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On the union of graded prime ideals

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Abstract: In this paper we investigate graded compactly packed rings, which is defined as; if any graded ideal I of R is contained in the union of a family of graded prime ideals of R , then I is actually contained in one of the graded prime ideals of the family. We give some characterizations of graded compactly packed rings. Further, we examine this property on $h - Spec(R)$. We also define a generalization of graded compactly packed rings, the graded coprimely packed rings. We show that R is a graded compactly packed ring if and only if R is a graded coprimely packed ring whenever R be a graded integral domain and $h - \dim R = 1$.

Keywords: graded ring; graded prime ideal

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1 Introduction

Throughout this paper, R will be a commutative ring with identity 1_R . R is a \mathbb{Z} -graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in \mathbb{Z}$ such that $R = \bigoplus_{g \in \mathbb{Z}} R_g$ and satisfies $R_g R_h \subseteq R_{gh}$ for all $g, h \in \mathbb{Z}$. The elements of R_g are homogeneous elements of R of degree g , and all homogeneous elements of the ring R are denoted by $h(R)$, i.e. $h(R) = \bigcup_{g \in \mathbb{Z}} R_g$. If $ab = 0$ implies $a = 0$ or $b = 0$ for nonzero homogeneous elements $a, b \in h(R)$, then R is called graded integral domain. A subset S of $h(R)$ is called homogeneous multiplicatively closed subset or shortly multiplicatively closed if $a, b \in S$ implies $ab \in S$. Then $S^{-1}R$, the ring of fraction is a graded ring with $S^{-1}R = \bigoplus_{g \in \mathbb{Z}} (S^{-1}R)_g$; where, $(S^{-1}R)_g = \{ \frac{r}{s} : r \in R, s \in S \text{ and } g = (\deg r) - (\deg s) \}$. Let

I be an ideal of R . If $I = \bigoplus_{g \in \mathbb{Z}} I_g$ where $I_g = I \cap R_g$, then I is called graded ideal of R . A graded ideal I is a graded prime ideal of R if $I \neq R$ and whenever $ab \in I$, then either $a \in I$ or $b \in I$, for $a, b \in h(R)$. The set of all graded prime ideals is denoted by $h - Spec(R)$. The maximal elements with respect to the inclusion in the set of all proper graded ideals are graded maximal ideals and the set of all graded maximal ideals is denoted by $h - Max(R)$. A graded ring with finite number of graded maximal ideals is a graded semilocal ring. Further the set of all minimal graded prime ideals is denoted by $h - Min(R)$. The graded height of a graded prime ideal P denoted by $h - htP$, is defined as the length of the longest chain of graded prime ideals contained in P . The Krull dimension of a graded ring R is denoted by $h - \dim(R)$ and defined as $h - \dim(R) = \max \{ h - htP \mid P \in h - Spec(R) \}$ [1].

The finite union of graded prime submodules are studied in [2]. For more details of graded prime submodules refer [3]. Moreover, the finite union of ideals are studied by Quatororo and Butts with the notion of u -ideal in [4]. With these motivations we investigate some properties of the finite union of graded ideals in Section 2. For this, we define graded u -ideal as follow: A graded ideal I is graded u -ideal if it is contained in the finite union of a family of graded ideals of R , then I is actually contained in one of the graded ideal of the family.

In Section 3, we examine some properties of graded compactly packed rings. Compactly packed rings have been studied by various authors, see, for example, [5–7]. The ring R is compactly packed if for any ideal I of R , $I \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$ where $\{P_\alpha\}_{\alpha \in \Delta}$ is a family of prime ideals of R with the index set Δ , then $I \subseteq P_\alpha$ for some $\alpha \in \Delta$. This concept was pointed out by C. Reis and Viswanathan in [5]. They also characterized on Noetherian rings that are compactly packed by prime ideal of R if and only if every prime ideal is the radical of a principal ideal in R . After this work, Smith [7] shows for this property that the ring need not be a Noetherian ring. Moreover Pakala and Shores [5] showed that for a compactly packed Noetherian ring the maximal ideal is the radical of a principal ideal. In [7], Principal Ideal Theorem of Krull was proven for graded rings and using this theorem we show that if R is a graded compactly packed ring, then $h - \dim R \leq 1$ whenever R be a graded Noetherian ring.

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In Section 4, we define graded coprimely packed rings that Erdođdu [8] defined coprimely packed rings as a generalization of compactly packed rings. An ideal I is coprimely packed if for an index set Δ and $\alpha \in \Delta$, $I + P_\alpha = R$ implies $I \not\subseteq \bigcup_{\alpha \in \Delta} P_\alpha$ where $\{P_\alpha\}_{\alpha \in \Delta}$ is a family of prime ideals of R . If every ideal of R is coprimely packed, then R is a coprimely packed ring. For the studies about coprimely packed rings the reader is referred to [8–10]. Finally we show that every graded compactly packed ring is a graded coprimely ring. Additionally, we also show that R is graded coprimely packed ring if and only if R is coprimely packed ring by $h - \text{Max}(R)$.

2 Finite union of graded ideals

Definition 1. Let R be a graded ring and I be a graded ideal of R . Then we say that I is a graded u -ideal if for any family of graded ideals $\{A_i\}_{i=1}^n$, $I \subseteq \bigcup_{i=1}^n A_i$ implies $I \subseteq A_j$ for some $j = 1, 2, \dots, n$. A graded ring R is graded u -ring if any graded ideal is a graded u -ideal.

Proposition 1. Let R be a graded ring and I be a graded ideal of R . Then the following conditions are equivalent;

- (i) R is a graded u -ring,
- (ii) Each finitely generated graded ideal of R is graded u -ideal,
- (iii) If $I = \bigcup_{i=1}^n A_i$ is finitely generated, then $I = A_j$ for some j ,
- (iv) If $I = \bigcup_{i=1}^n A_i$, then $I = A_j$ for some j .

Proof. (i) \Rightarrow (ii) is trivial.

(ii) \Rightarrow (iii) Since $I = \bigcup_{i=1}^n A_i$ we get $I \subseteq \bigcup_{i=1}^n A_i$ and so $I \subseteq A_j$

for some $j \in \{1, 2, \dots, n\}$. Then $\bigcup_{i=1}^n A_i = I \subseteq A_j \subseteq \bigcup_{i=1}^n A_i$ and so $I = A_j$.

(iii) \Rightarrow (iv) Suppose that $I = \bigcup_{i=1}^n A_i$ and assume that $I \neq A_j$ for all $j \in \{1, 2, \dots, n\}$. Then there exists an element $a_j \in I \setminus A_j$ and set $J = (a_1, \dots, a_n)$. Then $J = J \cap I = J \cap (\bigcup_{i=1}^n A_i) = \bigcup_{i=1}^n (J \cap A_i)$ by (iii) we have $J = J \cap A_j$ for some $j \in \{1, 2, \dots, n\}$. Hence $J \subseteq A_j$, which is a contradiction.

(iv) \Rightarrow (i) It follows from, $I \subseteq \bigcup_{i=1}^n A_i$ implies $I = \bigcup_{i=1}^n (I \cap A_i)$. □

Proposition 2. Every homomorphic image of a graded u -ring is a graded u -ring.

Proof. It is explicit. □

Proposition 3. If R is a graded u -ring, then $S^{-1}R$ is a graded u -ring.

Proof. It follows from [[2], Proposition 2.7]. □

3 Graded compactly packed rings

Definition 2. Let R be a graded ring and Δ an index set. If for any graded ideal I and any family of graded prime ideals $\{P_\alpha\}_{\alpha \in \Delta}$, $I \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$ implies $I \subseteq P_\beta$ for some $\beta \in \Delta$, then R is a graded compactly packed ring with $h - \text{Spec}(R)$.

Proposition 4. Every homomorphic image of a graded compactly packed ring is a graded compactly packed ring.

Proof. Let R be a graded compactly packed ring and S be any graded ring. Let $f : R \rightarrow S$ be an epimorphism. Assume that $\{P'_\alpha\}_{\alpha \in \Delta}$ be a family of graded prime ideals of S and I' be a graded ideal of S such that $I' \subseteq \bigcup_{\alpha \in \Delta} P'_\alpha$. Since f is an epimorphism, there exist graded prime ideals P_α and graded ideal I of R such that $\text{Ker} f \subseteq P_\alpha$, $\text{Ker} f \subseteq I$ and $f(P_\alpha) = P'_\alpha$, $f(I) = I'$. It follows that $f(I) \subseteq \bigcup_{\alpha \in \Delta} f(P_\alpha) \subseteq f(\bigcup_{\alpha \in \Delta} P_\alpha)$. Therefore $I \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$, and so $I \subseteq P_\beta$ for some $\beta \in \Delta$. Thus $I' = f(I) \subseteq f(P_\beta) = P'_\beta$ for some $\beta \in \Delta$. □

Now recall the following well known Lemma.

Lemma 1. [3, Lemma 2.1] Let R be a graded ring, $a \in h(R)$ and I, J be graded ideals of R . Then $aR, I + J$ and $I \cap J$ are graded ideals.

Note that the graded ideal aR is denoted by (a) .

Theorem 1. Let R be a graded ring, I be a graded ideal and $S \subseteq h(R)$ be a multiplicatively closed subset. Then the set

$$\psi = \{J \mid J \text{ is a graded ideal of } R, S \cap J = \emptyset, I \subseteq J\}$$

has a maximal element and such maximal elements are graded prime ideals of R .

Proof. Since $I \in \psi$, we get $\psi \neq \emptyset$. The set ψ is partially ordered set with respect to set inclusion " \subseteq ". Now let Δ be a totally ordered subset of ψ . Then $\mathfrak{J} = \bigcup_{J \in \Delta} J$ is an ideal of R and $\mathfrak{J}_g = \mathfrak{J} \cap R_g = (\bigcup_{J \in \Delta} J) \cap R_g = \bigcup_{J \in \Delta} (J \cap R_g) = \bigcup_{J \in \Delta} J_g$. Thus $\mathfrak{J} = \bigoplus J_g$ and so \mathfrak{J} is graded ideal. Now let P be a maximal element of ψ and $a, b \in h(R)$ such that $a \notin P$ and $b \notin P$.

Then $P \subsetneq P + (a)$ and $P + (a)$ is a graded ideal. Therefore $(P + (a)) \cap S = \emptyset$ and so there exist $s \in S$ such that $s = x + ar$ for some $x \in P, r \in R$. Similarly there exist $s' \in S$ such that $s' = x' + br'$ for some $x' \in P, r' \in R$. Then $ss' = (x + ar)(x' + br') = xx' + arx' + br'x + abrr'$. Then $abrr' \notin P$ and so $ab \notin P$. Hence P is a graded prime ideal. \square

Theorem 2. *Let R be a graded ring. Then the following are equivalent:*

- (i) R is a graded compactly packed ring.
- (ii) For every graded prime ideal $P, P \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$ implies $P \subseteq P_\beta$ for some $\beta \in \Delta$.
- (iii) Every graded prime ideal of R is the radical of a graded principal ideal in R .

Proof. (i) \Rightarrow (ii) It follows from the definition of graded compactly packed ring.

(ii) \Rightarrow (iii) Suppose that P is a graded prime ideal of R . Assume that P is not the radical of a graded principal ideal of R . Then we get $\sqrt{(r)} = P$ for all $r \in P \cap h(R)$. Hence there is a prime ideal P_r such that $r \in P_r$ and $P \not\subseteq P_r$ for all $r \in P \cap h(R)$. Further we have $P \subseteq \bigcup_{r \in P \cap h(R)} P_r$. Then by (ii)

$P \subseteq P_{r'}, r' \in P \cap h(R)$ which is a contradiction.
 (iii) \Rightarrow (i) Suppose that $I \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$. Since $h(R) \setminus (\bigcup_{\alpha \in \Delta} P_\alpha)$

is a graded multiplicatively closed subset, there exists a graded prime ideal P such that $I \subseteq P$ and $P \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$.

Suppose that $P = \sqrt{r}$ for some $r \in h(R)$. Then we get that $\sqrt{r} \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$ and $r \in \bigcup_{\alpha \in \Delta} P_\alpha$. Hence there exists $\beta \in \Delta$ such that $r \in P_\beta$. Therefore $I \subseteq P = \sqrt{r} \subseteq P_\beta$. \square

Theorem 3. (Principal Ideal Theorem, [[1], Theorem 3.5]) *Let x be a nonunit homogeneous element in a graded Noetherian ring R and let P be a graded prime ideal minimal over (x) . Then $h - htP \leq 1$.*

Theorem 4. *Let R be a graded Noetherian ring. If R is a graded compactly packed ring, then $h - \dim R \leq 1$.*

Proof. Suppose that R is a graded compactly packed ring. Then there exists an $r \in h(R)$ such that $\sqrt{I} = P$ where $I = (r)$ for any graded prime ideal P of R by Theorem 2. From Principal Ideal Theorem we have $h - htP \leq 1$. Thus $h - \dim R \leq 1$. \square

Theorem 5. *Let R be a graded ring, I be a graded ideal and P be a graded prime ideal such that $I \subseteq P$. Then the following are equivalent:*

- (i) P is a graded minimal prime ideal of I ,
- (ii) $h(R) \setminus P$ is a graded multiplicatively closed subset that is

maximal with respect to missing I ,

- (iii) For each $x \in P \cap h(R)$, there is a $y \in h(R) \setminus P$ and a nonnegative integer i such that $yx^i \in I$.

Proof. (i) \Rightarrow (ii) Suppose that P is a graded minimal prime ideal of I . If we set $S = h(R) \setminus P$, then S is a graded multiplicatively closed subset and there exists a maximal element in the set of graded ideals containing I and disjoint from S . Assume Q is maximal then Q is graded prime ideal by Theorem 1. Since P is minimal, $P = Q$ and so S is maximal with respect to missing I .

(ii) \Rightarrow (iii) Let $0 \neq x \in P \cap h(R)$ and $S = \{yx^i \mid y \in h(R) \setminus P, i = 0, 1, 2, \dots\}$. Then $h(R) \setminus P \subsetneq S$. Since $h(R) \setminus P$ is maximal, there exist an element $y \in h(R) \setminus P$ and i nonnegative integer such that $yx^i \in I$.

(iii) \Rightarrow (i) Assume that $I \subset Q \subseteq P$, where Q is graded prime ideal. If there exists $x \in P \setminus Q$ where $x \in h(R)$, then there exist an element $y \in h(R) \setminus P$ such that $yx^i \in I$ for some $i = 0, 1, 2, \dots$. Therefore $yx^i \in Q, y \notin Q$. Thus $x^i \in Q$. It is a contradiction. \square

Recall that a graded ring R is reduced if its nilradical is zero, i.e.

$$\bigcap_{P \in h\text{-Spec}(R)} P = (0).$$

Corollary 1. *If R is a graded reduced ring and P is a graded prime ideal of R , then P is a graded minimal prime ideal of R if and only if for each $x \in P \cap h(R)$ there exists some $y \in h(R) \setminus P$ such that $xy = 0$.*

Theorem 6. *Let R be a graded ring and $h\text{-Min}(R) = \{P_\alpha\}$. If $P_\alpha \not\subseteq \bigcup_{\alpha=\beta} P_\beta$ for each α , then $h - \text{Min}(R)$ is finite.*

Proof. Without loss of generality, assume that R is a graded reduced ring. Then for $P \in h\text{-Min}R, R_P$ is a graded field. Let $R' = \prod R_{P_\alpha}$. Then $\psi : R \rightarrow R', r \mapsto \{\theta_\alpha(r)\}$ where $\theta_\alpha : R \rightarrow R_{P_\alpha}$ be the canonical homomorphism. If $h - \text{Min}R$ is infinite, then $\sum R_{P_\alpha}$ is a proper ideal in R' . Let \mathfrak{M} be a graded maximal ideal of R' containing the ideal $\sum R_{P_\alpha}$. Then $\mathfrak{M} \cap R$ contains a graded minimal prime ideal P_γ of R . Choose $a \in P_\gamma \cap h(R)$ where $a \notin \bigcup_{\beta=\gamma} P_\beta$. Since R is a graded subring of R', a is identified with $\{\theta_\alpha(a)\}$. For $\beta \neq \gamma, \theta_\beta(a)$ is a unit in the graded ring R_{P_β} . Now define an element $b = \{b_\beta\} \in R'$ where $b_\beta = \theta_\beta(a)^{-1}$ for $\beta \neq \gamma$ and $b_\beta = 0$ if $\beta = \gamma$. Then we have $ab \in \mathfrak{M}$ where only the γ component is 0 and other components are identity. And so \mathfrak{M} contains the identity of R' , since $\sum R_{P_\alpha} \subseteq \mathfrak{M}$. This gives us a contradiction. \square

Corollary 2. *Let R be a graded u -ring and $h - \text{Min}(R) = \{P_\alpha\}$. Then $P_\alpha \not\subseteq \bigcup_{\alpha=\beta} P_\beta$ for each α if and only if $h - \text{Min}(R)$ is finite .*

Now we will investigate the graded compactly packing property on graded spectrum of a graded ring and refer to this as the $(*)$ property. The topology of graded spectrum was studied in [11]. For a graded ring R its graded spectrum, $h - \text{Spec}(R)$, is a topology with the closed sets $V_G^R(I) = \{P \in h - \text{Spec}(R) \mid I \subseteq P\}$ where I is a graded ideal of R . This topology is called Zariski topology. For any homogeneous element $r \in h(R)$ define $D_r = \{P \in h - \text{Spec}(R) \mid r \notin P\}$, and so the set $\{D_r \mid r \in h(R)\}$ is a basis for the Zariski topology on $h - \text{Spec}(R)$ [[11], Theorem 2.3]. Further D_r is quasi-compact for all $r \in h(R)$.

Definition 3. *Let R be a graded ring, Λ an index set and $r, s_\alpha \in h(R) \setminus \{0\}$ for all $\alpha \in \Lambda$. Then we say that R has property $(*)$ if $D_r \subseteq \bigcup_{\alpha \in \Lambda} D_{s_\alpha}$ implies $D_r \subseteq D_{s_\beta}$ for some $\beta \in \Lambda$.*

Theorem 7. *Let R be a graded ring. If R satisfies property $(*)$ then R has at most two graded maximal ideals.*

Proof. Suppose that R has property $(*)$ and assume that $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$ are three distinct graded maximal ideals of R . Then we have $a \in \mathfrak{M}_1 \cap h(R)$ and $b \in \mathfrak{M}_2 \cap h(R)$ such that $a + b = 1_R$. Now let $c \in (\mathfrak{M}_3 \cap h(R)) \setminus (\mathfrak{M}_1 \cup \mathfrak{M}_2)$. Since $c = ca + cb$, we get $D_c \subseteq D_{ac} \cup D_{bc}$. Since R satisfies $(*)$ property we get $D_c \subseteq D_{ac}$ or $D_c \subseteq D_{bc}$. Both of them is a contradiction. \square

Corollary 3. *Let R be a graded ring and every nonzero graded prime ideal is a graded maximal ideal. Then R has at most two nonzero graded prime ideals if and only if R satisfies $(*)$ property.*

Proof. Suppose that R has at most two nonzero graded prime ideals. If r is a nonzero nonunit homogeneous element of R then $D_r \setminus \{(0)\}$ is an empty set or single point set. Then R satisfies $(*)$ property. For the converse, if R satisfies $(*)$ property then by Theorem 7, R has at most two graded maximal ideals. Therefore, this completes the proof. \square

4 Graded coprimely packed rings

Definition 4. *Let R be a graded ring and I be a graded ideal. I is said to be graded coprimely packed ring if $I + P_\alpha = R$ where P_α ($\alpha \in \Delta$) are graded prime ideals of R ; then $I \not\subseteq \bigcup_{\alpha \in \Delta} P_\alpha$. If every graded ideal of R is a graded coprimely*

packed ring, then R is a graded coprimely packed ring by $h - \text{Spec}(R)$.

Proposition 5. *Every homomorphic image of a graded coprimely packed ring is a graded coprimely packed ring.*

Proof. Let R be a graded coprimely packed ring and S be a ring. Let $f : R \rightarrow S$ be an epimorphism. Assume that J be a graded ideal of S and $\{P'_\alpha\}_{\alpha \in \Delta}$ be a family of graded prime ideals of S such that $J + P'_\alpha = S$. Since f is an epimorphism, there exists a graded ideal I and graded prime ideals P_α of R such that $\text{Ker}f \subseteq I, \text{Ker}f \subseteq P_\alpha, f(I) = J$ and $f(P_\alpha) = P'_\alpha$. Thus we obtain $J + P'_\alpha = f(I + P_\alpha) = f(R) = S$. To show that $I + P_\alpha = R$, let $r \in R$. Then $f(r) \in f(R) = f(I + P_\alpha)$. Then there exists $m \in I + P_\alpha$ such that $f(r) = f(m)$, that is $r - m \in \text{Ker}(f) \subseteq I + P_\alpha$. So $r \in I + P_\alpha$. Since R is a graded coprimely packed ring, we have $I \not\subseteq \bigcup_{\alpha \in \Delta} P_\alpha$. Then $f(I) \not\subseteq f(\bigcup_{\alpha \in \Delta} P_\alpha)$. Indeed, if $f(I) \subseteq f(\bigcup_{\alpha \in \Delta} P_\alpha)$, then we have $I \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$ since $\text{Ker}(f) \subseteq I$, this gives us a contradiction. Thus we get $J = f(I) \not\subseteq \bigcup_{\alpha \in \Delta} f(P_\alpha) = \bigcup_{\alpha \in \Delta} P'_\alpha$. Hence S is a graded coprimely packed ring. \square

Proposition 6. *Let R be a graded u -ring. If R is a graded semilocal ring, then R is a graded coprimely packed ring.*

Proof. Suppose that R is a graded semilocal ring and $h - \text{Max}(R) = \{\mathfrak{M}_1, \dots, \mathfrak{M}_k\}$. Let I be a graded ideal of $R, \{P_\alpha\}_{\alpha \in \Delta}$ is a family of graded prime ideals of R such that for $\alpha \in \Delta, I + P_\alpha = R$. Then there exists a subset $\{i_1, \dots, i_t\}$ of $\{1, \dots, k\}$, for all $\alpha \in \Delta$ there exists $i_j \in \{i_1, \dots, i_t\}$ such that $P_\alpha \subseteq \mathfrak{M}_{i_j}$. Therefore we get $I + \mathfrak{M}_{i_j} = R$ for all $j = 1, \dots, t$. Assume that $I \subseteq \bigcup_{j=1}^t \mathfrak{M}_{i_j}$. Since R is a graded u -ring, we get $I \subseteq \mathfrak{M}_{i_l}$ for some $i_l \in \{i_1, \dots, i_t\}$. And so $I + \mathfrak{M}_{i_l} = R$ is a contradiction. Thus $I \not\subseteq \bigcup_{j=1}^t \mathfrak{M}_{i_j}$ and so $I \not\subseteq \bigcup_{\alpha \in \Delta} P_\alpha$. \square

Proposition 7. *Every graded compactly packed ring is a graded coprimely packed ring.*

Proof. Suppose that R is a graded compactly packed ring. Let I be graded ideal and $\{P_\alpha\}_{\alpha \in \Delta}$ be a family of graded prime ideals of R such that $I + P_\alpha = R$ for every $\alpha \in \Delta$. Assume that $I \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$. Since R is graded compactly packed ring, we get $I \subseteq P_\beta$ for some $\beta \in \Delta$. And so $I + P_\beta = P_\beta = R$, which is a contradiction. Thus $I \not\subseteq \bigcup_{\alpha \in \Delta} P_\alpha$. \square

Theorem 8. *Let R be a graded integral domain and $h - \dim R = 1$. Then R is a graded compactly packed ring if and only if R is a graded coprimely packed ring.*

Proof. It is clear that every graded compactly packed ring is a graded coprimely packed ring by Proposition 7. Now suppose that R is a graded coprimely packed ring and $I \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$, $P_{\alpha'} = 0$ for $\alpha \in \Delta$. Assume that $I + P_{\beta'} = R$ for some $\beta \in \Delta$. Then there exists a graded maximal ideal \mathfrak{M} such that $I + P_\beta = \mathfrak{M}$. Since $h - \dim R = 1$, we get $P_\beta = \mathfrak{M}$ and so $I \subseteq P_\beta$. \square

Theorem 9. *Let R be a graded ring. Then R is a graded coprimely packed ring if and only if R is coprimely packed ring by $h - \text{Max}(R)$.*

Proof. Suppose that R is a graded coprimely packed ring. Since $h - \text{Max}(R) \subseteq h - \text{Spec}(R)$, it is clear that R is coprimely packed ring by $h - \text{Max}(R)$. Now assume that R is a coprimely packed ring by $h - \text{Max}(R)$. Let I be graded ideal and $\{P_\alpha\}_{\alpha \in \Delta}$ be a family of graded prime ideals of R such that $I + P_\alpha = R$ for every $\alpha \in \Delta$. Then there exist $\mathfrak{M}_\alpha \in h - \text{Max}(R)$ such that $P_\alpha \subseteq \mathfrak{M}_\alpha$. Since $I + \mathfrak{M}_\alpha = R$ for every $\alpha \in \Delta$, then by our assumption we get $I \not\subseteq \bigcup_{\alpha \in \Delta} \mathfrak{M}_\alpha$. Hence $I \not\subseteq \bigcup_{\alpha \in \Delta} P_\alpha$. \square

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