Received: 31 January 2021 | Revised: 17 October 2021 | Accepted: 6 January 2022
DOI: 10.1002/asjc.2788

RESEARCH ARTICLE

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Stability analysis and controller design for consensus in discrete-time networks with heterogeneous agent dynamics

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Abstract

In this paper, we study the heterogeneous consensus problem in directed networks consisting of first- and second-order agents that can only receive the position states of their neighbors. Necessary and sufficient conditions on the controller parameters are obtained in order to achieve consensus in the network. The mathematical expressions of the consensus equilibria are given for two different scenarios. Furthermore, we propose a systematic method for choosing controller parameters to ensure stability in a network of agents with heterogeneous dynamics. Several numerical examples are also provided to illustrate the theoretical results.

KEYWORDS
consensus, controller design, directed networks, heterogeneous dynamics, multi-agent systems

1 | INTRODUCTION

In the last two decades, multi-agent systems (MASs) have received substantial research attention in a variety of fields including sensor networks [1], computer networks [2], clock synchronization [3], distributed decision making [4], distributed optimization [5], social networks [6], load balancing in power systems [7], formation-containment control [8, 9], persistent monitoring [10], and robotics [11, 12]. The consensus problem was investigated for the networks with first-order [13–16] and second-order agent dynamics [17–20] extensively. However, the agents in the distributed network may not have the same dynamics in practical applications [21]. For instance, in a network of mobile robots, the robots may have different shapes and abilities and consequently different dynamics [22]. Consensus of the MASs in networks consisting of both first- and second-order agents has drawn considerable attention of the researchers [22–35].

The vast majority of the literature on the consensus problem with heterogeneous agent dynamics has considered continuous-time networks [21–32]. Zheng and Wang, Liu et al., and Yin et al. [23, 26, 30] investigated the problem in undirected networks. Zheng and Wang [23] proposed two consensus protocols and give sufficient conditions for the stability of these protocols. A consensus protocol is proposed in Liu et al. [26] when the parameters of the heterogeneous network are known. Furthermore, by using Lyapunov theory and Barbalat's lemma, an adaptive-PD controller was designed for a heterogeneous system with unknown parameters. Yin et al. examined heterogeneous networks with linear and nonlinear second-order agents and derived sufficient conditions for stability of the proposed protocols [30]. Feng et al. studied the heterogeneous consensus problem in directed networks and obtained conditions on the controller parameters which ensure stability [25]. Zheng and Wang considered networks consisting of first- and second-order agents with and without velocity measurements and showed that finite time consensus can be achieved provided that the graph is strongly connected [27]. As a special case of the heterogeneous consensus problem, the leader–follower consensus was studied by several authors [24, 28, 31, 32]. Zheng and Wang [24] considered the containment control problem as the application of the consensus algorithm for two different cases (first-order agents are leaders and the second-order agents are followers, or vice-versa). Shi et al. [32] investigated the containment control in MASs by converting the consensus problem into a convergence problem and giving some sufficient conditions for stability where the leaders are stationary and the followers have first- or second-order dynamics in the network. Du et al. examined fixed-time heterogeneous consensus in networks with nonlinear agents [28]. Han et al. utilized the output regulation approach to show that an output consensus protocol can be used for heterogeneous networks.

The heterogeneous consensus problem is also investigated in discrete-time networks [33–35]. Cheng et al. [33] studied networks with first- and second-order agents where the topology is switching and link failures exist in the network. Yan et al. [34] considered formation control problem in heterogeneous networks and developed a control protocol for this purpose. Zhao and Fei [35] examined heterogeneous networks with first- and second-order agents under certain assumptions of the graph topology. While some conditions on the controller parameters were obtained for heterogeneous consensus in the MAS, the existence of a set of stable controller parameters and a method for finding them were not investigated. Furthermore, the effect of controller parameters on the consensus equilibria was not considered in Zhao and Fei [35].

The heterogeneous consensus is extensively examined in the literature by utilizing output regulation theory [8, 29, 36, 37]. Wang et al. investigated the output-containment problem and proposed a distributed hybrid active controller for leader-following MASs with heterogeneous and linear agent dynamics [8]. In Han et al. [29], leader-following consensus problem of MASs is studied where the followers are able to determine the local controller parameters provided that the dynamics of the leader and the minimum eigenvalue of the Laplacian matrix are known by each agent. Chen et al. used the output regulation theory to design controllers that guarantee cluster consensus in a heterogeneous network. Yaghmaie et al.[37] formulated the output regulation in a graphical game framework and provided offline and online algorithms for the solution of the output regulation problem. While output regulation theory is widely used for heterogeneous consensus problem, the advantage of the control approach investigated in the paper is that it is always possible to find a set of stable controller parameters systematically and the computational complexity for finding them is quite low. Furthermore, output regulation theory approach is most suitable for leader-following consensus problem, whereas the methods proposed in the paper are applicable to MASs without a leader.

In this paper, we study the stability analysis and controller design problems for consensus algorithms in MASs consisting of first- and second-order agents. In particular, the stability of a control law consisting of two parameters is examined by using matrix theory and bilinear transformation. The stability conditions on the controller parameters are determined by applying Routh–Hurwitz criteria for characteristic polynomials with complex coefficients. By investigating the eigenstructure of the system matrix, the equilibrium states are expressed as a function of controller parameters. In order for the control law studied in the paper to be applicable in practice, lower and upper bounds on the controller parameters are given, and it is mathematically proven that such a choice guarantees stability.

The main contributions of the paper can be summarized as follows:

- Necessary and sufficient conditions on the controller parameters are obtained in order to achieve consensus in a directed network with heterogeneous agent dynamics.
- The mathematical expressions of the consensus equilibria are given for two different scenarios. It is proven that one of the controller parameters does not affect the consensus equilibria.
- In order to show that the set of controller parameters that satisfy the stability conditions is nonempty, a systematic method has been proposed for choosing controller parameters. With this systematic method, not only the stability of the consensus algorithm is ensured, but also the consensus equilibria can be adjusted by the control engineer under given assumptions.

The remainder of the paper is organized as follows. Necessary preliminaries and the heterogeneous consensus model is given in Section 2. The stability analysis of the consensus algorithm is provided in Section 3. In Section 4, we propose a systematic method for choosing controller parameters which ensures consensus. In Section 5, numerical results are given to illustrate the theoretical results. Finally, we conclude the paper in Section 6.

2 | MATHEMATICAL PRELIMINARIES AND HETEROGENEOUS CONSENSUS MODEL

2.1 | Mathematical preliminaries

We represent the information flow in the network by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{v_1, \dots, v_n\}$ is the finite set of vertices and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges representing the links between the vertices. We denote by $A = [a_{ij}] \in R^{n\times n}$ a non-negative weighted adjacency matrix associated with the graph $\mathcal{G}$. If $(v_j, v_i) \in \mathcal{E}$ and $j \neq i$, a_{ij} is positive; and for all other cases, $a_{ij} = 0$. The Laplacian matrix $L = [l_{ij}] \in R^{n\times n}$ of a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined as

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{\substack{k=1 \\ k\neq i}}^{n} a_{ik}, & i = j. \end{cases}$$

Let $\mathcal{I}_s = \{1, \dots, n\}$ and $\mathcal{I}_f = \{n+1, \dots, m\}$ denote the set of indices of the first-order and second-order agents in the graph $\mathcal{G}$, respectively. We say that a directed path between v_1 and v_n exists if there is a finite sequence of ordered edges of the form $(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n)$ such that $(v_i, v_{i+1}) \in \mathcal{E}$, $i = 1, \dots, n-1$. A directed graph consists of a spanning tree if each vertex has one parent vertex except for one vertex that has a directed path to the rest of the vertices in the graph. We denote by I_m the identity matrix of dimension $m \times m$ and by $0_{m\times n}$ the $m \times n$ zero matrix. Let $e_n = [1, \dots, 1]^T$ denote the vector of ones with length n. Let $Re(\cdot)$ and $Im(\cdot)$, respectively, denote the real and the imaginary parts of their arguments.

2.2 | Dynamical model of heterogeneous consensus

Consider a multi-agent network consisting of n agents where the dynamics of the agents are given by

$$\begin{aligned} &x_i(k+1) = x_i(k) + u_i(k), && i \in \mathcal{I}_f, \\ &\begin{aligned} x_j(k+1) &= x_j(k) + v_j(k) \\ v_j(k+1) &= v_j(k) + u_j(k) \end{aligned}, && j \in \mathcal{I}_s, \end{aligned} \tag{1}$$

where $x_i(k) \in R$, $v_i(k) \in R$, and $u_i(k) \in R$ are the position state, velocity state, and the input signal of agent i at time step k, respectively. Note that the difference equations given in (1) are discrete-time integrator dynamics that are analogous to the continuous-time dynamics $\dot{x}_i(t) = u_i(t)$ $(i \in \mathcal{I}_f)$, $\dot{x}_j(t) = v_j(t)$, and $\dot{v}_j(t) = u_j(t)$ $(j \in \mathcal{I}_s)$. In order to obtain the discrete-time equivalent of a continuous-time algorithm, the sampling time must be chosen properly since an improper choice may lead to instability. For a detailed discussion on discretization methods and the choice of sampling time, we refer the reader to Santina et al. [38].

In this paper, we consider the control input of the agents as

$$\begin{aligned} u_i(k) &= k_1 \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)) \text{ for all } i \in \mathcal{I}_f, \\ u_j(k) &= k_1 \sum_{l\in\mathcal{N}_j} a_{jl}(x_l(k) - x_j(k)) - k_2 v_j(k) \text{ for all } j \in \mathcal{I}_s, \end{aligned} \tag{2}$$

where $k_1, k_2 > 0$ are controller parameters.[1] Note here that the control law (2) is also considered in Zhao and Fei [35] and suitable for systems where the agents are unable to measure the velocities of their neighbors.

In this paper, we are interested in the consensus problem in a network of agents with the dynamics (1) whose definition is given as follows.

Definition 1 (Consensus)**.** We say that the heterogeneous system with dynamics (1) achieves consensus if

$$\begin{aligned} &\lim_{k\to\infty} ||x_i(k) - x_j(k)|| = 0, \text{ for all } i, j \in \mathcal{I}_f \cup \mathcal{I}_s, \\ &\lim_{k\to\infty} ||v_i(k) - v_j(k)|| = 0, \text{ for all } i, j \in \mathcal{I}_s, \end{aligned} \tag{3}$$

where $||\cdot||$ denotes the Euclidean norm.

Note from Definition 1 that we require the position states of all agents and the velocity states of the second-order agents to converge to the same functions as $t \to \infty$.

For analysis purposes, we introduce the following notation to express the dynamics (1) and the control inputs (2) in matrix form. Let $y(k) = [X_1(k)^T, X_2(k)^T, V_1(k)^T]^T$ be the augmented state vector where $X_1(k) = [x_1(k), \dots, x_m(k)]^T$, $X_2(k) = [x_{m+1}(k), \dots, x_n(k)]^T$ and $V_1(k) = [v_1(k), \dots, v_m(k)]^T$. Then, we can express the system with dynamics (1) and controller inputs (2) as

$$y(k+1) = \Gamma y(k), \tag{4}$$

where

$$\Gamma = \begin{bmatrix} I_m & 0_{m\times(n-m)} & I_m \\ -k_1 L_{21} & I_{n-m} - k_1 L_{22} & 0_{(n-m)\times m} \\ -k_1 L_{11} & -k_1 L_{12} & (1-k_2) I_m \end{bmatrix}$$

$$\text{and } L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}_{n\times n},$$

with

$$L_{11} = \begin{bmatrix} l_{11} & \dots & l_{1m} \\ \vdots & \dots & \vdots \\ l_{m1} & \dots & l_{mm} \end{bmatrix}_{m\times m}, L_{12} = \begin{bmatrix} l_{1(m+1)} & \dots & l_{1n} \\ \vdots & \dots & \vdots \\ l_{m(m+1)} & \dots & l_{mn} \end{bmatrix}_{m\times(n-m)},$$

$$L_{21} = \begin{bmatrix} l_{(m+1)1} & \dots & l_{(m+1)m} \\ \vdots & \dots & \vdots \\ l_{n1} & \dots & l_{nm} \end{bmatrix}_{(n-m)\times m},$$

$$L_{22} = \begin{bmatrix} l_{(m+1)(m+1)} & \dots & l_{(m+1)n} \\ \vdots & \dots & \vdots \\ l_{n(m+1)} & \dots & l_{nn} \end{bmatrix}_{(n-m)\times(n-m)}.$$

Throughout the paper, we make the following assumptions.

Assumption 1.

- The underlying graph of L has a spanning tree.

[1] We prove in Theorem 1 that the controller parameters have to be positive.

- The information flow between the agents with first-order dynamics and second-order dynamics is one directional, i.e., either $L_{12} = 0_{m\times(n-m)}$ or $L_{21} = 0_{(n-m)\times m}$

Remark 1. Assumption 1(i) implies that the multiplicity of the eigenvalue 0 of L is 1 and it is a necessary condition since multiple groups will be formed when the network graph does not contain a spanning tree [14]. While Assumption 1(ii) is not necessary for achieving consensus in a heterogeneous network, the literature on heterogeneous consensus has similar assumptions (see for instance Zhao and Fei and Zhao et al. [35, 39] for heterogeneous networks consisting of first- and second-order agents, and Zhao and Fei [40] for those consisting of second and third order agents). This will be a key assumption in the stability analysis of (4).

3 | STABILITY ANALYSIS

In this section, we investigate the stability properties of the system (4) and obtain necessary and sufficient conditions on the controller parameters k_1 and k_2 to guarantee consensus in terms of Definition 1. Furthermore, we derive the mathematical expressions of the consensus equilibria as functions of the initial states and the controller parameters. To this end, we utilize the following lemmas.

Lemma 1. *The real parts of the nonzero eigenvalues of L_{11} and L_{22} are positive.*

Lemma 1 follows from Gershgorin's theorem [41]. The following lemma states the stability conditions of a second-order characteristic polynomial with complex coefficients, which will be utilized to prove Theorem 1.

Lemma 2. *[42] The roots of the polynomial $f(\lambda) = \lambda^2 + a_1\lambda + a_2$ has negative real parts if and only if*

$$Re(a_1) > 0 \ \ and \ \ Re(a_1)Re(a_1\bar{a}_2) - Im^2(a_2) > 0,$$

where a_1 and a_2 are complex numbers and $\bar{a}_2$ is the complex conjugate of a_2.

The following theorem states necessary and sufficient conditions on the controller parameters to guarantee consensus in a heterogeneous network.

Theorem 1. *The MAS with dynamics (1) and the controller inputs (2) achieves consensus under Assumption 1 if and only if there exist controller parameters $k_1, k_2 > 0$ such that*

$$k_1 < \frac{2Re(\mu_{2j})}{|\mu_{2j}|^2}, \tag{5}$$

$$k_2 > \frac{k_1|\mu_{1i}|^2}{Re(\mu_{1i})}, \tag{6}$$

$$\begin{aligned} &k_1\left(\frac{k_2}{k_1}\frac{Re(\mu_{1i})}{|\mu_{1i}|^2} - 1\right)^2\left((2-k_2)\frac{2Re(\mu_{1i})}{|\mu_{1i}|^2} + k_1\right) \\ &> (2-k_2)\left(2 - \frac{2k_2^2Re(\mu_{1i})}{k_1|\mu_{1i}|^2} + k_2\right)\frac{Im^2(\mu_{1i})}{|\mu_{1i}|^4}, \end{aligned} \tag{7}$$

hold where μ_{1i} and μ_{2j} denote the nonzero eigenvalues of L_{11} and L_{22}, respectively.

Proof. For the network to achieve consensus, we need all eigenvalues of Γ to be inside the unit circle except one eigenvalue at 1. We express the characteristic polynomial of Γ as

$$\begin{aligned} \det(\lambda I_{n+m} - \Gamma) &= \det \\ &\left(\begin{bmatrix} (\lambda-1)I_m & 0_{m\times(n-m)} & -I_m \\ k_1L_{21} & (\lambda-1)I_{n-m} + k_1L_{22} & 0_{(n-m)\times m} \\ k_1L_{11} & k_1L_{12} & (\lambda-(1-k_2))I_m \end{bmatrix}\right) \\ &= \det((\lambda-(1-k_2))I_m) \\ &\times \det\left(\begin{bmatrix} (\lambda-1)I_m & 0_{m\times(n-m)} \\ k_1L_{21} & (\lambda-1)I_{n-m} + k_1L_{22} \end{bmatrix}\right. \\ &\left. + \begin{bmatrix} I_m \\ 0_{(n-m)\times m} \end{bmatrix} \frac{1}{\lambda-(1-k_2)} I_m [k_1L_{11} \ \ k_1L_{12}]\right) \\ &= (\lambda-(1-k_2))^m \det \\ &\left(\begin{bmatrix} (\lambda-1)I_m + \frac{k_1}{\lambda-(1-k_2)}L_{11} & \frac{k_1}{\lambda-(1-k_2)}L_{12} \\ k_1L_{21} & (\lambda-1)I_{n-m} + k_1L_{22} \end{bmatrix}\right). \end{aligned}$$

From Assumption 1(ii), either $L_{12} = 0_{m\times(n-m)}$ or $L_{21} = 0_{(n-m)\times m}$ and therefore we have

$$\begin{aligned} &\det(\lambda I_{n+m} - \Gamma) = (\lambda-(1-k_2))^m \det \\ &\left((\lambda-1)I_m + \frac{k_1}{\lambda-(1-k_2)}L_{11}\right)\det((\lambda-1)I_{n-m} + k_1L_{22}) \\ &= \prod_{i=1}^{m}\left(\lambda^2 + (k_2-2)\lambda + 1 - k_2 + k_1\bar{\mu}_{1i}\right)\prod_{j=1}^{n-m}\left(\lambda - (1-k_1\bar{\mu}_{2j})\right), \end{aligned}$$

where $\bar{\mu}_{1i}$ and $\bar{\mu}_{2j}$ denote the eigenvalues of L_{11} and L_{22}, respectively. From Assumption 1(i), the multiplicity of the eigenvalue 0 of L is 1. Furthermore, $\sigma(L) = \sigma(L_{11}) \cup \sigma(L_{22})$ from Assumption 1(ii). If $L_{12} = 0_{m\times(n-m)}$, we conclude that 0 is an eigenvalue of L_{11} and if $L_{21} = 0_{(n-m)\times m}$, 0 is an eigenvalue of L_{22}.

Note that the system (4) achieves consensus if and only if $g_1(\lambda) = \lambda^2 + (k_2-2)\lambda + 1 - k_2 + k_1\mu_{1i}$ and $h_1(\lambda) = \lambda - (1 - k_1\mu_{2j})$ are Schur stable where μ_{1i} and μ_{2j} are nonzero eigenvalues of L_{11} and L_{22}, respectively. $h_1(\lambda)$ is Schur stable if and only if $|1 - k_1\mu_{2j}| < 1$ or equivalently

$$k_1^2|\mu_{2j}|^2 < 2k_1Re(\mu_{2j}). \tag{8}$$

We now show by contradiction that k_1 must be positive. Suppose that $k_1 < 0$. Then, we have $k_1 > 2Re(\mu_{2j})/|\mu_{2j}|^2 > 0$ which leads to a contradiction.

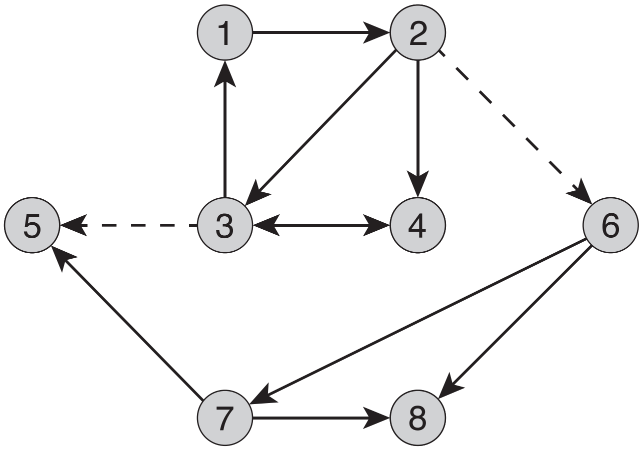

FIGURE 1 The directed graph under consideration in Example 1

Therefore, $k_1 > 0$, and (8) can be rewritten as (5). We now need to show Schur stability of $g_1(\lambda)$. Note that Lemma 2 states the conditions for Hurwitz stability of a second-order polynomial with complex coefficients. Since $\bar{\mu}_{1i}$ may be complex, we apply the bilinear transformation $\sigma = \varphi(\lambda) = \frac{\lambda+1}{\lambda-1}$ to $g_1(\lambda)$ to get

$$\begin{aligned}\theta_1(\sigma) &= (\sigma-1)^2 g_1\left(\frac{\sigma+1}{\sigma-1}\right)\\ &= (\sigma-1)^2\left(\left(\frac{\sigma+1}{\sigma-1}\right)^2 + (k_2-2)\left(\frac{\sigma+1}{\sigma-1}\right)\right.\\ &\quad \left. +1 - k_2 + k_1\mu_{1i}\right)\\ &= k_1\mu_{i1}\sigma^2 + (2k_2 - 2k_1\mu_{1i})\sigma - 2k_2 + k_1\mu_{1i} + 4.\end{aligned}$$

Then the polynomial

$$\vartheta_1 = \frac{\theta_1(\sigma)}{k_1\mu_{1i}} = \sigma^2 + \frac{2k_2 - 2k_1\mu_{1i}}{k_1\mu_{1i}}\sigma + \frac{-2k_2 + k_1\mu_{1i} + 4}{k_1\mu_{1i}} \tag{9}$$

is Hurwitz stable if and only if $g_1(\lambda)$ is Schur stable. From Lemma 2, we conclude that (9) is Hurwitz stable if and only if the following inequalities hold:

$$\frac{2k_2}{k_1}\frac{Re(\mu_{1i})}{|\mu_{1i}|^2} - 2 > 0, \tag{10}$$

$$\begin{aligned}&\left(\frac{2k_2}{k_1}\frac{Re(\mu_{1i})}{|\mu_{1i}|^2} - 2\right)\left(\left(\frac{2k_2}{k_1}\frac{Re(\mu_{1i})}{|\mu_{1i}|^2} - 2\right)\right.\\ &\left.\left(\frac{4-2k_2}{k_1}\frac{Re(\mu_{1i})}{|\mu_{1i}|^2} + 1\right) + \frac{4k_2(2-k_2)}{k_1^2}\frac{Im^2(\mu_{1i})}{|\mu_{1i}|^4}\right)\\ &> \frac{4(2-k_2)^2}{k_1^2}\frac{Im^2(\mu_{1i})}{|\mu_{1i}|^4}.\end{aligned} \tag{11}$$

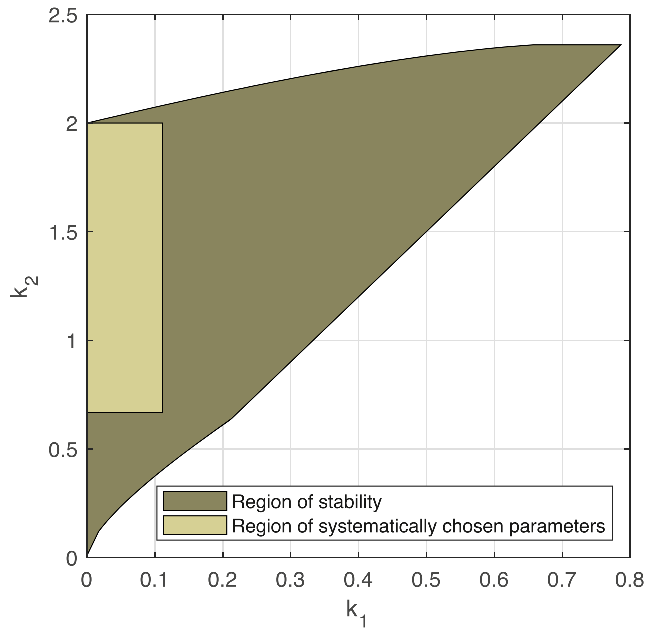

FIGURE 2 The region of stability and the region of systematically chosen controller parameters in Example 1

Since $k_1 > 0$, (10) and (6) are equivalent. Furthermore, (6) implies that $k_2 > 0$. Finally, we multiply (11) by $k_1^2/4$ to obtain (7) and conclude the proof. □

Remark 2. Theorem 1 states necessary and sufficient conditions on the controller parameters to ensure consensus. In this paper, we further aim to address the following questions: (i) If and when consensus is achieved, what would be the consensus equilibria, and how do the controller parameters affect the consensus equilibria? (ii) Does there always exist controller parameter for a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ such that the heterogeneous network achieves consensus? (iii) If so, how to obtain such parameters systematically?

The following lemma will be utilized to prove Theorem 2.

Lemma 3. *Let A be an $n \times n$ grounded Laplacian matrix (i.e., a matrix formed by removing certain rows and columns of a Laplacian matrix) and B be an $n \times m$ non-positive matrix. Then, the equality $[B \ \ A]e_{m+n} = 0_{n\times1}$ (or equivalently $[A \ \ B]e_{m+n} = 0_{n\times1}$) implies $-A^{-1}Be_m = e_n$.*

Proof. By multiplying both sides of $[B \ \ A]e_{m+n} = 0_{n\times1}$ by $-A^{-1}$, we have $-A^{-1}Be_m = e_n$. □

Theorem 2. *Suppose that the MAS with dynamics (1) and the controller inputs (2) achieves consensus under Assumption 1. Then,*

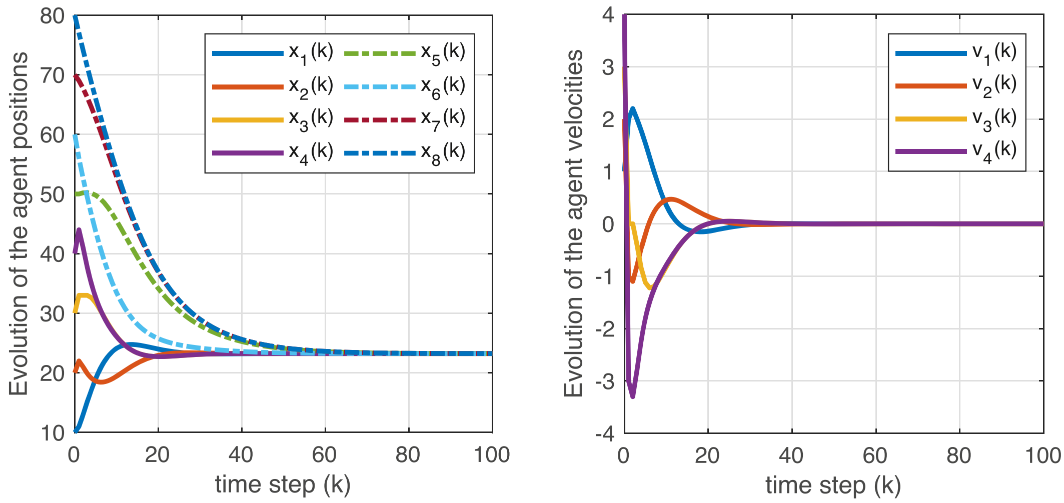

FIGURE 3 The evolution of the agent states for the choice of $k_1 = 0.1$ and $k_2 = 1$ in Example 1

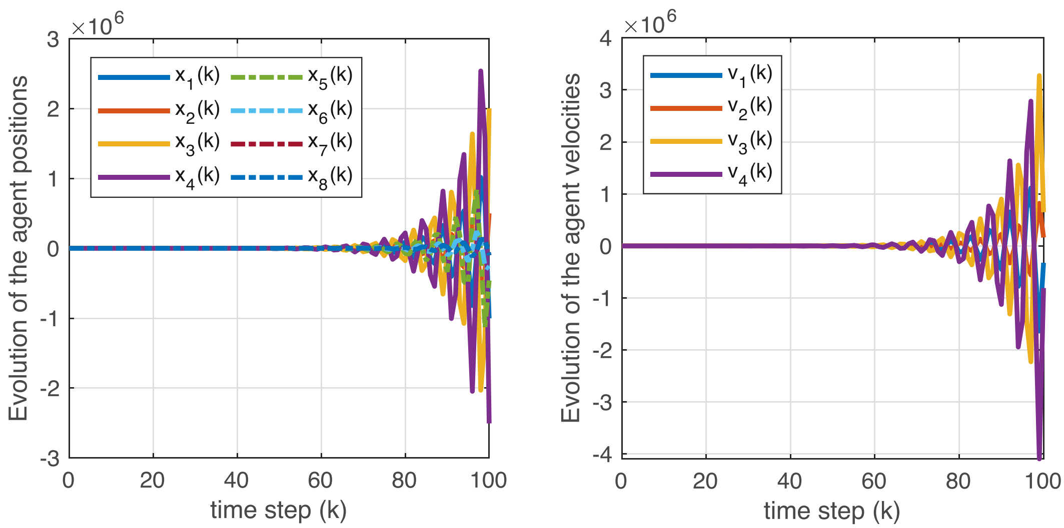

FIGURE 4 The evolution of the agent states for the choice of $k_1 = 0.6$ and $k_2 = 1.5$ in Example 1

- *If* $L_{12} = 0_{m\times(n-m)}$, *then* $\lim_{k\to\infty} y(k) \to \left(k_2\bar{v}_1^T X_1(0) + \bar{v}_1^T V_1(0)\right)\begin{bmatrix} e_n \\ 0_{m\times 1}\end{bmatrix}$,
- *If* $L_{21} = 0_{(n-m)\times m}$, *then* $\lim_{k\to\infty} y(k) \to \begin{bmatrix} \bar{v}_2^T X_2(0) e_n \\ 0_{m\times 1}\end{bmatrix}$,

where $\bar{v}_1$ *and* $\bar{v}_2$*, respectively, are the left eigenvectors of* L_{11} *and* L_{22} *corresponding to the eigenvalue 0 such that* $e_m^T\bar{v}_1 = 1/k_2$ *and* $e_{n-m}^T\bar{v}_2 = 1$ *holds.*

Proof. The multiplicity of the eigenvalue 1 of Γ is 1. Therefore, the Jordan form of Γ can be expressed as $J_1 = \begin{bmatrix} 1 & 0_{1\times(n+m-1)} \\ 0_{(n+m-1)\times 1} & \bar{J}_1 \end{bmatrix}$ where $\bar{J}_1$ is the Jordan block matrix corresponding to the eigenvalues of Γ that are less than one.

Case (i): Suppose that $L_{12} = 0_{m\times(n-m)}$ and $L_{21} \neq 0_{(n-m)\times m}$ so that L_{22} is invertible and L_{11} is a Laplacian matrix. Then, $\xi_1 = [e_m^T, -(L_{22}^{-1}L_{21}e_m)^T, 0_{m\times 1}^T]^T$ is a right eigenvector of Γ corresponding to the eigenvalue 1 where $e = [1, \ldots, 1]^T$. Note from Lemma 3 that ξ_1 can be rewritten as $\xi_1 = [e_n^T, 0_{m\times 1}^T]^T$. Let $\eta_1 = [k_2\bar{v}_1^T, 0_{(n-m)\times 1}^T, \bar{v}_1^T]^T$ be the left eigenvector of Γ corresponding to the eigenvalue 1 where $\bar{v}_1$ is the left eigenvector of L_{11} corresponding to the eigenvalue 0 satisfying $e_m^T\bar{v}_1 = 1/k_2$. Note that we have $\xi_1^T\eta_1 = 1$. Let ξ_i and η_i $(i = 2, \ldots, n+m)$ denote the right and left eigenvectors of Γ corresponding to the eigenvalues that are less than 1, respectively. Then, Γ can be rewritten as

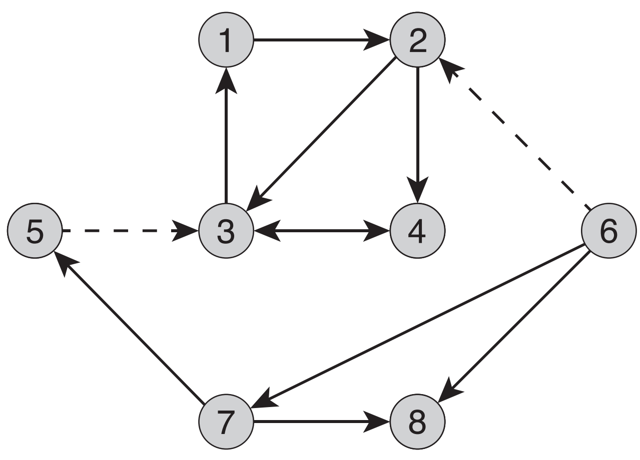

FIGURE 5 The directed graph under consideration in Example 2

$$\Gamma = [\xi_1, \dots, \xi_{n+m}] \begin{bmatrix} 1 & 0_{1\times(n+m-1)} \\ 0_{(n+m-1)\times 1} & \bar{J}_1 \end{bmatrix} \begin{bmatrix} \eta_1^T \\ \vdots \\ \eta_{n+m}^T \end{bmatrix}. \tag{12}$$

From (4), we have $y(k) = \Gamma^k y(0)$. Let $\bar{y} = \left(k_2 \bar{v}_1^T X_1(0) + \bar{v}_1^T V_1(0)\right) \begin{bmatrix} e_n \\ 0_{m\times 1} \end{bmatrix}$. Since $\lim_{k\to\infty} \bar{J}_1^k = 0_{(n+m-1)\times(n+m-1)}$, we can compute $\lim_{k\to\infty} ||y(k) - \bar{y}||$ as

$$\begin{aligned}
\lim_{k\to\infty} ||y(k) - \bar{y}|| &= \lim_{k\to\infty} \left\| \Gamma^k \begin{bmatrix} X_1(0) \\ X_2(0) \\ V_1(0) \end{bmatrix} - \bar{y} \right\| \\
&= \lim_{k\to\infty} \left\| [\xi_1, \dots, \xi_{n+m}] \begin{bmatrix} 1 & 0_{1\times(n+m-1)} \\ 0_{(n+m-1)\times 1} & \bar{J}_1^k \end{bmatrix} \begin{bmatrix} \eta_1^T \\ \vdots \\ \eta_{n+m}^T \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_2(0) \\ V_1(0) \end{bmatrix} - \bar{y} \right\| \\
&= \left\| \xi_1 \eta_1^T \begin{bmatrix} X_1(0) \\ X_2(0) \\ V_1(0) \end{bmatrix} - \bar{y} \right\| \\
&= \left\| \begin{bmatrix} k_2 e_m \bar{v}_1^T & 0_{m\times(n-m)} & e_m \bar{v}_1^T \\ k_2 e_{n-m} \bar{v}_1^T & 0_{(n-m)\times(n-m)} & e_{n-m} \bar{v}_1^T \\ 0_{m\times m} & 0_{m\times(n-m)} & 0_{m\times m} \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_2(0) \\ V_1(0) \end{bmatrix} - \bar{y} \right\| \\
&= \left\| \left(k_2 \bar{v}_1^T X_1(0) + \bar{v}_1^T V_1(0)\right) \begin{bmatrix} e_n \\ 0_{m\times 1} \end{bmatrix} - \bar{y} \right\| \\
&= 0.
\end{aligned}$$

Case (ii): Now suppose that $L_{21} = 0_{(n-m)\times m}$ and $L_{12} \neq 0_{m\times(n-m)}$ for which L_{11} is invertible and L_{22} is a Laplacian matrix. Then, $\xi_1 = [-(L_{11}^{-1} L_{12} e_{n-m})^T, e_{n-m}^T, 0_{m\times 1}^T]^T$ is a right eigenvector of Γ corresponding to the eigenvalue 1. Note from Lemma 3 that ξ_1 can be rewritten as $\xi_1 = [e_n^T, 0_{m\times 1}^T]^T$. Let $\eta_1 = [0_{m\times 1}^T, \bar{v}_2^T, 0_{m\times 1}^T]^T$ denote the left eigenvector of Γ corresponding to the eigenvalue 1 where $\bar{v}_2$ is the left eigenvector of L_{22} corresponding to the eigenvalue 0 satisfying $e_{n-m}^T \bar{v}_2 = 1$. Note that we have $\xi_1^T \eta_1 = 1$. Let ξ_i and η_i ($i = 2, \dots, n+m$) denote the right and left eigenvectors of Γ corresponding to the eigenvalues that are less than 1, respectively. Let $\bar{y} = \begin{bmatrix} \bar{v}_2^T X_2(0) e_n \\ 0_{m\times 1} \end{bmatrix}$. Following the same procedure as in Case (i), we have

$$\begin{aligned}
\lim_{k\to\infty} ||y(k) - \bar{y}|| &= \lim_{k\to\infty} \left\| \Gamma^k \begin{bmatrix} X_1(0) \\ X_2(0) \\ V_1(0) \end{bmatrix} - \bar{y} \right\| \\
&= \left\| \xi_1 \eta_1^T \begin{bmatrix} X_1(0) \\ X_2(0) \\ V_1(0) \end{bmatrix} - \bar{y} \right\| \\
&= \left\| \begin{bmatrix} 0_{m\times m} & e_m \bar{v}_2^T & 0_{m\times m} \\ 0_{(n-m)\times m} & e_{n-m} \bar{v}_2^T & 0_{(n-m)\times m} \\ 0_{m\times m} & 0_{m\times(n-m)} & 0_{m\times m} \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_2(0) \\ V_1(0) \end{bmatrix} - \bar{y} \right\| \\
&= \left\| \begin{bmatrix} \bar{v}_2^T X_2(0) e_n \\ 0_{m\times 1} \end{bmatrix} - \bar{y} \right\| \\
&= 0,
\end{aligned}$$

which concludes the proof. □

Remark 3. Theorem 2 implies that the consensus value of the MAS is independent of k_1 in both cases. Furthermore, when the second-order agents do not receive information from the first-order agents, the consensus value of the MAS depends only on the initial values of the second-order agents.

Remark 4. The analysis given in the proof of Theorem 2 show that the equilibrium states depend on the controller parameter k_2 in Case (i). It will be further shown in Section 4 that the parameter k_2 can be chosen in a fixed range to ensure stability of the system. This allows the control engineer to design the controller parameters that yield desired equilibrium states.

4 | CONTROLLER PARAMETER SELECTION IN HETEROGENEOUS NETWORKS

In this section, we provide a systematic method for choosing controller parameters which guarantees consensus in a heterogeneous network.

Theorem 3. *Let* ϵ_1 *and* ϵ_2 *be parameters such that* $0 < \epsilon_1 \leq \min_i \frac{Re(\mu_{1i})}{|\mu_{1i}|^2}$ *and* $0 < \epsilon_2 \leq \min_j \frac{Re(\mu_{2j})}{|\mu_{2j}|^2}$ *hold where* μ_{1i} *and* μ_{2j} *are nonzero eigenvalues of* L_{11} *and* L_{22}*, respectively. Then, the choice of* $0 < k_1 < \min\left\{\frac{\epsilon_1}{3}, \epsilon_2\right\}$ *and* $\frac{2}{3} < k_2 < 2$ *satisfies the conditions of Theorem 1.*

Proof. The inequality (5) directly holds since we have

$$0 < k_1 < \epsilon_2 \le \min_j \frac{Re(\mu_{2j})}{|\mu_{2j}|^2} < \min_j \frac{2Re(\mu_{2j})}{|\mu_{2j}|^2} \le \frac{2Re(\mu_{2j})}{|\mu_{2j}|^2}$$

for all μ_{2j}. Furthermore, (6) is satisfied since

$$\frac{k_1|\mu_{1i}|^2}{Re(\mu_{1i})} = \frac{k_1}{\frac{Re(\mu_{1i})}{|\mu_{1i}|^2}} \le \frac{k_1}{\epsilon_1} < \frac{\epsilon_1/3}{\epsilon_1} = \frac{1}{3},$$

and the choice of $\frac{2}{3} < k_2 < 2$ yields

$$k_2 > \frac{2k_1|\mu_{1i}|^2}{Re(\mu_{1i})}. \tag{13}$$

In order to show that (7) holds, we need the following claim.

Claim: $2 - \frac{2k_2^2 Re(\mu_{1i})}{k_1|\mu_{1i}|^2} + k_2 < 0$ holds since $2 - \frac{2k_2^2 Re(\mu_{1i})}{k_1|\mu_{1i}|^2} + k_2 < 2 - 4k_2 + k_2 = 2 - 3k_2 < 0$ by inequality (13) and the choice of $\frac{2}{3} < k_2 < 2$.

Finally, we conclude that (7) holds since the left-hand side of (7) is positive and the right-hand side is non-positive due to the claim. □

Remark 5. Theorem 3 provides a systematic method for choosing controller parameters which ensures the stability conditions given in Theorem 1. The positive parameters ϵ_1 and ϵ_2 can be determined by utilizing the known lower bounds in the literature (see for instance Berman and Zhang [43]).

Remark 6. Note from Theorem 2 that when $L_{21} = 0_{m\times(n-m)}$, the equilibrium states of the network depend on the parameter k_2 while they are independent of k_1. In this case, we conclude from Theorem 3 that the consensus values of the multi-agent network can be adjusted by the controller designer by choosing k_2 properly in the range $\frac{2}{3} < k_2 < 2$ without any knowledge of the network graph, provided that k_1 is small enough.

5 | NUMERICAL RESULTS

In this section, we verify the theoretical results given in Sections 4 by considering two cases investigated in the paper: (i) $L_{12} = 0_{m\times(n-m)}$ and (ii) $L_{21} = 0_{(n-m)\times m}$.

Example 1. (Case[i]: $L_{12} = 0_{m\times(n-m)}$) Consider the directed network of eight agents as depicted in Figure 1 where the sets of indices of the first- and second-order agents are given by $\mathcal{I}_f = \{5,6,7,8\}$ and $\mathcal{I}_s = \{1,2,3,4\}$, respectively.

Note that the links between the agents with the same order dynamics are illustrated by solid arrows whereas the links between first- and second-order agents are shown by dashed arrows. From Theorem 1, the stability region for the controller parameters and the region of systematic controller parameter choices are illustrated in Figure 2.

As can be seen from the figure, systematic choices correspond to a rectangular region inside the stability region in the $k_1 - k_2$ space. For the initial conditions $x_i(0) = 10i$ $(i \in \{1, \ldots, 8\})$, $v_i(0) = i$ $(i \in \{1,2,3,4\}$ and the choice of controller parameters $k_1 = 0.1$ and $k_2 = 1$ which satisfies the conditions of Theorem 3, the evolution of the agent states is given in Figure 3. Since the left eigenvector corresponding to the eigenvalue 0 of L_{11} is $\bar{v}_1 = \frac{1}{9}[3,\ 3,\ 2,\ 1]^T$, Figure 3 verifies that the consensus equilibria are computed by $\lim_{k\to\infty} \left(k_2\bar{v}_1^T X_1(0) + \bar{v}_1^T V_1(0)\right) \begin{bmatrix} e_n \\ 0_{m\times 1}\end{bmatrix} = \frac{209}{9}[e_n^T,\ 0_{m\times1}^T]^T$.

On the other hand, for the choice of $k_1 = 0.6$ and $k_2 = 1.5$ which is not inside the stability region, the agents in the network do not achieve consensus as illustrated in Figure 4.

Example 2. (Case[ii]: $L_{21} = 0_{(n-m)\times m}$) Consider the directed network of eight agents as depicted in Figure 5 where the sets of indices of the first- and second-order agents are given by $\mathcal{I}_f = \{5,6,7,8\}$ and $\mathcal{I}_s = \{1,2,3,4\}$, respectively. Note that the communication between the agents with same order dynamics is illustrated by solid arrows whereas communication

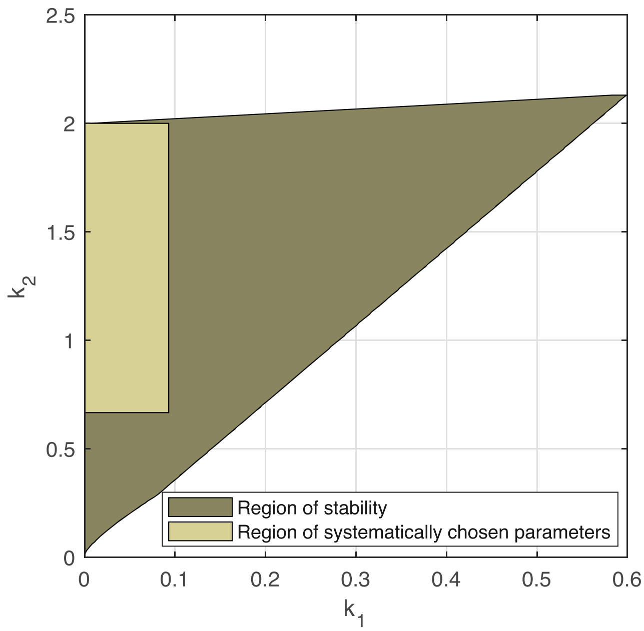

FIGURE 6 The region of stability and the region of systematically chosen controller parameters in Example 2

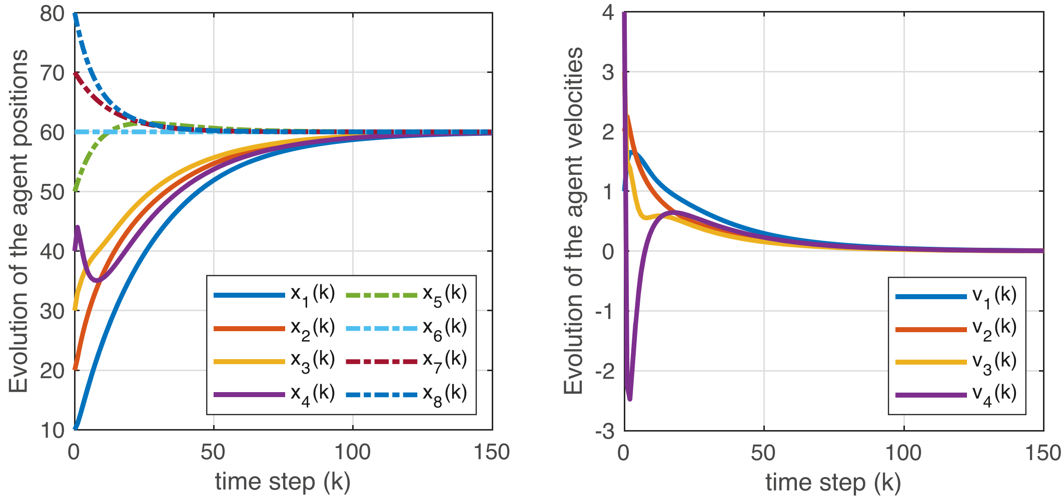

FIGURE 7 The evolution of the agent states for the choice of $k_1 = 0.075$ and $k_2 = 1$ in Example 2

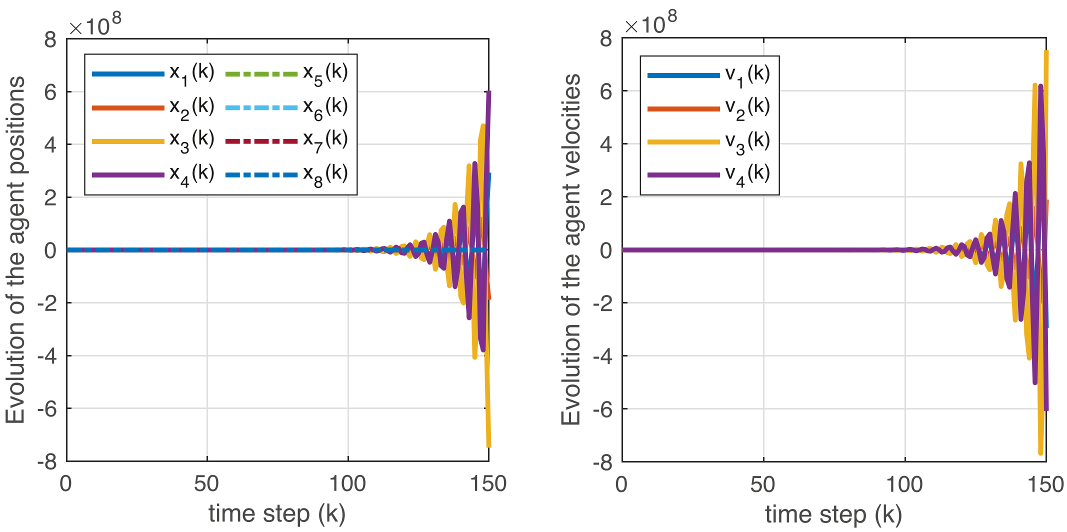

FIGURE 8 The evolution of the agent states for the choice of $k_1 = 0.5$ and $k_2 = 1.5$ in Example 2

between first- and second-order agents are shown by dashed arrows.

From Theorem 1, the stability region for the controller parameters and the region of systematic controller parameter choices are illustrated in Figure 6.

For the initial conditions $x_i(0) = 10i$ ($i \in \{1, \dots, 8\}$), $v_i(0) = i$ ($i \in \{1, 2, 3, 4\}$ and the choice of controller parameters $k_1 = 0.075$ and $k_2 = 1$ which satisfies the conditions of Theorem 3, the evolution of the agent states is given in Figure 7. Since the left eigenvector corresponding to the eigenvalue 0 of L_{22} is $\bar{v}_2 = [0, 1, 0, 0]^T$, Figure 7 verifies that the consensus equilibria are computed by $\lim_{k\to\infty} \begin{bmatrix} \bar{v}_2^T X_2(0) e_n \\ 0_{m\times 1} \end{bmatrix} = 60 \, [e_n^T, \, 0_{m\times 1}^T]^T$.

On the other hand, for the choice of $k_1 = 0.5$ and $k_2 = 1.5$ which is not a point inside the stability region, the agents in the network do not achieve consensus as illustrated in Figure 8.

Remark 7. We observe from Figures 2 and 6 that the region of controller parameters that can be chosen systematically by using the method discussed in Section 4 is a subset of the region of stable controller parameters. While this may seem conservative, this systematic choice is quite useful since (i) the choice of k_2 does not depend on the network graph and (ii) although the choice of k_1 depends on the minimum nonzero eigenvalue of the Laplacian matrix, it is possible to utilize the well-known bounds in the literature to determine a proper k_1 [43]. Furthermore, the proposed systematic method can be used to determine the controller parameters for any graph satisfying Assumption 1(i).

Remark 8. Note that the computational burden of the proposed controller parameter selection procedure is less than those in the literature since there is no requirement of solving an algebraic Riccati equation [44], a linear matrix inequality [8] or coupled Hamilton–Jacobi equations [37]. Furthermore, unlike the existing methods, the equilibrium states can be determined by the controller designer with the stability analysis given in the paper.

6 | CONCLUSIONS

In this paper, the heterogeneous consensus problem was investigated in MASs where the agents are unable to measure the velocities of their neighbors. For a directed network consisting of both first- and second-order agents, necessary and sufficient conditions on the controller parameters were presented so as to achieve consensus. The consensus equilibria were expressed as functions of a controller parameter and the initial states of the agents. Moreover, a systematic method was provided for choosing controller parameters that guarantees to satisfy the stability conditions. Finally, theoretical results were verified via some numerical examples. Further research should be undertaken to relax the assumptions made on the Laplacian matrix.

ACKNOWLEDGMENTS

We would like to thank the anonymous reviewers for their constructive comments which have improved the clarity and presentation of this paper.

CONFLICT OF INTEREST

The author declares no potential conflict of interests.

FUNDING INFORMATION

None reported.

AUTHOR CONTRIBUTIONS

Onur Cihan: conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, software, supervision, validation, visualization.

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How to cite this article: O. Cihan, *Stability analysis and controller design for consensus in discrete-time networks with heterogeneous agent dynamics*, Asian J Control **25** (2023), 77–87. https://doi.org/10.1002/asjc.2788