Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2023) 117:76
https://doi.org/10.1007/s13398-023-01402-0

ORIGINAL PAPER

Extension procedures for lattice Lipschitz operators on Euclidean spaces

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Received: 5 October 2022 / Accepted: 3 February 2023

Abstract
We present a new class of Lipschitz operators on Euclidean lattices that we call lattice Lipschitz maps, and we prove that the associated McShane and Whitney formulas provide the same extension result that holds for the real valued case. Essentially, these maps satisfy a (vector-valued) Lipschitz inequality involving the order of the lattice, with the peculiarity that the usual Lipschitz constant becomes a positive real function. Our main result shows that, in the case of Euclidean space, being lattice Lipschitz is equivalent to having a diagonal representation, in which the coordinate coefficients are real-valued Lipschitz functions. We also show that in the linear case the extension of a diagonalizable operator from the values in their eigenvectors coincide with the operator obtained both from the McShane and the Whitney formulae. Our work on such extension/representation formulas is intended to follow current research on the design of machine learning algorithms based on the extension of Lipschitz functions.

Keywords Lipschitz · Operator · Banach lattice · Eigenvalue · Extension · Diagonal · Euclidean space

Mathematics Subject Classification Primary 26A16 · 47J10; Secondary 15A42 · 46B42

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1 Introduction

The diagonal representation of symmetric linear operators is a classical and powerful tool in the mathematical treatment of many problems in all fields of scientific and technical research. Following this pattern, it has been a common interest of many researchers in mathematics to extend the ideas that allow to obtain such diagonal representations to other classes of maps, such as bilinear operators (see for example [9, 10, 12, 22] and references therein), and Lipschitz operators [3, 11, 14]. We are interested in developing new techniques to extend Lipschitz operators on finite-dimensional normed spaces from subsets of eigenvectors to the whole space to obtain explicit representation formulas for them, based on the previous computation of subsets of eigenvectors of the operator.

In this paper we introduce a new class of Lipschitz-type maps on Euclidean lattices which we call lattice Lipschitz operators. We provide a technique for the extension of such maps from a set of vectors in the space that satisfy eigenvector equations, proving the error formulas that are needed in any approximation framework. The reason is that the set of eigenvectors is known to be a convenient set to start extending a linear operator in Euclidean spaces. In this case the computations to obtain the pointwise evaluation of the map are just additions and multiplications by scalars, once the values of the operator in this set are known. However, for more general maps this rule would not work, even when these maps have a reasonably good linear approximation.

This opens the door to the project of defining new extension rules. Essentially, our operator should be able to be "reconstructed" from the knowledge of its "fixed points" (except for a multiplicative constant). Such points are easily described: an eigenvector of an operator T is an element x of the space for which the projection of $T(x)$ onto the subspace generated by x coincides with its multiplication by an scalar. Consequently, a useful extension rule is one that derives some benefit from this easy description, in terms of facilitating the mathematical representation of the operator by means of a simple formula. We will show that, for the class of lattice Lipschitz operators, the adaptations of the McShane and Whitney formulas to the lattice valued case give the desired representations. In fact, our construction relies on the fundamental fact that, for the case of "diagonalizable" Lipschitz operators on Euclidean spaces, our two extension formulae coincide with the original map when applied using the set of eigenvector of the operator as primary set. Thus, we show that the extension of a lattice Lipschitz map T from its set of eigenvalues using any of these formulae give as a result the original map. Moreover, as a consequence of the minimal and maximal character of these extension formulas, we get that any other lattice Lipschitz extension give again the operator T. This is proved in Theorem 4 (see also Corollary 5 for the linear diagonalizable operators).

Our results are motivated by the potential use of Lipschitz extension formulas in Machine Learning algorithms. It has been shown that Lipschitz extensions are useful to predict value functions in Reinforcement Learning (see for example [5, 13]). Although the use of the theory of Lipschitz functions goes back to the origin of Artificial Intelligence, recently some new results have pointed out the power of this mathematical setting to provide competitive algorithms in different fields of machine learning (for example, [2, 4, 8, 24]). Following this guide, we intend in this paper to provide a mathematical foundation for the safe use of extension formulas for Lipschitz maps in this applied setting. An important key here is that they can be applied for vector valued functions, instead of the classical case that works for real valued function. Far from being a formal extension of the classical case, we are interested in constructing the mathematical setting in which the relevant functions appearing in, for example, machine learning (value functions, reward functions, oracle functions…), are vector-valued. This would allow to transfer the ideas supporting artificial intelligence algorithms that are based on evaluation of real functions to the vector valued setting. This also provides a uniform treatment of all the coordinates in the same Lipschitz type theoretical framework.

This is also the reason why we present in this paper the simplest case, which consists of operators defined on finite dimensional lattices. Most of our results could be formulated and studied in the more general context of order complete Banach lattices, including abstract ordered normed spaces, as well as Banach function lattices of measurable functions. However, we are interested in showing the results as clean and directly applicable as possible, in order to facilitate the direct construction of new machine learning algorithms.

The article is organized as follows. After this introduction, we present in Sect. 2 the class of lattice Lipschitz operators. Section 3 is devoted to show the main results on the extension of lattice Lipschitz maps. After adapting the classical formulas of McShane and Whitney, we demonstrate the main results on how these extensions work in the lattice setting. Finally, in Sect. 4 we show the role played by the set of eigenvectors of lattice Lipschitz maps in the representation of these operators from their eigenvalues and eigenvectors. In this sense, the coincidence of the lattice extension formulas with the linear extension shows that essentially, the linear representation of diagonalizable maps coincides with the lattice representation by the McShane and Whitney formulas. The interest of our results lies in the fact that our technique can also be applied to a broader class than linear maps, or even to operators that can be reasonably approximated in order by lattice Lipschitz maps. Some examples illustrating this fact are also explained in Sect. 4.

Throughout the paper we will use standard notation and concepts of normed lattices. The symbols $\wedge$, $\vee$ and $|\cdot|$ stand for the infimum, the supremum and the modulus of the elements of the lattice. Recall that $|x| = x \vee (-x)$ for every element of the normed lattice E. We refer to [19] for general notions on normed lattices, as order completeness and regular operator.

Let (D, d) and (M, ρ) be two metric spaces. A function $T : D \to M$ that satisfies that there exists a constant $K > 0$ such that

$$\rho\big(T(a), T(b)\big) \le K d(a, b), \quad a, b \in D,$$

is called a Lipschitz function. The Lipschitz constant of T is the infimum of all constants K satisfying this inequality.

If $T : B \to \mathbb{R}$ is a real-valued Lipschitz function defined on a subset $B \subseteq D$, it can be extended by the McShane–Whitney Theorem [18, 25] to the whole space D preserving the Lipschitz constant. Moreover, the extension can be explicitly computed using the following formulas, called McShane and Whitney extensions of T respectively:

$$T^M(b) := \sup_{a \in B} \Big\{T(a) - K\, d(b, a)\Big\}, \quad b \in D,$$
$$T^W(b) := \inf_{a \in B} \Big\{T(a) + K\, d(b, a)\Big\}, \quad b \in D.$$

This result, together with the celebrated Kirszbraun Theorem [16] which proves the existence of extension of Lipschitz maps on subsets of Hilbert spaces, form the classical core of the topic. A comprehensive study of Lipschitz functions and its extensions can be seen on the book by Cobzaş, Miculescu and Nicolae [7].

In this paper we will modify these definitions to consider the pointwise evaluation of functions $T : E \to E$, where E is an Euclidean normed lattice $(E, \|\cdot\|, \le)$. We will show that these extension formulas make sense also in this case. Let us recall the definition of (finite dimensional) normed lattice. This is a vector lattice—with the order relation "$\le$"—

and a norm $\|\cdot\|$ that satisfies that $\|x\| \le \|y\|$ when $|x| \le |y|$, $x, y \in E$. Once a basis $\mathcal{B}$ is fixed for E, we can associate to it the order in the space, that coincides with the coordinates order: that is, for two vectors of the space $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, $x \le y$ if and only if $x_i \le y_i$ for $i = 1, \dots, n$ (see Sect. 2).

Consider a Lipschitz operator $T : E \to E$. As in the linear case, we will say that a vector v is an eigenvector of T if there is a real number λ (we consider only real eigenvalues) such that $T(v) = \lambda\, v$. Note that, in the Lipschitz case, the ray defined by such a vector v need not define a vector subspace of eigenvalues. Recall that a linear map T is diagonalizable if there is a basis of eigenvectors. We will adapt this definition to the Lipschitz setting. However—as in the linear case—, the set of eigenvalues of the Lipschitz map T will play a relevant role in the representation results that we show in this paper.

2 Lattice Lipschitz operators on Euclidean spaces

In this section we define the particular class of Lipschitz maps to which this paper is devoted. In the case we consider, the finite dimensional normed space E is enriched by an order relation to become a normed lattice, which is defined by a cone associated to a given basis.

Fix $E = \mathbb{R}^n$ and let $\mathcal{B} = \{x_1, \dots, x_n\}$ be a basis for $\mathbb{R}^n$. Consider, as usual, the addition and scalar multiplication defined by its components. We will consider the lattice order on $\mathbb{R}^n$ provided by the basis, that is, if $x = (\alpha_1, \dots, \alpha_n)$ and $y = (\beta_1, \dots, \beta_n)$ are representations of two vectors of E given by their coordinates in $\mathcal{B}$, we define

$$x \le y \quad \text{if and only if} \quad \alpha_i \le \beta_i \text{ for every } i = 1, \dots, n.$$

This order is given by the positive cone

$$C = \left\{ x = \sum_{i=1}^{n} \alpha_i x_i \,:\, \alpha_i \ge 0 \text{ for } i = 1, \dots, n \right\},$$

so $x \le y$ if and only if $y - x \in C$. Each vector $x = \sum_{i=1}^{n} \alpha_i x_i = (\alpha_1, \dots, \alpha_n)$ of the space E can be considered as a function $f_x : \Omega = \{1, \dots, n\} \to \mathbb{R}$ in the usual way, $f_x(w) = \alpha_w$. Thus, E can be seen as a function lattice with the pointwise order. We will use "function notation" for simplicity, that is, if $z \in E$ we will write $z(w)$ for the coordinate of z that corresponds to the element of $\mathcal{B}$ with label $w \in \Omega$. Therefore, for $x, y \in E$,

$$x \le y \quad \text{if and only if} \quad x(w) \le y(w) \text{ for every } \; w \in \{1, \dots, n\}.$$

The supremum, the infimum and the modulus in this space are those given by their coordinates,

$$\left| \sum_{i=1}^{n} \alpha_i x_i \right| = \sum_{i=1}^{n} |\alpha_i| x_i .$$

In what follows, we will assume that $\mathcal{B}$ is a fixed basis of E with the order defined above. Therefore, we consider the Euclidean space as a normed lattice of functions on the set Ω that contains as many elements as the dimension of E.

Definition 1 Let $\mathcal{B}$ be a basis for E, and consider the normed lattice with the order and the norm provided by $\mathcal{B}$. Let E_0 be a subset of E. We say that an operator $T : E_0 \to E$ is lattice Lipschitz if there is a non-negative bounded function $K : \Omega \to \mathbb{R}$ such that for every $x, y \in E_0$,

$$\big|T(x) - T(y)\big|(w) \le K(w)\big|x - y\big|(w), \quad w \in \Omega. \tag{1}$$

The pointwise infimum of all the functions K satisfying the inequality above will be called the associate function to T.

If no further information is given we will assume throughout the paper that the function $K(w)$ appearing in the Lipschitz inequality for a Lipschitz operator of the lattice is its associate function. For easy of reading, we will not make an explicit reference to the basis $\mathcal{B}$ when we say that an operator is Lipschitz lattice; however, note that we need to fix this basis beforehand for the definition to make sense.

Our definition can be seen as a particular case of the one in [17]—that is given for the case when E is a Banach algebra—, considering E as a cone metric space on itself with the cone metric $\rho : E \times E \to E$,

$$\rho(x, y)(w) = \big|x - y\big|(w), \quad w \in \Omega,$$

for $x, y \in E$. Recall that a cone metric on a set X—see for example [17]—is a map $\rho : X \times X \to Y$, where Y is a Banach lattice, satisfying that for all $x_1, x_2, x_3 \in X$,

(1) $0 < \rho(x_1, x_2)$ if $x_1 \neq x_2$, and $\rho(x_1, x_1) = 0$,
(2) $\rho(x_1, x_2) = \rho(x_2, x_1)$,
(3) $\rho(x_1, x_2) \leq \rho(x_1, x_3) + \rho(x_3, x_2)$.

This notion has been applied mainly in the context of fixed point theory (see, for example, [23] and the references therein). To the best of our knowledge, no results on operator extension satisfying the above definition are known in the existing mathematical literature.

Some other Lipschitz-type inequalities in ordered spaces can be found on Papageorgiou [21], Németh [20] or Li [15]. For the specific case of Banach function spaces, the recent paper [6] provides a concrete adaptation of such kind of inequalities for spaces of integrable functions. An overview on classical results on the topic can be found in [7, Ch.4.1].

Remark 1 Note that lattice Lipschitz operators are in particular Lipschitz: if T is lattice Lipschitz with associate function $K(w)$, it is a Lipschitz operator with norm less than or equal to $\sup_w |K(w)|$. Indeed, for $x, y \in E$, we have by the definition that

$$\big|T(x) - T(y)\big|(w) \leq K(w)\big|x - y\big|(w), \quad w \in \Omega.$$

Thus, taking into account the relation of the order and the norm on any normed lattice and the specific formula of the Euclidean norm, a direct computation gives

$$\big\|T(x) - T(y)\big\| \leq \Big\|K(w)\big|x - y\big|(w)\Big\| \leq \Big(\sup_w |K(w)|\Big)\, \|x - y\|.$$

3 Extension theorems for lattice Lipschitz operators

In what follows we will show the extension results for lattice Lipschitz operators on Euclidean lattices. As usual in mathematical research, in order to prove that our method makes sense we show that our new extension rule provides the same fundamental results that hold for diagonalizable linear operators and for non-linear operators (Theorem 4 and Corollary 5).

We will write E_0 for *any* subset of the Euclidean space E, and we will consider a lattice Lipschitz operator $T : E_0 \to E$.

Definition 2 Let $E_0 \subseteq E$. For a lattice Lipschitz operator $T : E_0 \to E$ with associated bounded function $K : \Omega \to \mathbb{R}$, we consider the formulas

$$T^M(x)(w) := \bigvee \Big\{T(z)(w) - K(w)|x - z|(w) : z \in E_0\Big\}, \quad x \in E,$$

and

$$T^W(x)(w) := \bigwedge \Big\{ T(z)(w) + K(w)|x - z|(w) : z \in E_0 \Big\}, \quad x \in E.$$

Remark 2 Previous expressions are well defined in the sense that the supremum $\bigvee$ and the infimum $\bigwedge$ exist for all $w \in \Omega$, and so T^M, T^W are operators from E to E. Indeed, given $x \in E$, let $y \in E_0$ be fixed and let $w \in \Omega$. Then, for any $z \in E_0$,

$$\begin{aligned} T(z)(w) - T(y)(w) &\leq \big|T(z) - T(y)\big|(w) \leq K(w)\big|z - y|(w) \\ &\leq K(w)|z - x|(w) + K(w)|x - y|(w). \end{aligned}$$

It follows that

$$T(z)(w) - K(w)|x - z|(w) \leq T(y)(w) + K(w)|x - y|(w) =: M.$$

Since M does not depend on z, $\{T(z)(w) - K(w)|x - z|(w) : z \in E_0\}$ is a set of real numbers bounded from above by M, so its supremum exists. Therefore $T^M(x) \in E$. For the case of T^W we get the same result; to see it just consider the following remark.

Remark 3 Notice that the extension formulas T^M and T^W are related by the equation

$$\begin{aligned} T^W(x)(w) &= \bigwedge \Big\{ T(z)(w) + K(w)|x - z|(w) : z \in E_0 \Big\} \\ &= \bigwedge \Big\{ -\Big((-T)(z)(w) - K(w)|x - z|(w)\Big) : z \in E_0 \Big\} \\ &= -\bigvee \Big\{ (-T)(z)(w) - K(w)|x - z|(w) : z \in E_0 \Big\} = -(-T)^M(x)(w). \end{aligned}$$

Moreover, it can be directly seen that $-T : E_0 \to E$ is also a lattice Lipschitz operator with the same associate function K. This fact will allow us to prove some properties only for one of the formulas, and then apply the previous identity for the other case.

Observe that the only information of the space E required for defining both extension formulas given above is the one related to the order in E and the values of the lattice Lipschitz function T in E_0. No information on metric or linear structure is required.

The following result extends the McShane–Whitney Theorem for the case of lattice Lipschitz operators. The calculations are similar to those proving this classical result. However, the reader should note that the objects for which the result is proved are functions and, in general, elements of a normed lattice. In fact, the same calculations should give the corresponding result for abstract Banach lattices under the assumption of order completeness, to guarantee the existence of the infimums and supremums appearing in the formulae. Even for the case of Dedekind complete Riesz spaces [1], the results of the present paper could make sense.

Proposition 1 *Let $E_0 \subseteq E$. Let $T : E_0 \to E$ be a lattice Lipschitz operator with associated bounded function $K : \Omega \to \mathbb{R}$. Then T^M and T^W are suitable extension formulas from E_0 to E which preserve the lattice Lipschitz inequality with the same associate function K.*

Proof First we show that T^M is an extension of T. For any $x \in E_0$ and $w \in \Omega$, clearly one has

$$T^M(x)(w) \geq T(x)(w) - K(w)|x - x|(w) = T(x)(w).$$

In addition, for every $y \in E_0$,

$$T(y)(w) - T(x)(w) \leq |T(y) - T(x)|(w) \leq K(w)|y - x|(w),$$

so $T(y)(w) - K(w)|x-y|(w) \leq T(x)(w)$ and, taking supremum, $T^M(x)(w) \leq T(x)(w)$. We conclude that $T(x) = T^M(x)$ for every $x \in E_0$.

In order to justify that T^M satisfies the lattice Lipschitz inequality with the same function K, let $x, y \in E$. For any $w \in \Omega$, if $z \in E_0$

$$\begin{aligned} T^M(x)(w) &\geq T(z)(w) - K(w)|x-z|(w) \\ &\geq T(z)(w) - K(w)|y-z|(w) - K(w)|x-y|(w). \end{aligned}$$

It follows that

$$\begin{aligned} T^M(x)(w) &\geq \bigvee \left\{ T(z)(w) - K(w)|y-z|(w) - K(w)|x-y|(w) : z \in E_0 \right\} \\ &= T^M(y)(w) - K(w)|x-y|(w), \end{aligned}$$

so $K(w)|x-y|(w) \geq T^M(y)(w) - T^M(x)(w)$. Interchanging the roles of x and y, one obtains

$$K(w)|y-x|(w) \geq T^M(x)(w) - T^M(y)(w),$$

so

$$\left| T^M(x)(w) - T^M(y)(w) \right| \leq K(w) \left| y-x \right|(w).$$

The case of T^W can be proved similarly. Alternatively, it can be shown just by applying the previous case to $-T$ taking into account Remark 3, as in Remark 2. □

The following result gives information on the extremal properties of the McShane and Whitney lattice extensions. It is relevant to note that, as in the case of real functions, these lattice extensions are minimal and maximal, respectively, but in this case among all suitable operators providing admissible Lipschitz-type extensions. Extremality is therefore to be understood as a vector-valued property, which works for a wide class of Lipschitz operators. This is one of the properties that make these extremal formulas especially relevant for applications in machine learning, as in the real-valued case (see for example [4]).

Note that the notation $T \leq S$ used below does not follow the convention for regular linear operators on Banach lattices (see for example [19]). In this linear case, this means that $T(z) \leq S(z)$ for *positive* lattice elements $0 \leq z \in E$. However, in our case it means that $T(z) \leq S(z)$ *for all* elements $z \in E$, which is of course only compatible with linearity when $T = S$.

Proposition 2 *Let $E_0 \subseteq E$. Let $T : E_0 \to E$ be a lattice Lipschitz operator with associated bounded function $K : \Omega \to \mathbb{R}$. If $\widehat{T} : E \to E$ is an extension of T which preserve the lattice Lipschitz inequality with the same function K, then $T^M \leq \widehat{T} \leq T^W$.*

Proof Let $x \in E$ and $w \in \Omega$. Since $\widehat{T}$ is a lattice Lipschitz operator with associate function K, for every $z \in E_0$

$$-K(w)|x-z|(w) \leq \widehat{T}(x)(w) - \widehat{T}(z)(w) \leq K(w)|x-z|(w).$$

Notice that $\widehat{T}(z)(w) = T(z)(w)$, and so

$$T(z)(w) - K(w)|x-z|(w) \leq \widehat{T}(x)(w) \leq T(z)(w) + K(w)|x-z|(w).$$

Now, taking supremum on the left and infimum on the right, we obtain that $T^M(x)(w) \leq \widehat{T}(x)(w) \leq T^W(x)(w)$, as we wanted to show. □

Remark 4 In particular, Proposition 2 shows that $T^M \leq T^W$. But, in order to control the error that is made when we approximate a given extension of T by any of these operators, we are also interested in studying how much the two extensions differ from each other. Let us show that

$$0 \leq T^W(x) - T^M(x) \leq 2K \bigwedge \Big\{|x-z| : z \in E_0\Big\},$$

for any $x \in E$. Indeed, if $w \in \Omega$,

$$\begin{aligned} &T^W(x) - T^M(x) \\ &\quad = \bigwedge \Big\{T(z) + K|x-z| : z \in E_0\Big\} - \bigvee \Big\{T(z) - K|x-z| : z \in E_0\Big\} \\ &\quad = \bigwedge \Big\{T(z) + K|x-z| : z \in E_0\Big\} + \bigwedge \Big\{ - T(z) + K|x-z| : z \in E_0\Big\} \\ &\quad \leq \bigwedge \Big\{T(z) + K|x-z| - T(z) + K|x-z| : z \in E_0\Big\} \\ &\quad = 2K \bigwedge \Big\{|x-z| : z \in E_0\Big\}. \end{aligned}$$

4 Lattice Lipschitz and diagonal operators on Euclidean lattices

Mimicking the linear setting, we are interested in studying to what extent lattice Lipschitz operators can obtain a diagonal representation. By definition, a diagonalizable linear operator T can always be written as a diagonal operator with respect to a basis of eigenvectors; in this case, the restriction of the operator to a subspace generated by an eigenvector is simply a multiplication by a real number. It is not difficult to see (we will show in what follows) that if T is diagonalizable, it is in particular Lipschitz lattice if one chooses an eigenvector basis $\mathcal{B}$ to define the order in E. In this section we analyze whether a similar result exists for the case of lattice Lipschitz operators, showing that, in fact, the existence of a kind of diagonal representation (with real-valued functional "eigenvalues") is equivalent to being lattice Lipschitz, thus closing the parallelism with the linear case satisfactorily. We will illustrate this idea with some examples in this section too.

Definition 3 We say that an operator $T : E \to E$ is *diagonal* with respect to a basis $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$ of $E = \mathbb{R}^n$ if there exist real functions $f_i : \mathbb{R} \to \mathbb{R}$ for $1 \leq i \leq n$ such that

$$T\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{i=1}^{n} f_i(\alpha_i) x_i, \qquad \text{for every } \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}. \tag{2}$$

In this case, we call the functions f_i the *coordinate functions* of T with respect to the basis $\mathcal{B}$.

Using the standard vector/coordinates notation, we will write this representation also as

$$T\Big((\alpha_1, \alpha_2, \ldots, \alpha_n)\Big) = \Big(f_1(\alpha_1), f_2(\alpha_2), \ldots, f_n(\alpha_n)\Big),$$

where the corresponding coordinate in the basis $\mathcal{B}$ is written at each position of the vector, both in the domain and in the range of the operator. We will use this notation in the examples of this section.

Theorem 3 *Let $T : E \to E$ be an operator. Consider on E the order provided by the basis $\mathcal{B} = \{x_1, \ldots, x_n\}$. Then, T is a lattice Lipschitz operator with associate function $K : \Omega \to \mathbb{R}$*

if and only if T is diagonal with respect to the basis $\mathcal{B}$ with coordinate functions being real Lipschitz functions with associated Lipschitz constants $K(i)$, $i = 1, \ldots, n$.

Proof Suppose first that T is diagonal on the basis $\mathcal{B}$ and the coordinate functions f_i are real Lipschitz functions with Lipschitz constants $K(i)$ for $1 \leq i \leq n$. Consider two elements of E, $x = \sum_{i=1}^n \alpha_i x_i$ and $y = \sum_{i=1}^n \beta_i x_i$. Then, taking into account the Lipschitz inequality for each coordinate function, we get the following domination,

$$\big|T(x) - T(y)\big| = \sum_{i=1}^n \Big|f_i(\alpha_i) - f_i(\beta_i)\Big| x_i \leq \sum_{i=1}^n K(i)\big|\alpha_i - \beta_i\big| x_i = K|x - y|,$$

what gives the desired result.

For the converse, let T be a lattice Lipschitz function with associate function K. Note that for every function $T : E \to E$, just by the coordinate representation of the vectors of E, there are functions $T_i : E \to \mathbb{R}$ that satisfy

$$T(x) = \sum_{i=1}^n T_i(x) x_i, \qquad x \in E. \tag{3}$$

Define now the real functions $f_j : \mathbb{R} \to \mathbb{R}$ for $1 \leq j \leq n$ by

$$f_j(\alpha) = T_j(\alpha x_j).$$

Let us see that

$$T_j\left(\sum_{i=1}^n \alpha_i x_i\right) = f_j(\alpha_j). \tag{4}$$

Indeed, fix $1 \leq j \leq n$. If $x = \sum_{i=1}^n \alpha_i x_i$, then

$$\begin{aligned}\big|T_j(x) - f_j(\alpha_j)\big| &= \big|T_j(x) - T_j(\alpha_j x_j)\big| = \big|T(x) - T(\alpha_j x_j)\big|(j)\\ &\leq K(j)\big|x - \alpha_j x_j\big|(j) = K(j)\big|\alpha_j - \alpha_j\big| = 0.\end{aligned}$$

In other words, the function T_j only depends on the j component of x. As a consequence of (3) and (4),

$$T\left(\sum_{i=1}^n \alpha_i x_i\right) = \sum_{j=1}^n T_j\left(\sum_{i=1}^n \alpha_i x_i\right) x_j = \sum_{j=1}^n f_j(\alpha_j) x_j.$$

To finish the proof we show that each f_j is a (real) Lipschitz function, and we compute its Lipschitz constant. Indeed, let $\alpha, \beta \in \mathbb{R}$. Then we have

$$\begin{aligned}\big|f_j(\alpha) - f_j(\beta)\big| &= \big|T_j(\alpha x_j) - T_j(\beta x_j)\big| = \big|T(\alpha x_j) - T(\beta x_j)\big|(j)\\ &\leq K(j)\big|\alpha x_j - \beta x_j\big|(j) = K(j)|\alpha - \beta|,\end{aligned}$$

and the constants $K(j)$ clearly give the smallest real numbers that satisfy the inequalities. □

Remark 5 The above result can also be rewritten in terms of a domination by functionals acting on the dual space E^* of E, which would give the key to a more abstract formulation of the Lipschitz operators on general Banach lattices. Let us write this reformulation for the sake of completeness. Consider on E the order provided by the basis $\mathcal{B} = \{x_1, \ldots, x_n\}$. For an operator $T : E \to E$ the following statements are equivalent.

(1) T is a lattice Lipschitz operator with the order given by $\mathcal{B}$.
(2) T is a diagonal operator with coordinate functions being real Lipschitz functions.

(3) T satisfies that for every $1 \leq i \leq n$ there exists a constant $K_i > 0$ such that

$$\left|\langle T(x) - T(y), x_i^* \rangle\right| \leq K_i \left|\langle x - y, x_i^* \rangle\right|,$$

where $x_i^* \in E^*$ is the functional defined as $x_i^*(x_i) = 1$, and $x_i^*(x_j) = 0$ if $i \neq j$.

The following examples show the relevance of the order in the space on the study of lattice Lipschitz—or diagonal—mappings.

Example 1 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be $f(x, y) = (y, x)$. Clearly it is not a lattice Lipschitz mapping—with respect to the usual order of $\mathbb{R}^2$—because

$$\left|f(1, 0) - f(0, 0)\right| = (0, 1) \nleq K \cdot \left|(1, 0) - (0, 0)\right| = (K_1, 0),$$

for any $K = (K_1, K_2) \in \mathbb{R}^2$. But it is a lattice Lipschitz function on $\mathbb{R}^2$ when an appropriate order is considered. Indeed, let us consider the basis $\mathcal{B} = \left\{v_1 = (1, 1), v_2 = (1, -1)\right\}$ and take the order provided by $\mathcal{B}$. Then,

$$f(\alpha v_1 + \beta v_2) = f(\alpha + \beta, \alpha - \beta) = (\alpha - \beta, \alpha + \beta) = f_1(\alpha)v_1 + f_2(\beta)v_2,$$

where $f_1(t) = t$, $f_2(t) = -t$, that are real Lipschitz functions. Therefore we obtain that it is a lattice Lipschitz mapping by using Theorem 3.

Example 2 Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping on $\mathbb{R}^2$ endowed with the usual order given by

$$\phi(x, y) = \begin{cases} \left(2x^2 + 2y^2, 4xy\right) & \text{if } |x + y| \leq 2, |x - y| \leq 2 \\ \left((x + y)^2 + 4, (x + y)^2 - 4\right) & \text{if } |x + y| \leq 2, |x - y| > 2 \\ \left(4 + (x - y)^2, 4 - (x - y)^2\right) & \text{if } |x + y| > 2, |x - y| \leq 2 \\ (8, 0) & \text{if } |x + y| > 2, |x - y| > 2. \end{cases}$$

Clearly it is not a lattice Lipschitz function, since

$$\left|\phi(0, 3) - \phi(0, 0)\right|(1) = 8 \nleq K|(0, 3) - (0, 0)|(1) = 0,$$

for any K. Let us see that it is a diagonal function when considering on $\mathbb{R}^2$ the order induced by the basis $\mathcal{B} = \left\{v_1 = (1, 1), v_2 = (1, -1)\right\}$. For $\alpha, \beta \in \mathbb{R}$, it is easy to see that

$$\phi(\alpha v_1 + \beta v_2) = \begin{cases} (4\alpha^2 + 4\beta^2, 4\alpha^2 - 4\beta^2) & \text{if } |\alpha| \leq 1, |\beta| \leq 1 \\ (4\alpha^2 + 4, 4\alpha^2 - 4) & \text{if } |\alpha| \leq 1, |\beta| > 1 \\ (4 + 4\beta^2, 4 - 4\beta^2) & \text{if } |\alpha| > 1, |\beta| \leq 1 \\ (8, 0) & \text{if } |\alpha| > 1, |\beta| > 1, \end{cases}$$

so $\phi(\alpha v_1 + \beta v_2) = f_1(\alpha)v_1 + f_2(\beta)v_2$, where $f_1(t) = f_2(t) = 4t^2$ for $|t| \leq 2$ and 4 otherwise. Since ϕ is diagonal on the basis $\mathcal{B}$ with coordinate functions f_1 and f_2, that are real Lipschitz functions with Lipschitz constant equal to 8, we obtain by Theorem 3 that ϕ it is a lattice Lipschitz function with the associate function K given by $K(1) = K(2) = 8$.

Theorem 4 *Let $E = \mathbb{R}^n$ and consider the order provided by a basis $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$ of E, and let $T : E \to E$ be a lattice Lipschitz function with associate function K and coordinate functions f_i (the decomposition given in Eq. (2) of Definition 3 can be obtained by Theorem 3). Consider the "axis" set $\Sigma = \left\{\alpha x_i : \alpha \in \mathbb{R}, 1 \leq i \leq n\right\}$. Then,*

(1) *the Whitney and McShane extensions of $T|_\Sigma$ from Σ to E satisfy*

$$(T|_\Sigma)^M = T \quad \text{and} \quad (T|_\Sigma)^W = T.$$

If, in addition, $T(0)=0$,

(2) *the set of eigenvectors of* T *contains* Σ, *and*
(3) *if* λ *is an eigenvalue of* T, *then* $|\lambda| \leq \sup_{w\in\Omega} K_w$.

Proof (1) Consider the McShane extension of $T|_\Sigma$ at x,

$$\big(T|_\Sigma\big)^M(x) = \bigvee_{s\in\Sigma}\Big(T(s) - K|x-s|\Big) = \bigvee_{i=1}^{n}\left(\bigvee_{\alpha\in\mathbb{R}}\big(T(\alpha x_i) - K|x-\alpha x_i|\big)\right).$$

Suppose now that $x = \sum_{i=1}^n \alpha_i x_i$ and observe that, for each $1 \leq i \leq n$, we have $x - \alpha x_i = (\alpha_i - \alpha)x_i + \sum_{j\neq i}\alpha_j x_j$. Fix $1 \leq w \leq n$ and assume first that $w \neq i$. Then

$$\left(\bigvee_{\alpha\in\mathbb{R}}\big(T(\alpha x_i) - K|x-\alpha x_i|\big)\right)(w) = \bigvee_{\alpha\in\mathbb{R}}\big(f_w(0) - K_w|\alpha_w|\big) = f_w(0) - K_w|\alpha_w|.$$

Since $f_w(0) - f_w(\alpha_w) \leq |f_w(0) - f_w(\alpha_w)| \leq K_w|\alpha_w|$, then $f_w(0) - K_w|\alpha_w| \leq f_w(\alpha_w)$, and so

$$\left(\bigvee_{\alpha\in\mathbb{R}}\big(T(\alpha x_i) - K|x-\alpha x_i|\big)\right)(w) \leq f_w(\alpha_w). \tag{5}$$

For the case $w = i$, note that

$$\left(\bigvee_{\alpha\in\mathbb{R}}\big(T(\alpha x_w) - K|x-\alpha x_w|\big)\right)(w) = \bigvee_{\alpha\in\mathbb{R}}\big(f_w(\alpha) - K_w|\alpha_w - \alpha|\big).$$

Observe that $f_w(\alpha) - f_w(\alpha_w) \leq |f_w(\alpha) - f_w(\alpha_w)| \leq K_w|\alpha - \alpha_w|$ for every $\alpha \in \mathbb{R}$, so $f_w(\alpha) - K_w|\alpha_w - \alpha| \leq f_w(\alpha_w)$. Hence the supremum is attained at $\alpha = \alpha_w$. Therefore

$$\left(\bigvee_{\alpha\in\mathbb{R}}\big(T(\alpha x_w) - K|x-\alpha x_w|\big)\right)(w) = f_w(\alpha_w). \tag{6}$$

As a consequence of (5) and (6),

$$(T|_\Sigma)^M(x)(w) = \bigvee_{i=1}^{n}\left(\bigvee_{\alpha\in\mathbb{R}}\big(T(\alpha e_i) - K|x-\alpha e_i|\big)\right)(w) = f_w(\alpha_w) = T(x)(w),$$

which proves that $(T|_\Sigma)^M(x) = T(x)$.

The proof for the Whitney formula is immediate by using the just proved case with $-T$.

(2) Clearly, $x = 0$ is an eigenvector of T. Observe that the condition $T(0) = 0$ implies that $f_i(0) = 0$ for all $1 \leq i \leq n$. Let αx_i in Σ with $\alpha \neq 0$, and note that in this case

$$T(\alpha x_i) = \sum_{j\neq i}\big(f_j(0)x_j + f_i(\alpha)x_i\big) = \frac{f_i(\alpha)}{\alpha}\alpha x_i,$$

so αx_i is an eigenvector of T with eigenvalue $f_i(\alpha)/\alpha$.

(3) If $T(x) = \lambda x$ (with $x \neq 0$), the Eq. (2) implies that $f_i(\alpha_i) = \lambda\alpha_i$ for each i. Since $x \neq 0$, at least one α_{i_0} is not 0, so

$$|\lambda| = \left|\frac{f_{i_0}(\alpha_{i_0})}{\alpha_{i_0}}\right| = \left|\frac{f_{i_0}(\alpha_{i_0}) - f_{i_0}(0)}{\alpha_{i_0}}\right| \leq \left|\frac{K_{i_0}|\alpha_{i_0} - 0|}{\alpha_{i_0}}\right| = K_{i_0} \leq \sup_{w\in\Omega} K_w.$$

□

An immediate consequence of Theorem 4 is the following result for linear operators. It shows that for linear diagonalizable operators, the representation provided by the linear combination of the eigenvectors coincides with both the McShane and the Whitney extensions with the order defined by these eigenvectors.

Corollary 5 *Let $T : E \to E$ be a linear diagonalizable operator with real eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Let $\mathcal{B} = \{x_1, \dots, x_n\}$ be a basis for E of eigenvectors of T. Consider the complete set of eigenvectors E_0 and take the order in the lattice induced by $\mathcal{B}$. Then*

(1) *$T|_{E_0}$ is lattice Lipschitz with associate function $K(r) = |\lambda_r|$, $r \in \{1, \dots, n\}$.*
(2) *Both $(T|_{E_0})^M$ and $(T|_{E_0})^W$ provide lattice Lipschitz extensions of $T|_{E_0}$ from E_0 to E preserving the associate function K such that*

$$(T|_{E_0})^M = T \quad \text{and} \quad (T|_{E_0})^W = T.$$

Consequently, T is lattice Lipschitz with the associate function K written above.

The coincidence of the lattice Lipschitz extension rule and the linear rule opens the door to a general procedure for the representation of Lipschitz maps with a "large enough" set of eigenvectors. For the diagonalizable case, the minimum-maximum condition of the McShane and Whitney extensions (Proposition 2) can be considered, along with the extension behavior of diagonalizable mappings when applying these formulas (Theorem 4), to obtain the following results, which reveals a uniqueness property for the lattice extension of linear maps.

Corollary 6 *Under the hypothesis of Theorem 4, T is the unique lattice Lipschitz operator with associated function K that extends $T|_\Sigma$.*

Corollary 7 *Under the hypothesis of Theorem 4, if $T(0) = 0$ and E_0 is the set of eigenvectors of T, then the McShane and Whitney extensions of $T|_{E_0}$ are equal to T.*

Next we show the error bounds for the lattice extension formulas at a point $x \in E$ with respect to the original operator T. This expression is valid for lattice Lipschitz functions, and could be used to control the error committed when the reconstruction of the function is done from the information we have about it in a subset. This result is fundamental for the application of our extension tool in artificial intelligence.

Theorem 8 *Let $T : E \to E$ be a lattice Lipschitz operator with associated bounded function $K : \Omega \to \mathbb{R}$ and $E_0 \subseteq E$. Then, for any $x \in E$*

$$-2K \bigwedge\{|x - z| : z \in E_0\} \le (T|_{E_0})^M(x) - T(x) \le 0,$$
$$0 \le (T|_{E_0})^W(x) - T(x) \le 2K \bigwedge\{|x - z| : z \in E_0\}.$$

Proof For $T : E \to E$ as in the statement of the proposition, consider $T|_{E_0}$. If $x \in E$, applying that T is a lattice Lipschitz operator in the whole E,

$$\begin{aligned}(T|_{E_0})^W(x) - T(x) &= \bigwedge\{T(z) + K|x - z| : z \in E_0\} - T(x)\\ &= \bigwedge\{T(z) - T(x) + K|x - z| : z \in E_0\}\\ &\le \bigwedge\{|T(z) - T(x)| + K|x - z| : z \in E_0\}\\ &\le \bigwedge\{K|z - x| + K|x - z| : z \in E_0\}\\ &= 2K \bigwedge\{|z - x| : z \in E_0\}.\end{aligned}$$

In addition,

$$
\begin{aligned}
(T|_{E_0})^W(x) - T(x) &= \bigwedge\{T(z) - T(x) + K|x-z| : z \in E_0\} \\
&\geq \bigwedge\{-|T(z) - T(x)| + K|x-z| : z \in E_0\} \\
&\geq \bigwedge\{-K|z-x| + K|x-z| : z \in E_0\} = 0.
\end{aligned}
$$

The bounds for the McShane case can be proved by applying Remark 3. □

Remark 6 Theorem 8 can also be proved using Proposition 2 and Remark 4. Observe that the original map T is a suitable extension of $T|_{E_0}$ (with the same associate function). Thus, it is clear that for $x \in E$, $T^M(x) \leq T(x) \leq T^W(x)$. Also note that $|T^W(x) - T^M(x)| \leq 2K \bigwedge\{|x-z| : z \in E_0\}$, so $T(x)$ cannot differ from $T^M(x)$ and $T^W(x)$ more than $2K \bigwedge\{|x-z| : z \in E_0\}$.

Let us explain an example point by point to finish the paper. We will show the set of eigenvectors, the associate function, and the Whitney extension of the map.

Example 3 Consider the function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$
\phi(x, y) = \left(\frac{x^2}{1+x^2}, \frac{y^2}{1+y^2}\right), \quad (x, y) \in \mathbb{R}^2.
$$

The computation of the eigenvectors by means of the eigenvector equations provide the next formulas in terms of the eigenvalues λ; for $\lambda = 0$, the eigenvector is $(0, 0)$. The spectrum of the map is given by the values of $\lambda \in [-1/2, 1/2]$. The eigenvectors associated to each of these values are given by all the combination of values (x, y) for $\lambda \neq 0$ given by the formulas

$$
x = 0, \quad x = \frac{1}{2\lambda}\left(1 + \sqrt{1-4\lambda^2}\right) \quad \text{or} \quad x = \frac{1}{2\lambda}\left(1 - \sqrt{1-4\lambda^2}\right),
$$

and

$$
y = 0, \quad y = \frac{1}{2\lambda}\left(1 + \sqrt{1-4\lambda^2}\right) \quad \text{or} \quad y = \frac{1}{2\lambda}\left(1 - \sqrt{1-4\lambda^2}\right).
$$

Let us write E_0 for the set of all these eigenvectors. A graphical representation of this set can be found in Fig. 1.

In this case, the lattice structure is given by a function space defined on a two-point measurable space $\{w_1, w_2\}$. To show that this function is a lattice Lipschitz operator, consider $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and let ψ be the real function $\psi(t) = \frac{t^2}{1+t^2}$. Then, by the mean value theorem, there exists $\xi, \mu \in \mathbb{R}$ such that

$$
\begin{aligned}
|\phi(x_1, y_1) - \phi(x_2, y_2)| &= (|\psi(x_1) - \psi(x_2)|, |\psi(y_1) - \psi(y_2)|) \\
&= \left(|\psi'(\xi)| \cdot |x_1 - x_2|, |\psi'(\mu)| \cdot |y_1 - y_2|\right).
\end{aligned}
$$

A standard optimization procedure allows to obtain that $\sup_{t\in\mathbb{R}} |\psi'(t)| = \left|\psi'\left(\pm\frac{1}{\sqrt{3}}\right)\right| = \frac{3\sqrt{3}}{8}$. So the function that plays the role of the Lipschitz norm for the case of lattice Lipschitz functions, is the constant function

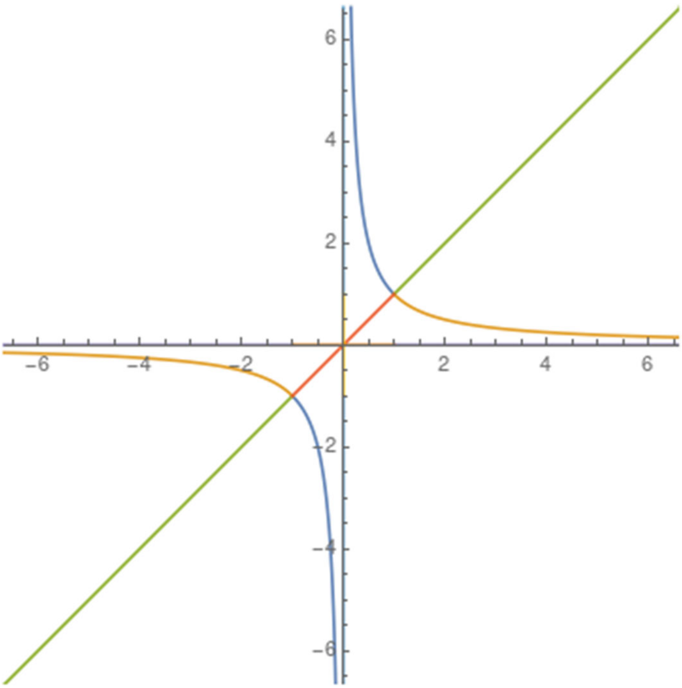

Fig. 1 Representation of the set of eigenvectors for the function ϕ. All the points in the curves and lines represented are eigenvectors, including the axis OX and OY

$$K(w_1) = K(w_2) = \frac{3\sqrt{3}}{8}.$$

Let us consider as subset for constructing the McShane and Whitney extensions the set E_0 of all the eigenvectors described above. We write only the Whitney formula. Using the parametrization provided by the eigenvalues λ and taking into account that the infimum can be computed separately for each coordinate, the formulas are, for every $(x_0, y_0) \in \mathbb{R}^2$,

$$\begin{aligned} P_1\big((\phi|_{E_0})^W(x_0, y_0)\big) &= \bigwedge_{\lambda \in [-\frac{1}{2}, \frac{1}{2}]} \Big(\lambda\, x(\lambda) + \frac{3\sqrt{3}}{8} \cdot \big|x(\lambda) - x_0\big|\Big) \wedge \frac{3\sqrt{3}}{8} \cdot |x_0| \\ &= \inf_{\lambda \in [-\frac{1}{2}, \frac{1}{2}]} \left\{ \frac{1}{2}\Big(1 \pm \sqrt{1 - 4\lambda^2}\Big) + \frac{3\sqrt{3}}{8} \cdot \left| \frac{1}{2\lambda}\Big(1 \pm \sqrt{1 - 4\lambda^2}\Big) - x_0 \right| \right\} \wedge \frac{3\sqrt{3}}{8} \cdot |x_0|, \end{aligned}$$

and

$$\begin{aligned} P_2\big((\phi|_{E_0})^W(x_0, y_0)\big) &= \bigwedge_{\lambda \in [-\frac{1}{2}, \frac{1}{2}]} \Big(\lambda\, y(\lambda) + \frac{3\sqrt{3}}{8} \cdot \big|y(\lambda) - y_0\big|\Big) \wedge \frac{3\sqrt{3}}{8} \cdot |y_0| \\ &= \inf_{\lambda \in [-\frac{1}{2}, \frac{1}{2}]} \left\{ \frac{1}{2}\Big(1 \pm \sqrt{1 - 4\lambda^2}\Big) + \frac{3\sqrt{3}}{8} \cdot \left| \frac{1}{2\lambda}\Big(1 \pm \sqrt{1 - 4\lambda^2}\Big) - y_0 \right| \right\} \wedge \frac{3\sqrt{3}}{8} \cdot |y_0|, \end{aligned}$$

where

$$(\phi|_{E_0})^W(x_0, y_0) = \big(P_1\big((\phi|_{E_0})^W(x_0, y_0)\big), P_2\big((\phi|_{E_0})^W(x_0, y_0)\big)\big).$$

As a consequence of Corollary 7, we know that this extension formula (and also the McShane formula) coincides with the original function. Note, however, that we can also use these formulas if we take a (proper) subset of eigenvectors. In case that such a subset does not contain the axis, the coincidence of the extension with the function need not occur.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data Availability It is not applicable

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