some } k \in \mathbb{N}\}$. In particular, $\sqrt{0} := \{r \in R\colon r^k = 0 \text{ for some } k \in \mathbb{N}\}$ is the set of all nilpotent elements of R. For a subset S of R and an ideal I of R, we define $(I :_R S) := \{r \in R\colon rS \subseteq I\}$. In particular, we use $\operatorname{Ann}(S)$ instead of $(0 :_R S)$. Moreover, for any $a \in R$ and any ideal I of R we use $(I : a)$ and $\operatorname{Ann}(a)$ to denote $(I :_R \{a\})$ and $\operatorname{Ann}(\{a\})$, respectively. An element $a \in R$ is called a *regular* (or *zerodivisor*) *element* if $\operatorname{Ann}(a) = (0)$ (or $\operatorname{Ann}(a) \neq (0)$). The set of all regular (or zerodivisor) elements of R is denoted by $r(R)$ (or $\operatorname{zd}(R)$).

In 2015, Mohamadian presented the notion of r-ideals in commutative rings with a nonzero identity as follows: an ideal I of a commutative ring with identity R

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is called *r-ideal* (or *pr-ideal*) if $ab \in I$ and a is regular element implies that $b \in I$ (or $b^n \in I$, for some natural number n) for each $a, b \in R$, see [9]. In 2017, the authors introduced the concept of n-ideals of a commutative ring with a nonzero identity R as follows: let I be a proper ideal of R. If whenever $ab \in I$ and $a \notin \sqrt{0}$, then $b \in I$, we say I is an n-ideal of R, see [11]. It is clear that every n-ideal is an r-ideal since $\sqrt{0} \subseteq \mathrm{zd}(R)$. In [10], Tamekkante and Bouba introduced a generalization of the class of n-ideals called $(2,n)$-*ideals*. A proper ideal I of R is said to be a $(2,n)$-ideal if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{0}$ or $bc \in \sqrt{0}$. They proved that an ideal I of R is a $(2,n)$-ideal if and only if I is 2-absorbing primary ideal and $I \subseteq \sqrt{0}$, see [10], Theorem 2.4.

On the other hand, the concept of quasi primary ideals in commutative rings was introduced and investigated by Fuchs in [7]. The author called an ideal I of R as a quasi primary ideal if $\sqrt{I}$ is a prime ideal. In [12], the notion of 2-absorbing quasi primary ideals is introduced as follows: a proper ideal I of R is 2-absorbing quasi primary if $\sqrt{I}$ is a 2-absorbing ideal of R.

In this paper, our aim is to introduce a generalization of the concepts of n-ideals in commutative rings with a nonzero identity. For this, firstly with Definition 2.1, we introduce the concept of quasi n-ideals of R as follows: let I be a proper ideal of R, if $\sqrt{I}$ is an n-ideal of R, then I is said to be a quasi n-ideal of R. In addition to giving main properties of quasi n-ideals, we give a characterization for them, see Theorem 2.1. At this point, we observe that quasi n-ideals exist in a ring R only when $\sqrt{0}$ is a prime ideal. On the other hand, we have the following figure with nonreversible arrows, see Examples 2.1 and 2.2

$$n\text{-ideal} \to \text{quasi } n\text{-ideal} \to (2,n)\text{-ideal}.$$

Moreover, we study the rings over which every proper ideal is a quasi n-ideal. Finally, we give an idea about quasi n-ideals of the localization of rings, the power series rings, the trivial ring extensions and the amalgamated of rings along an ideal.

2. Quasi n-ideals of commutative rings

Definition 2.1. Let R be a commutative ring with a nonzero identity and I be a proper ideal of R. If $\sqrt{I}$ is an n-ideal of R, then I is said to be a quasi n-ideal of R.

It can be easily seen that every n-ideal of a ring R is a quasi n-ideal. But the converse is not true. For this, we can give the following example, which is a *quasi n-ideal* but not *n-ideal*.

Example 2.1. Let $R = \mathbb{Z}[X,Y]/(Y^4)$. For $x = X + (Y^4)$ and $y = Y + (Y^4)$, consider $I = (xy, y^2)$. It is clear that $\sqrt{0_R} = (y)$. Since $(x+y)y \in I$ but $x+y \notin \sqrt{0_R}$ and $y \notin I$, we get that I is not an n-ideal of R. On the other hand, $\sqrt{0_R} = (y)$ is a prime ideal of R. By [11], Corollary 2.9 (i), we say $\sqrt{0_R}$ is an n-ideal. Moreover, $\sqrt{I} = \sqrt{0_R}$ as $I \subseteq \sqrt{0_R}$. Hence, $\sqrt{I}$ is an n-ideal, i.e., I is a quasi n-ideal of R.

The following theorem provides necessary and sufficient conditions for a proper ideal to be a quasi n-ideal.

Theorem 2.1. *Let R be a ring and I be a proper ideal of R. Then the following statements are equivalent:*
(1) *I is a quasi n-ideal.*
(2) *I is a quasi primary ideal and $I \subseteq \sqrt{0}$.*
(3) *For two ideals I_1, I_2 of R, if $I_1 I_2 \subseteq \sqrt{I}$ and $I_1 \cap (R - \sqrt{0}) \neq \emptyset$, then $I_2 \subseteq \sqrt{I}$.*

Proof. (1) $\Rightarrow$ (2): Let I be a quasi n-ideal of R. Suppose that $I \nsubseteq \sqrt{0}$, then we can pick an element $a \in I - \sqrt{0}$ and we consider $a \cdot 1 \in I \subseteq \sqrt{I}$. As $\sqrt{I}$ is an n-ideal and $a \notin \sqrt{0}$, we must have $1 \in \sqrt{I}$, a contradiction. Thus, $I \subseteq \sqrt{0}$ and hence $\sqrt{I} = \sqrt{0}$ is a prime ideal.

(2) $\Rightarrow$ (3): Let $I_1 I_2 \subseteq \sqrt{I}$ and $I_1 \cap (R - \sqrt{0}) \neq \emptyset$ for two ideals I_1, I_2 of R. There exists $a \in I_1 - \sqrt{0}$. Then we say $aI_2 \subseteq \sqrt{I}$, i.e., $I_2 \subseteq (\sqrt{I} : a)$. By assumption, we have $I_2 \subseteq (\sqrt{I} : a) = \sqrt{I}$, as needed.

(3) $\Rightarrow$ (1): Choose $a, b \in R$ such that $ab \in \sqrt{I}$ and $a \notin \sqrt{0}$. Consider $I_1 = (a)$ and $I_2 = (b)$. By our hypothesis, $(b) \subseteq \sqrt{I}$, that is, $b \in \sqrt{I}$. □

Corollary 2.1. *Let R be a ring.*
(1) *(0) is a quasi n-ideal of R if and only if $\sqrt{0}$ is a prime ideal of R.*
(2) *Let R be a reduced ring. Then R is an integral domain if and only if (0) is the only quasi n-ideal of R.*

Proof. (1) It is clear.

(2) Suppose that R is an integral domain, then as $\sqrt{0} = (0)$ is prime, (0) is a quasi n-ideal by (1). On the other hand, if I is a quasi n-ideal of R, then $I \subseteq \sqrt{0} = (0)$ by Theorem 2.1. For the converse, one can see that if (0) is a quasi n-ideal, then R is an integral domain. □

Remark 2.1. It should not be surprising that a ring R does not have a quasi n-ideal. For instance, $R = \mathbb{Z}_6$ has no quasi n-ideals. Indeed, let I be a quasi n-ideal. By Theorem 2.1, we say $I \subseteq \sqrt{\bar{0}} = (\bar{0})$, so $I = (\bar{0})$. Moreover, since $\bar{2} \cdot \bar{3} \in \sqrt{\bar{0}}$, $\bar{2} \notin \sqrt{\bar{0}}$ and $\bar{3} \notin \sqrt{\bar{0}}$, we conclude $(\bar{0})$ is not a quasi n-ideal.

As an immediate consequence of Theorem 2.1, we give a characterization of rings that admit quasi n-ideals.

Corollary 2.2. *Let R be a ring. There is a quasi n-ideal of R if and only if $\sqrt{0}$ is a prime ideal of R.*

The following proposition shows that the class of quasi n-ideals is a subclass of $(2,n)$-ideals.

Proposition 2.1. *Every quasi n-ideal of a ring R is a $(2,n)$-ideal.*

Proof. Let I be a quasi n-ideal, then $\sqrt{I} = \sqrt{0}$ is a prime. By Theorem 2.8 of [2], I is a 2-absorbing primary ideal and hence I is a $(2,n)$-ideal of R by Theorem 2.4 of [10], as needed. □

The following example proves that the converse of the previous proposition is not true, in general.

Example 2.2. Consider the ideal $I := (\bar{0})$ of the ring $R = \mathbb{Z}_6$. Then, by Example 2.3 of [10], I is a $(2,n)$-ideal. However, R has no quasi n-ideals by Remark 2.1.

Note that similarly to the concept of quasi n-ideals, we can define the concept of "quasi r-ideals" of R as follows: if $\sqrt{I}$ is an r-ideal, we say I is a quasi r-ideal of R. On the other hand, Mohamadian proved that I is a pr-ideal if and only if $\sqrt{I}$ is an r-ideal, see [9], Proposition 2.16. Thus, we conclude the two concepts, quasi r-ideals and pr-ideals, are identical. In this study for this concept, we will use "quasi r-ideals" to catch the similarity of the concept of "quasi n-ideals".

Proposition 2.2. *Let I be a proper ideal of R. If I is a quasi n-ideal, then I is a quasi r-ideal.*

Proof. Suppose that I is a quasi n-ideal, so $\sqrt{I}$ is an n-ideal. Since every n-ideal is an r-ideal, $\sqrt{I}$ is also an r-ideal. It is done. □

As $\sqrt{0} \subseteq \mathrm{zd}(R)$, one can easily show that if (0) is a primary ideal of R, then $\sqrt{0} = \mathrm{zd}(R)$. Thus, the n-ideals and r-ideals are identical in any commutative ring such that (0) is primary. By the help of the same argument, one can see the following remark.

Remark 2.2. The quasi n-ideals and quasi r-ideals are identical in any commutative ring, where (0) is a primary ideal.

Proposition 2.3. *The intersection of any family of quasi n-ideals of R is a quasi n-ideal of R.*

Proof. Let $\{I_\alpha\}_{\alpha\in\Delta}$ be a family of quasi n-ideals of R. We will show that $\sqrt{\bigcap\limits_{\alpha\in\Delta} I_\alpha}$ is an n-ideal of R. As I_α is a quasi n-ideal of R, we know $\sqrt{I_\alpha}$ is an n-ideal of R. Thus, $\sqrt{\bigcap\limits_{\alpha\in\Delta} I_\alpha} = \bigcap\limits_{\alpha\in\Delta} \sqrt{I_\alpha}$ implies that $\sqrt{\bigcap\limits_{\alpha\in\Delta} I_\alpha}$ is an n-ideal by [11], Proposition 2.4. □

Proposition 2.4. *Let R be a ring. If I is a proper ideal of R and P is a prime ideal of R such that $I \cap P$ is a quasi n-ideal, then either I is a quasi n-ideal or $P = \sqrt{0}$.*

Proof. If $I \subseteq P$, then $I = I \cap P$ is a quasi n-ideal. Now, we suppose that $I \nsubseteq P$ and take $a, b \in R$ with $ab \in P$ and $a \notin \sqrt{0}$. By hypothesis, we can pick an element $x \in I - P$, hence $abx \in I \cap P$. The fact that $I \cap P$ is a quasi n-ideal and $a \notin \sqrt{0}$ implies that $bx \in \sqrt{I \cap P}$. Thus, $b \in P$ and so P is an n-ideal of R, which shows that $P = \sqrt{0}$. This completes the proof. □

Theorem 2.2. *Let R be a ring and $I_1, \ldots, I_n$ be ideals of R, where $n \geqslant 2$. If I_i and I_j are co-primes for each $i \neq j$, then $\bigcap\limits_{k=1}^{n} I_k$ is not a quasi-n-ideal of R.*

Proof. Suppose that $\bigcap\limits_{k=1}^{n} I_k$ is a quasi-n-ideal. We will prove that I_j is a quasi n-ideal for each j. Let $a, b \in R$ such that $ab \in \sqrt{I_j}$ and $a \notin \sqrt{0}$. Since I_j and I_k are co-primes for each $k \neq j$, we have that I_j and $\bigcap\limits_{k \neq j} I_k$ must be co-primes. Then there exist $x \in I_j$ and $y \in \bigcap\limits_{k \neq j} I_k$ such that $1 = x + y$. Thus, $aby \in \sqrt{\bigcap\limits_{k=1}^{n} I_k}$, which implies that $b^m y^m \in \bigcap\limits_{k=1}^{n} I_k$ for a positive integer m. So, $b^m y^{m-1} = b^m y^{m-1} x + b^m y^m \in I_j$. By induction, we can prove that $b \in \sqrt{I_j}$. It follows that I_j is a quasi n-ideal. By Theorem 2.1, we obtain $1 \in \sqrt{0}$, a desired contradiction. □

Proposition 2.5. *Let R be a ring and S be a nonempty subset of R. If I is a quasi n-ideal of R with $S \nsubseteq \sqrt{I}$, then $(I : S)$ is a quasi n-ideal of R.*

Proof. It suffices to show that $\sqrt{I} \subseteq \sqrt{(I : S)} \subseteq (\sqrt{I} : S) = \sqrt{I}$. This, in turn, follows from the fact that I is a quasi n-ideal of R and $S \nsubseteq \sqrt{0}$, as needed. □

Let R be a ring. We call a quasi n-ideal I of R a maximal quasi n-ideal if there is no quasi n-ideal which contains I properly. We observe that $\sqrt{0}$ is the unique maximal quasi n-ideal in a ring R.

Theorem 2.3. *Let R be a ring. If I is a maximal quasi n-ideal of R, then $I = \sqrt{0}$.*

Proof. Let I be a maximal quasi n-ideal. We claim that I is an n-ideal. Choose $a, b \in R$ such that $ab \in I$ and $a \notin \sqrt{0}$. Then, by Proposition 2.5, $(I : a)$ is a quasi n-ideal of R. Since I is a maximal quasi n-ideal of R, it must be $(I : a) = I$, hence $b \in I$. Consequently, I is a maximal n-ideal, that is, $I = \sqrt{0}$ by [11], Theorem 2.11. □

Proposition 2.6. *Let R be a zero dimensional ring. Then R admits a quasi n-ideal if and only if $(R, \sqrt{0})$ is a local ring.*

Proof. Let R be a zero dimensional ring which admits a quasi n-ideal. Then, by Theorem 2.2, $\sqrt{0}$ is a prime ideal. Moreover, if P is a prime ideal of R, then $\sqrt{0} = P$ by maximality of $\sqrt{0}$. Hence, R is a local ring. For the converse, it can be easily seen that if $(R, \sqrt{0})$ is a local ring, then $\sqrt{0}$ is the unique prime ideal of R. Thus, every proper ideal of R is an n-ideal (so a quasi n-ideal), as desired. □

Corollary 2.3. *Let R be a ring. Then the following statements are equivalent:*
(1) *R is a field.*
(2) *R is a Boolean ring and (0) is a quasi n-ideal.*
(3) *R is a von Neumann regular ring and (0) is a quasi n-ideal.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.
(3) $\Rightarrow$ (1): Assume that R is a von Neumann regular ring and (0) is a quasi n-ideal. So, R is a reduced ring and is zero dimensional. Hence, R is a field by Proposition 2.6. □

Corollary 2.4 ([11], Proposition 3.1)**.** *Let m be a positive integer. Then the following statements are equivalent:*
(1) *$\mathbb{Z}_m$ has a quasi n-ideal.*
(2) *$\mathbb{Z}_m$ has an n-ideal.*
(3) *$m = p^k$ for some $k \in \mathbb{Z}^+$, where p is a prime number.*

According to [3], a ring R is called an UN-*ring* if every nonunit element a of R is a product of a unit and a nilpotent element.

Proposition 2.7. *Let R be a ring. Then the following statements are equivalent:*
(1) *R is an UN-ring.*
(2) *Every proper principal ideal of R is a quasi n-ideal.*
(3) *Every proper ideal of R is a quasi n-ideal.*

Proof. (1) $\Rightarrow$ (2) follows from Proposition 2.25 of [11].
(2) $\Rightarrow$ (3): Let I be a proper ideal of R. Assume that $ab \in I$ for some elements $a \in R - \sqrt{0}$ and $b \in R$. Then, by assumption, $b \in \sqrt{(ab)} \subseteq \sqrt{I}$. Thus, I is a quasi n-ideal.
(3) $\Rightarrow$ (1): Let P be a prime ideal of R, then P is a quasi n-ideal and so $P = \sqrt{0}$, which implies that $\sqrt{0}$ is the unique prime ideal of R. It follows that R is an UN-ring by [3], Proposition 2 (3). □

Theorem 2.4. *Let $I, I_1, I_2, \ldots, I_m$ be ideals of R such that $I \subseteq I_1 \cup I_2 \cup \ldots \cup I_m$. If I_i is a quasi n-ideal and the others have nonnilpotent elements such that $I \nsubseteq \bigcup_{j\neq i} I_j$, then $I \subseteq \sqrt{I_i}$.*

P r o o f. Without loss of generality, let $i = 1$. By our hypothesis, $I \nsubseteq I_2 \cup \ldots \cup I_m$. Thus, there is $x \in I$ but $x \notin I_2 \cup \ldots \cup I_m$. This means that $x \in I_1$. Now, we claim $I \cap \bigcap_{k=2}^{m} I_k \subseteq I_1$. Choose $\alpha \in I \cap \bigcap_{k=2}^{m} I_k$. Note that $x \notin I_k$ and $\alpha \in I_k$ for $k = 2, \ldots, m$. This implies $x + \alpha \notin I_k$. Thus, $x + \alpha \in I - \bigcup_{j=2}^{m} I_j$, which implies $x + \alpha \in I_1$. Then we conlclude $\alpha \in I_1$. On the other hand, by Theorem 2.2, $\sqrt{0}$ is a prime ideal of R. Hence, $R - \sqrt{0}$ is a multiplicatively closed subset of R, so the product of nonnilpotent elements is a nonnilpotent element. This means that $\prod_{k=2}^{m} I_k \cap (R - \sqrt{0}) \neq \emptyset$. Now, note that $I\Big(\prod_{k=2}^{m} I_k\Big) \subseteq I \cap \Big(\prod_{k=2}^{m} I_k\Big) \subseteq I_1$. Consider $I\Big(\prod_{k=2}^{m} I_k\Big) \subseteq \sqrt{I_1}$ and $\prod_{k=2}^{m} I_k \cap (R - \sqrt{0}) \neq \emptyset$. By Theorem 2.1, we conclude $I \subseteq \sqrt{I_1}$. □

Proposition 2.8. *Let R be a ring and J be an ideal of R such that $J \cap (R - \sqrt{0}) \neq \emptyset$. Then:*

(1) *If I_1 and I_2 are two quasi n-ideals of R such that $\sqrt{I_1}J = \sqrt{I_2}J$, then $\sqrt{I_1} = \sqrt{I_2}$.*
(2) *If $\sqrt{I}J$ is a quasi n-ideal of R, then $\sqrt{IJ} = \sqrt{I}$.*

P r o o f. (1) Consider $\sqrt{I_1}J \subseteq \sqrt{I_2}$. By Theorem 2.1, $\sqrt{I_1} \subseteq \sqrt{I_2}$. Similarly, we conclude $\sqrt{I_2} \subseteq \sqrt{I_1}$.

(2) By the assumption, $\sqrt{I}J$ is a quasi n-ideal and also consider $\sqrt{I}J \subseteq \sqrt{\sqrt{I}J}$. By Theorem 2.1, we have $\sqrt{I} \subseteq \sqrt{\sqrt{I}J}$. As $\sqrt{\sqrt{I}J} = \sqrt{\sqrt{I} \cap \sqrt{J}} = \sqrt{IJ}$, we obtain $\sqrt{I} \subseteq \sqrt{IJ}$, as required. □

Theorem 2.5. *Let $f\colon R \to S$ be a homomorphism. Then:*

(1) *Suppose f is an epimorphism. If I is a quasi n-ideal of R such that* $\mathrm{Ker}(f) \subseteq I$, *then $f(I)$ is a quasi n-ideal of S.*
(2) *Suppose f is a monomorphism. If J is a quasi n-ideal of S, then $f^{-1}(J)$ is a quasi n-ideal of S.*

P r o o f. (1) Choose $x, y \in S$ such that $xy \in \sqrt{f(I)}$ and $x \notin \sqrt{0_S}$. Then there are $a, b \in R$ with $x = f(a)$ and $y = f(b)$. It is clear that $f(ab) \in \sqrt{f(I)}$. Also, $\mathrm{Ker}(f) \subseteq I$ implies $ab \in \sqrt{I}$. Note that $a \notin \sqrt{0_R}$ as $x \notin \sqrt{0_S}$. Thus, as I is a quasi n-ideal, we conclude $b \in \sqrt{I}$, that is, $y \in \sqrt{f(I)}$.

(2) Take $a, b \in R$ with $ab \in \sqrt{f^{-1}(J)}$ and $a \notin \sqrt{0_R}$. Then there is $k \in \mathbb{N}$ such that $(ab)^k \in f^{-1}(J)$, that is, $f(ab)^k \in J$. On the other hand, as f is a monomorphism, $a \notin \sqrt{0}$ means $f(a) \notin \sqrt{0_S}$. Then we get $f(a)^k \notin \sqrt{0_S}$. Thus, by hypothesis, we obtain $f(b)^k \in J$, i.e., $b \in \sqrt{f^{-1}(J)}$, which completes the proof. □

Corollary 2.5. *Let I and J be two ideals of R such that $J \subseteq I$.*

(1) *If I is a quasi n-ideal of R, then I/J is a quasi n-ideal of R/J.*

(2) *If I/J is a quasi n-ideal of R/J and $J \subseteq \sqrt{0_R}$, then I is a quasi n-ideal of R.*

(3) *If I/J is a quasi n-ideal of R/J and J is a quasi n-ideal of R, then I is a quasi n-ideal of R.*

Proof. (1) Let $\pi\colon R \to R/J$ be the natural homomorphism. Since $\mathrm{Ker}(f) = J \subseteq I$, by Theorem 2.5, we say $\pi(I) = I/J$ is a quasi n-ideal of R/J.

(2) Choose $a, b \in R$ with $ab \in \sqrt{I}$ and $a \notin \sqrt{0_R}$. This implies that $(a+J)(b+J) \in \sqrt{I}/J = \sqrt{I/J}$. Also, note that $a + J \notin \sqrt{0_{R/J}}$, otherwise it would contradict with $a \notin \sqrt{0_R}$ since $J \subseteq \sqrt{0_R}$. Hence, $b+J \in \sqrt{I/J}$, so $b \in \sqrt{I}$. Consequently, I is a quasi n-ideal of R.

(3) Since J is a quasi n-ideal, by Theorem 2.1, $J \subseteq \sqrt{0_R}$. Thus, with item (2), it is done. $\square$

Corollary 2.6. *Let S be a subring of R. If I is a quasi n-ideal of R such that $S \nsubseteq I$, then $I \cap S$ is a quasi n-ideal of S.*

Proof. Let $i\colon S \to R$ be the injection homomorphism. Clearly, $i^{-1}(I) = I \cap S$. By Theorem 2.5, $I \cap S$ is a quasi n-ideal of S. $\square$

Proposition 2.9. *Let R be a ring and S be a multiplicatively closed subset of R. Then the following statements hold:*

(1) *If I is a quasi n-ideal of R, then $S^{-1}I$ is a quasi n-ideal of $S^{-1}R$.*

(2) *Suppose that $S = r(R)$. If J is a quasi n-ideal of $S^{-1}R$, then J^c is a quasi n-ideal of R.*

Proof. (1) Choose $a/s, b/t \in S^{-1}R$ such that $(a/s)(b/t) \in \sqrt{S^{-1}I} = S^{-1}\sqrt{I}$ and $a/s \notin \sqrt{0_{S^{-1}R}}$. Then we have $uab \in \sqrt{I}$ for some $u \in S$. Also, $a/s \notin \sqrt{0_{S^{-1}R}}$ implies that $a \notin \sqrt{0_R}$. Since I is a quasi n-ideal of R, we conclude $ub \in \sqrt{I}$, i.e., $b/t = ub/(ut) \in S^{-1}\sqrt{I}$.

(2) Take $a, b \in R$ with $ab \in \sqrt{J^c}$ and $a \notin \sqrt{0_R}$. Then $(ab)^k \in J^c$ for some $k \in \mathbb{N}$. Consider $(a/1)^k(b/1)^k \in J$. Now, we claim $(a/1)^k \notin \sqrt{0_{S^{-1}R}}$. Let $(a/1)^k \in \sqrt{0_{S^{-1}R}}$. There exists $t \in \mathbb{N}$ such that $(a/1)^{kt} = 0_{S^{-1}R}$. Thus, for some $u \in S = r(R)$, we have $ua^{kt} = 0_R$. This implies that $a^{kt} \in \mathrm{Ann}(u) = 0_R$, i.e., $a \in \sqrt{0_R}$. This gives us a contradiction. Thus, as J is a quasi n-ideal of $S^{-1}R$, we conclude $(b/1)^k \in J$. Consequently, $b \in \sqrt{J^c}$. $\square$

Theorem 2.6. *Let R and S be two commutative rings. Then $R \times S$ has no quasi n-ideals.*

P r o o f. Let $I \times J$ be a quasi n-ideal of $R \times S$. Then $\sqrt{I \times J} = \sqrt{I} \times \sqrt{J}$ is an n-ideal of $R \times S$. But this result contradicts with Proposition 2.26 of [11]. $\square$

Proposition 2.10. *Let R be a ring and I be an ideal. Then:*

(1) *R has a quasi n-ideal if and only if $R[X]$ has a quasi n-ideal.*

(2) *If $I[X]$ is a quasi n-ideal of $R[X]$, then I is a quasi n-ideal of R.*

(3) *(I, X) is never a quasi n-ideal of $R[X]$.*

P r o o f. (1) Combine Theorem 2.2 with Lemma 3.6 of [10].

(2) Assume that $I[X]$ is a quasi n-ideal of $R[X]$. Then, by Corollary 2.6, $I = I[X] \cap R$ is a quasi n-ideal of R.

(3) It follows from the fact that $\sqrt{(I, X)} \not\subseteq \sqrt{0_{R[X]}}$. $\square$

Recall that an ideal I of a ring is said to be a strong finite type (or an SFT-ideal) if there exist a natural number k and a finitely generated ideal $J \subseteq I$ such that $x^k \in J$ for each $x \in I$.

Proposition 2.11. *Let R be a ring and I be an ideal of R. Then the following statements hold:*

(1) *If $R[[X]]$ admits a quasi n-ideal, then R admits a quasi n-ideal. The converse holds provided that $\sqrt{0_R}$ is an SFT-ideal.*

(2) *If $I[[X]]$ is a quasi n-ideal of $R[[X]]$, then $I[X]$ is a quasi n-ideal of $R[X]$ (so I is a quasi n-ideal of R).*

P r o o f. (1) If $R[[X]]$ has a quasi n-ideal, then $\sqrt{0_R} = \sqrt{0_{R[[X]]}} \cap R$ is an n-ideal of R and so $\sqrt{0_R}$ is a prime ideal of R. For the converse, we assume that $\sqrt{0_R}$ is an SFT-ideal. Then, by [8], Corollary 2.4, $\sqrt{0_{R[[X]]}} = \sqrt{0_R}[[X]]$. On the other hand, since R admits a quasi n-ideal, then $\sqrt{0_{R[[X]]}}$ is a prime ideal, which implies that $R[[X]]$ admits a quasi n-ideal.

(2) Suppose that $I[[X]]$ is a quasi n-ideal of $R[[X]]$, then $I[X] = I[[X]] \cap R[X]$ is a quasi n-ideal by Corollary 2.6. Hence, I is a quasi n-ideal. $\square$

Let R be a commutative ring with a nonzero identity and M be an R-module. Then the idealization $R(+)M = \{(a, m)\colon a \in R,\ m \in M\}$ is a commutative ring with componentwise addition and multiplication $(a, m)(b, n) = (ab, an + bm)$ for each $a, b \in R$ and $m, n \in M$. In addition, if I is an ideal of R and N is a submodule of M, then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$, see [1].

Theorem 2.7. *Let R be a ring and M be an R-module.*

(1) *A proper ideal J of $R(+)M$ is a quasi n-ideal if and only if J_R is a quasi n-ideal of R, where $J_R = \{r \in R\colon (r, m) \in J$ for some $m \in M\}$.*

(2) *I is a quasi n-ideal of R if and only if $I(+)N$ is a quasi n-ideal of $R(+)M$ for each submodule N of M such that $IM \subseteq N$.*

Proof. (1) Let J be a proper ideal of $R(+)M$. It is well known from [1], Theorem 3.2 (3) that $\sqrt{J} = \sqrt{J_R}(+)M$, where $J_R = \{r \in R\colon (r,m) \in J$ for some $m \in M\}$. On the other hand, by Theorem 2.1, J is a quasi n-ideal if and only if $\sqrt{J_R}(+)M = \sqrt{0}(+)M$ is a prime ideal if and only if J_R is a quasi n-ideal of R. It is done.

(2) It follows from (1). □

The following is an example of a quasi n-ideal that is not an n-ideal.

Example 2.3. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{pq}$, where p and q are prime numbers. Then, the zero ideal of $R(+)M$ is a quasi n-ideal which is not an n-ideal. Indeed, $\sqrt{0_{R(+)M}} = 0(+)M$ is prime. However, $(p,0)(0,q) \in (0,0)$ but $(p,0) \notin \sqrt{0_{R(+)M}}$ and $(0,q) \notin (0,0)$.

Let R and S be two rings, J be an ideal of S and $f\colon R \to S$ be a ring homomorphism. In this setting, we can consider the subring of $R \times S$

$$R \bowtie^f J = \{(r, f(r)+j)\colon r \in R \text{ and } j \in J\}$$

called the amalgamation of R with S along J with respect to f. This construction has been first indroduced and studied by D'Anna, Finocchiaro, and Fontana in [6], [4]. In particular, if I is an ideal of R and $\mathrm{id}_R\colon R \to R$ is the identity homomorphism on R, then $R \bowtie J = R \bowtie^{\mathrm{id}_R} J = \{(r, r+j)\colon r \in R \text{ and } j \in J\}$ is the amalgamated duplication of R along J (introduced and studied by D'Anna and Fontana in [5]). For all ideals I of R and ideals K of S, set:

$$\begin{aligned} I \bowtie^f J &= \{(r, f(r)+j)\colon r \in I \text{ and } j \in J\},\\ \overline{K}^f &= \{(r, f(r)+j)\colon r \in R,\ j \in J \text{ and } f(r)+j \in K\}. \end{aligned}$$

Theorem 2.8. *Let R and S be a pair of rings, J be an ideal of S and $f\colon R \to S$ be a ring homomorphism. Let I be an ideal of R and K be an ideal of S. The following statements hold:*

(1) *If $I \bowtie^f J$ is a quasi n-ideal (or n-ideal) of $R \bowtie^f J$, then I is a quasi n-ideal (or n-ideal) of R. The converse is true if $J \subseteq \sqrt{0_S}$.*

(2) *Assume that f is an epimorphism. Then the fact that $\overline{K}^f$ is a quasi n-ideal (or n-ideal) of $R \bowtie^f J$ implies that K is a quasi n-ideal (or n-ideal) of S. The converse holds provided that $f^{-1}(J) \subseteq \sqrt{0_R}$.*

Proof. (1) Assume that $I \bowtie^f J$ is a quasi n-ideal of $R \bowtie^f J$. Let $a, b \in R$ such that $ab \in \sqrt{I}$ and $a \notin \sqrt{0_R}$. Then $(a, f(a))(b, f(b)) \in \sqrt{I \bowtie^f J}$ with $(a, f(a)) \notin \sqrt{0_{R\bowtie^f J}}$. This implies that $(b, f(b)) \in \sqrt{I \bowtie^f J}$ and hence $b \in \sqrt{I}$. Now, we will prove the converse under additional hypothesis that $J \subseteq \sqrt{0_S}$. Suppose that $(a, f(a) + j)(b, f(b) + j') \in \sqrt{I \bowtie^f J}$ for some $(a, f(a) + j) \notin \sqrt{0_{R\bowtie^f J}}$ and $(b, f(b) + j') \in R \bowtie^f J$. By hypothesis, we must have $a \notin \sqrt{0_R}$. Therefore, $b \in \sqrt{I}$ and thus $(b, f(b) + j') \in \sqrt{I \bowtie^f J}$. Similarly, one can prove that if $I \bowtie^f J$ is an n-ideal of $R \bowtie^f J$, then I is an n-ideal of R, and the converse is true if $J \subseteq \sqrt{0_S}$.

(2) Let $x, y \in S$ with $x = f(a)$ and $y = f(b)$. Suppose that $xy \in \sqrt{K}$ and $x \notin \sqrt{0_S}$. So, $(a, f(a))(b, f(b)) \in \sqrt{\overline{K}^f}$ and $(a, f(a)) \notin \sqrt{0_{R\bowtie^f J}}$. Since $\overline{K}^f$ is a quasi n-ideal, we then have $(b, f(b)) \in \sqrt{\overline{K}^f}$ and so $y = f(b) \in \sqrt{K}$. For the converse, suppose that K is a quasi n-ideal of S and $f^{-1}(J) \subseteq \sqrt{0_R}$. Let $(a, f(a) + j), (b, f(b) + j') \in R \bowtie^f J$ such that $(a, f(a) + j)(b, f(b) + j') \in \sqrt{\overline{K}^f}$ and $(a, f(a) + j) \notin \sqrt{0_{R\bowtie^f J}}$. Then $(f(a)+j)(f(b)+j') \in \sqrt{K}$. The fact that $(a, f(a)+j) \notin \sqrt{0_{R\bowtie^f J}}$ ensures that $f(a)+ j \notin \sqrt{0_S}$. Suppose, on the contrary, that $f(a) + j \in \sqrt{0_S}$. As f is an epimorphism, then there exists $c \in R$ such that $f(c) = j$. It is obvious that $c \in \sqrt{0_R}$ and hence $j \in \sqrt{0_S}$, which proves that $a^m \in \operatorname{Ker}(f)$ for a positive integer m. Moreover, $a \in \sqrt{0_R}$ since $f^{-1}(J) \subseteq \sqrt{0_R}$, that is, $(a, f(a) + j) \in \sqrt{0_{R\bowtie^f J}}$, a contradiction. We conclude that $(f(b) + j') \in \sqrt{K}$ since K is a quasi n-ideal of S. Thus, $\overline{K}^f$ is a quasi n-ideal of $R \bowtie^f J$. Similarly, one can prove that if $\overline{K}^f$ is an n-ideal of $R \bowtie^f J$, then K is an n-ideal of S, and the converse is true in the case, where $f^{-1}(J) \subseteq \sqrt{0_R}$. This completes the proof. $\square$

Corollary 2.7. *Let R be a ring and let I and J be ideals of R.*

(1) *If $I \bowtie J$ is a quasi n-ideal (or n-ideal) of $R \bowtie J$, then I is a quasi n-ideal (or n-ideal) of R. The converse is true if $J \subseteq \sqrt{0_R}$.*

(2) *If $\overline{I} := \{(a, a + i)\colon\ a \in R,\ j \in J \text{ and } a + j \in I\}$ is a quasi n-ideal (or n-ideal) of $R \bowtie J$, then I is a quasi n-ideal (or n-ideal) of R. The converse is true if $J \subseteq \sqrt{0_R}$.*

The following example shows that the converse of Theorem 2.8 (1) fails if one deletes the hypothesis that $J \subseteq \sqrt{0_S}$.

Example 2.4. Let $R = \mathbb{Z}(+)\mathbb{Z}_{pq}$, where p and q are prime numbers, and let $J = p\mathbb{Z}(+)\mathbb{Z}_{pq}$. It is clear that $I = 0(+)\mathbb{Z}_{pq}$ is an n-ideal (and so is a quasi n-ideal) of R. However, $I \bowtie J$ is not a quasi n-ideal (and so is not an n-ideal). Indeed, $((0, \bar{1}), (p, \bar{1}))((1, \bar{0}), (1, \bar{0})) = ((0, \bar{1}), (p, \bar{1})) \in I \bowtie J$. But $((0, \bar{1}), (p, \bar{1})) \notin \sqrt{0_{R\bowtie J}}$ and $((1, \bar{0}), (1, \bar{0})) \notin \sqrt{I \bowtie J}$.

In the following example, we prove that the condition $f^{-1}(J) \subseteq \sqrt{0_R}$ cannot be discarded in the proof of the converse of Theorem 2.8 (2).

Example 2.5. Let $R=\mathbb{Z}(+)\mathbb{Z}_{pq}$, where p and q are prime numbers, $S=\mathbb{Z}$, and let $J=p\mathbb{Z}$. Consider the canonical epimorphism $f\colon R\to S$ defined by $f(r,m)=r$. Note that $f^{-1}(J)=p\mathbb{Z}(+)\mathbb{Z}_{pq}\nsubseteq\sqrt{0_R}$. On the other hand, $K=(0)$ is an n-ideal of S. However, $\overline{K}^f$ is not a quasi n-ideal of $R\bowtie^f J$ because $((p,\bar{0}),0)((1,\bar{0}),1)\in\overline{K}^f$, $((p,\bar{0}),0)\notin\sqrt{0_{R\bowtie^f J}}$ and $((1,\bar{0}),1)\notin\sqrt{\overline{K}^f}$.

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