

Research Article

q -Sumudu Transforms of q -Analogues of Bessel Functions

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The main purpose of this paper is to evaluate q -Sumudu transforms of a product of q -Bessel functions. Interesting special cases of theorems are also discussed. Further, the results proved in this paper may find certain applications of q -Sumudu transforms to the solutions of the q -integrodifferential equations involving q -Bessel functions. The results may help to extend the q -theory of orthogonal functions.

Dedicated to Professor Yusuf Avcı on the occasion of his 65th birthday

1. Introduction

The Sumudu transform introduced by Watugala [1] has a close resemblance with the Laplace transform but has a wider frequency domain and is better suited to solve intricate problems in engineering and applied sciences. The main advantage of the Sumudu transform is that it may be used to solve problems without resorting to a new frequency domain, because it preserves scale and unit properties (see [2]). For further details, readers may refer to the recent papers, for example, [3–5], on this subject. It is well known that in the literature there are many q -extensions of the Bessel function rearranged by Ismail [6]. Here we are concerned with q -Sumudu transform of q -analogues of these q -Bessel functions. Purohit and Kalla [7] evaluated the q -Laplace transforms of a product of basic analogues of the Bessel functions. They gave several useful special cases as application. Recently, Albayrak et al. [8] have investigated the fundamental properties of the q -Sumudu transforms and established several theorems related to q -images of some elementary functions. Subsequently, the same authors evaluated the q -Sumudu images of a number of q -polynomials and q -hypergeometric functions (see [9]).

2. Definitions and Preliminaries

In this section, we purpose to add one more dimension to this study by giving some theorems which give rise to q -Sumudu

images of a product of q -Bessel functions. q -Bessel functions were introduced by Jackson [10] and are therefore referred to as Jackson's q -Bessel functions. Some q -analogues of the Bessel functions are given by

$$J_\nu^{(1)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \left(\frac{z}{2}\right)^\nu \times {}_2\Phi_1 \left[\begin{matrix} 0 & 0 \\ q^{\nu+1} \end{matrix}; q, -\frac{z^2}{4} \right], \quad |z| < 2 \quad (1)$$

$$= \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-z^2/4)^n}{(q; q)_{\nu+n} (q; q)_n},$$

$$J_\nu^{(2)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \left(\frac{z}{2}\right)^\nu \times {}_0\Phi_1 \left[\begin{matrix} - \\ q^{\nu+1} \end{matrix}; q, -\frac{q^{\nu+1} z^2}{4} \right], \quad (z \in \mathbb{C}) \quad (2)$$

$$= \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{q^{n(n+\nu)} (-z^2/4)^n}{(q; q)_{\nu+n} (q; q)_n}$$

$$\begin{aligned}
 &= \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \left(\frac{z}{2}\right)^{\nu} \\
 &\times \sum_{k=0}^{\infty} \frac{q^{n(n-1)}}{(q^{\nu+1}; q)_n (q; q)_n} \left(-\frac{q^{\nu+1} z^2}{4}\right)^n.
 \end{aligned} \tag{3}$$

Hahn [11] showed that these q -Bessel functions are related by the following equality:

$$J_{\nu}^{(2)}(z; q) = \left(-\frac{z^2}{4}; q\right)_{\infty} J_{\nu}^{(1)}(z; q), \quad |z| < 2. \tag{4}$$

Both these q -analogues have been studied extensively by Ismail in [6, 12]. The third kind of q -analogue of the Bessel function is given by

$$\begin{aligned}
 J_{\nu}^{(3)}(z; q) &= \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} z^{\nu} \\
 &\times {}_1\Phi_1 \left[\begin{matrix} 0 \\ q^{\nu+1} \end{matrix}; q, qz^2 \right], \quad (z \in \mathbb{C}) \\
 &= z^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (qz^2)^n}{(q; q)_{\nu+n} (q; q)_n}.
 \end{aligned} \tag{5}$$

This third kind of q -Bessel function is also defined by Jackson and sometimes it is called Hahn-Exton q -Bessel function. The notation ${}_1\Phi_1$ in (5) is the standard in use of q -hypergeometric series [13]. These q -analogues of the Bessel function satisfy the following relations [14]:

$$\begin{aligned}
 \lim_{q \rightarrow 1^-} J_{\nu}^{(k)}((1-q)z; q) &= J_{\nu}(z), \quad (k = 1, 2), \\
 \lim_{q \rightarrow 1^-} J_{\nu}^{(3)}((1-q)z; q) &= J_{\nu}(2z).
 \end{aligned} \tag{6}$$

To make this work easy to read, we need some notations and preliminaries about the quantum theory. For any real number α , the q -analogue of α is defined by

$$[\alpha] := \frac{q^{\alpha} - 1}{q - 1}. \tag{7}$$

The following usual notations are very useful in the theory of q -calculus:

$$\begin{aligned}
 (a; q)_n &= \prod_{k=0}^{n-1} (1 - aq^k), \quad \text{for } n = 1, 2, \dots, \quad (a; q)_0 = 1, \\
 (a; q)_{\infty} &= \prod_{k=0}^{\infty} (1 - aq^k), \quad (a; q)_t = \frac{(a; q)_{\infty}}{(aq^t; q)_{\infty}} \quad (t \in \mathbb{R}).
 \end{aligned} \tag{8}$$

See [13] for all of the above definitions and the related formulas. Furthermore, q -hypergeometric functions are defined by [13]

$$\begin{aligned}
 &{}_r\Phi_s \left[\begin{matrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_s \end{matrix}; q, z \right] \\
 &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_{\infty}}{(b_1, b_2, \dots, b_s; q)_{\infty}} \left[(-1)^n q^{\binom{n}{2}} \right]^{s+1-r} \frac{z^n}{(q; q)_n},
 \end{aligned} \tag{9}$$

where the q -shifted factorial, for $a_i \in \mathbb{C}$ ($i = 1, 2, \dots, r$), is

$$(a_1, a_2, \dots, a_r; q)_{\infty} = \prod_{i=1}^r (a_i; q)_{\infty}. \tag{10}$$

Albayrak et al. [8] have defined q -analogues of the Sumudu transform by means of the following q -integrals:

$$\begin{aligned}
 &S_q \{f(t); s\} \\
 &= \frac{1}{(1-q)_s} \int_0^s E_q\left(\frac{q}{s}t\right) f(t) d_q t, \quad s \in (-\tau_1, \tau_2),
 \end{aligned} \tag{11}$$

over the set of functions

$$\begin{aligned}
 A &= \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, \right. \\
 &\quad \left. |f(t)| < M E_q\left(\frac{|t|}{\tau_j}\right), t \in (-1)^j \times [0, \infty) \right\},
 \end{aligned} \tag{12}$$

$$\begin{aligned}
 S_q \{f(t); s\} &= \frac{1}{(1-q)_s} \int_0^{\infty} e_q\left(-\frac{1}{s}t\right) f(t) d_q t \\
 &\quad s \in (-\tau_1, \tau_2),
 \end{aligned} \tag{13}$$

over the set of functions

$$\begin{aligned}
 B &= \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, \right. \\
 &\quad \left. |f(t)| < M e_q\left(\frac{|t|}{\tau_j}\right), t \in (-1)^j \times [0, \infty) \right\},
 \end{aligned} \tag{14}$$

where q -analogues of the classical exponential functions are defined by

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} = \frac{1}{(t; q)_{\infty}}, \quad (|t| < 1), \tag{15}$$

$$E_q(t) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} t^n}{(q; q)_n} = (t; q)_{\infty}, \quad (t \in \mathbb{C}).$$

Jackson [15] introduced the q -integrals as follows

$$\begin{aligned}
 \int_0^x f(t) d_q t &= x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k), \\
 \int_0^{\infty/A} f(t) d_q t &= (1-q) \sum_{k \in \mathbb{Z}} \frac{q^k}{A} f\left(\frac{q^k}{A}\right).
 \end{aligned} \tag{16}$$

By virtue of (16), q -Sumudu transforms can be expressed as

$$S_q \{f(t); s\} = (q; q)_\infty \sum_{k=0}^{\infty} \frac{q^k}{(q; q)_k} f(sq^k), \quad (17)$$

$$\mathbb{S}_q \{f(t); s\} = \frac{s^{-1}}{(-1/s; q)_\infty} \sum_{k \in \mathbb{Z}} q^k f(q^k) \left(-\frac{1}{s}; q\right)_k. \quad (18)$$

Integral representations of q -gamma function are defined by

$$\Gamma_q(\alpha) = \int_0^{1/(1-q)} x^{\alpha-1} E_q(q(1-q)x) d_q x, \quad (\alpha > 0),$$

$$\Gamma_q(\alpha) = K(A; \alpha) \int_0^{\infty/A(1-q)} x^{\alpha-1} e_q(-(1-q)x) d_q x \quad (19)$$

$$(\alpha > 0),$$

where $K(A; \alpha)$ is the following remarkable function [16, p. 15]:

$$K(A; \alpha) = A^{\alpha-1} \frac{(-q/\alpha; q)_\infty}{(-q^t/\alpha; q)_\infty} \frac{(-\alpha; q)_\infty}{(-\alpha q^{1-t}; q)_\infty} \quad (\alpha \in \mathbb{R}). \quad (20)$$

The q -gamma and $K(A; \alpha)$ function have the following properties [16, p. 15]:

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x} = \frac{(q; q)_{x-1}}{(1-q)^{x-1}}, \quad (21)$$

$$K(A; \alpha) = q^{\alpha-1} K(A; \alpha-1). \quad (22)$$

By virtue of (16), q -gamma function can be expressed as

$$\Gamma_q(\alpha) = \frac{(q; q)_\infty}{(1-q)^{\alpha-1}} \sum_{k=0}^{\infty} \frac{q^{k\alpha}}{(q; q)_k}, \quad (23)$$

$$\Gamma_q(\alpha) = \frac{K(A; \alpha)}{(1-q)^{\alpha-1} (-1/A; q)_\infty} \sum_{k \in \mathbb{Z}} \left(\frac{q^k}{A}\right)^\alpha \left(-\frac{1}{A}; q\right)_k. \quad (24)$$

3. Main Theorems

In this section we will evaluate q -Sumudu transforms of a product of q -Bessel functions.

Theorem 1. (a) Let $J_{2\mu_j}^{(1)}(2\sqrt{a_j t}; q)$, $j = 1, 2, \dots, n$, be n different q -Bessel functions and

$$f(t) = t^{v-1} \prod_{j=1}^n J_{2\mu_j}^{(1)}(2\sqrt{a_j t}; q). \quad (25)$$

Then, q -Sumudu transform of $f(t)$ is

$$S_q \{f(t); s\} = \frac{(1-q)^{v-M-1} a_1^{\mu_1} \dots a_n^{\mu_n} \Gamma_q(v+M) s^{v+M-1}}{\Gamma_q(2\mu_1+1) \dots \Gamma_q(2\mu_n+1)} \times \sum_{m_1, \dots, m_n=0}^{\infty} (q^{v+M}; q)_{m_j} \times \prod_{j=1}^n \frac{(-a_j s)^{m_j}}{(q^{2\mu_j+1}; q)_{m_j} (q; q)_{m_j}}, \quad (26)$$

where $m = m_1 + m_2 + \dots + m_n$ and $M = \mu_1 + \dots + \mu_n$, $\text{Re}(s) > 0$ and $\text{Re}(v+M) > 0$.

(b) Let $J_{2\mu_j}^{(2)}(2\sqrt{a_j t}; q)$, $j = 1, 2, \dots, n$, be n different q -Bessel functions and

$$f(t) = t^{v-1} \prod_{j=1}^n J_{2\mu_j}^{(2)}(2\sqrt{a_j t}; q). \quad (27)$$

Then, q -Sumudu transform of $f(t)$ is

$$S_q \{f(t); s\} = \frac{\Gamma_q(v+M) (1-q)^{v-M-1} a_1^{\mu_1} \dots a_n^{\mu_n} s^{v+M-1}}{\Gamma_q(2\mu_1+1) \dots \Gamma_q(2\mu_n+1)} \times \sum_{m_1, \dots, m_n=0}^{\infty} (q^{v+M}; q)_{m_j} \prod_{j=1}^n \frac{q^{m_j(m_j-1)} (-q^{2\mu_j+1} a_j s)^{m_j}}{(q^{2\mu_j+1}; q)_{m_j} (q; q)_{m_j}}, \quad (28)$$

where $m = m_1 + m_2 + \dots + m_n$ and $M = \mu_1 + \dots + \mu_n$, $\text{Re}(s) > 0$ and $\text{Re}(v+M) > 0$.

(c) Let $J_{2\mu_j}^{(3)}(\sqrt{q^{-1} a_j t}; q)$, $j = 1, 2, \dots, n$, be n different q -Bessel functions and

$$f(t) = t^{v-1} \prod_{j=1}^n q^{\mu_j} J_{2\mu_j}^{(3)}(\sqrt{q^{-1} a_j t}; q). \quad (29)$$

Then, q -Sumudu transform of $f(t)$ is

$$S_q \{f(t); s\} = \frac{(1-q)^{v-M-1} a_1^{\mu_1} \dots a_n^{\mu_n} \Gamma_q(v+M) s^{v+M-1}}{\Gamma_q(2\mu_1+1) \dots \Gamma_q(2\mu_n+1)} \times \sum_{m_1, \dots, m_n=0}^{\infty} (q^{v+M}; q)_{m_j} \prod_{j=1}^n \frac{q^{m_j(m_j-1)/2} (-a_j s)^{m_j}}{(q^{2\mu_j+1}; q)_{m_j} (q; q)_{m_j}}, \quad (30)$$

where $m = m_1 + m_2 + \dots + m_n$ and $M = \mu_1 + \dots + \mu_n$, $\text{Re}(s) > 0$ and $\text{Re}(v+M) > 0$.

Proof. We will only give the proof of (26), because the proof of (28) and (30) is the same. We put

$$f(t) = t^{v-1} J_{2\mu_1}^{(1)}(2\sqrt{a_1 t}; q) \dots J_{2\mu_n}^{(1)}(2\sqrt{a_n t}; q) \quad (31)$$

into the definition (17) and making use of (1) yields

$$\begin{aligned}
 S_q \{f(t); s\} &= (q; q)_\infty \sum_{j=0}^{\infty} \frac{q^j}{(q; q)_j} (sq^j)^{v-1} \\
 &\times \left\{ \frac{(q^{2\mu_1+1}; q)_\infty}{(q; q)_\infty} (a_1 sq^j)^{\mu_1} \cdots \frac{(q^{2\mu_n+1}; q)_\infty}{(q; q)_\infty} (a_n sq^j)^{\mu_n} \right\} \\
 &\times \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(-a_1 sq^j)^{m_1}}{(q^{2\mu_1+1}; q)_{m_1} (q; q)_{m_1}} \\
 &\times \cdots \frac{(-a_n sq^j)^{m_n}}{(q^{2\mu_n+1}; q)_{m_n} (q; q)_{m_n}}.
 \end{aligned} \tag{32}$$

On interchanging the order of summations, which is valid under the conditions given by the theorem, we obtain

$$\begin{aligned}
 S_q \{f(t); s\} &= (q; q)_\infty s^{v+M-1} \prod_{j=1}^n \frac{a_j^{\mu_j} (q^{2\mu_j+1}; q)_\infty}{(q; q)_\infty} \\
 &\times \sum_{m_1, \dots, m_n=0}^{\infty} \prod_{j=1}^n \frac{(-a_j sq^k)^{m_j}}{(q^{2\mu_j+1}; q)_{m_j} (q; q)_{m_j}} \sum_{j=0}^{\infty} \frac{q^{j(v+M+m)}}{(q; q)_j},
 \end{aligned} \tag{33}$$

where $m = m_1 + m_2 + \cdots + m_n$ and $M = \mu_1 + \cdots + \mu_n$. By using the property of (21) and then using the definition of q -exponential function, namely,

$$\begin{aligned}
 (q; q)_\infty \sum_{j=0}^{\infty} \frac{q^{j(v+M+m)}}{(q; q)_j} &= \frac{(q; q)_\infty}{(q^{v+M+m}; q)_\infty} \\
 &= \frac{(q; q)_\infty}{(q^{v+M}; q)_\infty} (q^{v+M}; q)_m,
 \end{aligned} \tag{34}$$

we get the desired result. By applying the similar procedure as of Theorem 1(a), one can easily establish Theorem 1(b) and (c). Therefore, we omit the further details of the proof of this theorem. \square

Theorem 2. (a) Let $J_{2\mu_j}^{(2)}(2\sqrt{a_j t}; q)$, $j = 1, 2, \dots, n$, be n different q -Bessel functions and

$$f(t) = t^{v-1} \prod_{j=1}^n J_{2\mu_j}^{(2)}(2\sqrt{a_j t}; q). \tag{35}$$

Then, q -Sumudu transform of $f(t)$ is

$$\begin{aligned}
 S_q \{f(t); s\} &= \frac{\Gamma_q(v+M)(1-q)^{v-M-1}}{\Gamma_q(2\mu_1+1) \cdots \Gamma_q(2\mu_n+1)} \frac{a_1^{\mu_1} \cdots a_n^{\mu_n}}{K(1/s; v+M)} s^{v+M-1} \\
 &\times \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(q^{v+M}; q)_m}{(q^{v+M})^m q^{m(m-1)/2}} \\
 &\times \prod_{k=1}^n \frac{q^{m_k(m_k-1)} (-q^{2\mu_k+1} a_k s)^{m_k}}{(q^{2\mu_k+1}; q)_{m_k} (q; q)_{m_k}},
 \end{aligned} \tag{36}$$

where $m = m_1 + m_2 + \cdots + m_n$ and $M = \mu_1 + \cdots + \mu_n$, $\text{Re}(s) > 0$ and $\text{Re}(v+M) > 0$.

(b) Let $J_{2\mu_j}^{(3)}(\sqrt{q^{-1} a_j t}; q)$, $j = 1, 2, \dots, n$, be n different q -Bessel functions and

$$f(t) = t^{v-1} \prod_{j=1}^n q^{\mu_j} J_{2\mu_j}^{(3)}(\sqrt{q^{-1} a_j t}; q). \tag{37}$$

Then, q -Sumudu transform of $f(t)$ is given by

$$\begin{aligned}
 S_q \{f(t); s\} &= \frac{\Gamma_q(v+M)(1-q)^{v+M-1}}{\Gamma_q(2\mu_1+1) \cdots \Gamma_q(2\mu_n+1)} \frac{a_1^{\mu_1} \cdots a_n^{\mu_n}}{K(1/s; v+M)} s^{v+M-1} \\
 &\times \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(q^{v+M}; q)_m}{(q^{v+M})^m q^{m(m-1)/2}} \\
 &\times \prod_{k=1}^n \frac{q^{m_k(m_k-1)/2} (-a_k s)^{m_k}}{(q^{2\mu_k+1}; q)_{m_k} (q; q)_{m_k}},
 \end{aligned} \tag{38}$$

where $m = m_1 + m_2 + \cdots + m_n$ and $M = \mu_1 + \cdots + \mu_n$, $\text{Re}(s) > 0$ and $\text{Re}(v+M) > 0$.

Proof. We will only give the proof of (36), because the proof of (38) is the same. We put

$$f(t) = t^{v-1} J_{2\mu_1}^{(2)}(2\sqrt{a_1 t}; q) \cdots J_{2\mu_n}^{(2)}(2\sqrt{a_n t}; q) \tag{39}$$

into the definition (18) and make use of (36); then we have

$$\begin{aligned} \mathbb{S}_q \{f(t); s\} &= \frac{s^{-1}}{(-1/s; q)_\infty} \sum_{j \in \mathbb{Z}} q^j \left(-\frac{1}{s}; q\right)_j (q^j)^{v-1} \\ &\times \left\{ \frac{(q^{2\mu_1+1}; q)_\infty}{(q; q)_\infty} (a_1 q^j)^{\mu_1} \dots \frac{(q^{2\mu_n+1}; q)_\infty}{(q; q)_\infty} (a_n q^j)^{\mu_n} \right\} \\ &\times \sum_{m_1, \dots, m_n=0}^{\infty} \frac{q^{m_1(m_1-1)} (-q^{2\mu_1+1} a_1 q^j)^{m_1}}{(q^{2\mu_1+1}; q)_{m_1} (q; q)_{m_1}} \\ &\times \dots \frac{q^{m_n(m_n-1)} (-q^{2\mu_n+1} a_n q^j)^{m_n}}{(q^{2\mu_n+1}; q)_{m_n} (q; q)_{m_n}}. \end{aligned} \tag{40}$$

On interchanging the order of summations, which is valid under the conditions given by the theorem, we obtain

$$\begin{aligned} \mathbb{S}_q \{f(t); s\} &= \frac{s^{-1}}{(-1/s; q)_\infty} \frac{(q^{2\mu_1+1}; q)_\infty \dots (q^{2\mu_n+1}; q)_\infty}{(q; q)_\infty} a_1^{\mu_1} \dots a_n^{\mu_n} \\ &\times \sum_{m_1, \dots, m_n=0}^{\infty} \frac{q^{m_1(m_1-1)} (-q^{2\mu_1+1} a_1)^{m_1}}{(q^{2\mu_1+1}; q)_{m_1} (q; q)_{m_1}} \\ &\times \dots \frac{q^{m_n(m_n-1)} (-q^{2\mu_n+1} a_n)^{m_n}}{(q^{2\mu_n+1}; q)_{m_n} (q; q)_{m_n}} \\ &\times \sum_{j \in \mathbb{Z}} (-s^{-1}; q)_j q^{j(v+\mu_1+\dots+\mu_n+m_1+\dots+m_n)}. \end{aligned} \tag{41}$$

In the series representation of (24), if we write $A = s$ and $\alpha = v + M + m$, where $M = \mu_1 + \dots + \mu_n$ and $m = m_1 + \dots + m_n$, we get

$$\begin{aligned} \mathbb{S}_q \{f(t); s\} &= \frac{(q^{2\mu_1+1}; q)_\infty \dots (q^{2\mu_n+1}; q)_\infty}{(q; q)_\infty} a_1^{\mu_1} \dots a_n^{\mu_n} \\ &\times \sum_{m_1, \dots, m_n=0}^{\infty} \frac{q^{m_1(m_1-1)} (-q^{2\mu_1+1} a_1)^{m_1}}{(q^{2\mu_1+1}; q)_{m_1} (q; q)_{m_1}} \\ &\times \dots \frac{q^{m_n(m_n-1)} (-q^{2\mu_n+1} a_n)^{m_n}}{(q^{2\mu_n+1}; q)_{m_n} (q; q)_{m_n}} \\ &\times \frac{(1-q)^{v+M+m-1} s^{v+M+m-1}}{K(s; v+M+m)} \Gamma_q(v+M+m). \end{aligned} \tag{42}$$

By using the properties of (21) and (22) and then using the following remarkable identity:

$$(q^{t+m}; q)_\infty = \frac{(q^t; q)_\infty}{(q^t; q)_m} \quad (m \in \mathbb{N}), \tag{43}$$

we have

$$\begin{aligned} \mathbb{S}_q \{f(t); s\} &= \frac{\Gamma_q(v+M) (1-q)^{v-M-1}}{\Gamma_q(2\mu_1+1) \dots \Gamma_q(2\mu_n+1)} \frac{a_1^{\mu_1} \dots a_n^{\mu_n}}{K(1/s; v+M)} s^{v+M-1} \\ &\times \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(q^{v+M}; q)_m}{(q^{v+M}; q)_m q^{m(m-1)/2}} \\ &\times \prod_{k=1}^n \frac{q^{m_k(m_k-1)} (-q^{2\mu_k+1} a_k s)^{m_k}}{(q^{2\mu_k+1}; q)_{m_k} (q; q)_{m_k}}. \end{aligned} \tag{44}$$

Again, by applying the similar procedure as of Theorem 2(a), one can easily establish Theorem 2(b). Therefore, we omit the further details of the proof of this theorem. \square

4. Illustrative Examples

In this section we evaluate the q -Sumudu transforms involving the q -Bessel functions as applications of our main results.

Corollary 3. *If one takes $n = 1$, $\mu_1 = v$, $v = \mu$, and $a_1 = a$ in above theorems, respectively, one has*

$$\begin{aligned} \mathbb{S}_q \{t^{\mu-1} J_{2v}^{(1)}(2\sqrt{at}; q); s\} &= \frac{\Gamma_q(\mu+v) (1-q)^{\mu-v-1}}{\Gamma_q(2v+1)} a^v s^{v+\mu-1} \\ &\times {}_2\Phi_1 \left[\begin{matrix} q^{\mu+v} & 0 \\ q^{2v+1} & \end{matrix}; q, -as \right], \end{aligned}$$

$$\begin{aligned} \mathbb{S}_q \{t^{\mu-1} J_{2v}^{(2)}(2\sqrt{at}; q); s\} &= \frac{\Gamma_q(\mu+v) (1-q)^{\mu-v-1}}{\Gamma_q(2v+1)} a^v s^{v+\mu-1} \\ &\times {}_1\Phi_2 \left[\begin{matrix} q^{\mu+v} \\ q^{2v+1} & 0 \end{matrix}; q, -q^{2v+1} as \right], \end{aligned}$$

$$\begin{aligned} \mathbb{S}_q \{t^{\mu-1} q^v J_{2v}^{(3)}(\sqrt{q^{-1}at}; q); s\} &= \frac{\Gamma_q(\mu+v) (1-q)^{\mu-v-1}}{\Gamma_q(2v+1)} a^v s^{v+\mu-1} \\ &\times {}_1\Phi_1 \left[\begin{matrix} q^{\mu+v} \\ q^{2v+1} \end{matrix}; q, as \right], \end{aligned}$$

$$\begin{aligned}
 & \mathbb{S}_q \left\{ t^{\mu-1} J_{2\nu}^{(2)} \left(2\sqrt{at}; q \right); s \right\} \\
 &= \frac{\Gamma_q(\mu + \nu)(1 - q)^{\mu-\nu-1}}{\Gamma_q(2\nu + 1)K(s; \mu + \nu)} a^\nu s^{\nu+\mu-1} \\
 & \quad \times {}_1\Phi_1 \left[\begin{matrix} q^{\mu+\nu} \\ q^{2\nu+1} \end{matrix}; q, \frac{q^{2\nu+1}}{q^{\mu+\nu}} as \right], \\
 & \mathbb{S}_q \left\{ t^{\mu-1} q^\nu J_{2\nu}^{(3)} \left(\sqrt{q^{-1}at}; q \right); s \right\} \\
 &= \frac{\Gamma_q(\mu + \nu)(1 - q)^{\mu-\nu-1}}{\Gamma_q(2\nu + 1)K(s; \mu + \nu)} a^\nu s^{\nu+\mu-1} \\
 & \quad \times {}_2\Phi_1 \left[\begin{matrix} q^{\mu+\nu} & 0 \\ q^{2\nu+1} \end{matrix}; q, -\frac{as}{q^{\mu+\nu}} \right],
 \end{aligned} \tag{45}$$

where $\text{Re}(s) > 0$ and $\text{Re}(\mu + \nu) > 0$.

Corollary 4. In Corollary 3, if one writes $(\nu/2) + 1$ and $(\nu/2)$ instead of μ and ν , respectively, one obtains

$$\begin{aligned}
 & \mathbb{S}_q \left\{ t^{\nu/2} J_\nu^{(1)} \left(2\sqrt{at}; q \right); s \right\} = a^{\nu/2} s^\nu e_q(-as), \\
 & \mathbb{S}_q \left\{ t^{\nu/2} J_\nu^{(2)} \left(2\sqrt{at}; q \right); s \right\} = a^{\nu/2} s^\nu {}_0\Phi_1 \left[\begin{matrix} - \\ 0 \end{matrix}; q, -q^{\nu+1} as \right], \\
 & \mathbb{S}_q \left\{ (qt)^{\nu/2} J_\nu^{(3)} \left(\sqrt{q^{-1}at}; q \right); s \right\} = a^{\nu/2} s^\nu E_q(as), \\
 & \mathbb{S}_q \left\{ t^{\nu/2} J_\nu^{(2)} \left(2\sqrt{at}; q \right); s \right\} = \frac{a^{\nu/2} s^\nu}{K(s; \nu + 1)} E_q(as), \\
 & \mathbb{S}_q \left\{ (qt)^{\nu/2} J_\nu^{(3)} \left(\sqrt{q^{-1}at}; q \right); s \right\} = \frac{a^{\nu/2} s^\nu}{K(s; \nu + 1)} e_q \left(-\frac{as}{q^{\nu+1}} \right),
 \end{aligned} \tag{46}$$

where $\text{Re}(s) > 0$ and $\text{Re}(\nu + 1) > 0$.

Corollary 5. In Corollary 4, if one chooses $\nu = 1$, respectively, one gets

$$\begin{aligned}
 & \mathbb{S}_q \left\{ t^{1/2} J_1^{(1)} \left(2\sqrt{at}; q \right); s \right\} = a^{1/2} s e_q(-as), \\
 & \mathbb{S}_q \left\{ t^{\nu/2} J_1^{(2)} \left(2\sqrt{at}; q \right); s \right\} = a^{1/2} s {}_0\Phi_1 \left[\begin{matrix} - \\ 0 \end{matrix}; q, -q^2 as \right], \\
 & \mathbb{S}_q \left\{ (qt)^{1/2} J_1^{(3)} \left(\sqrt{q^{-1}at}; q \right); s \right\} = a^{1/2} s E_q(as), \\
 & \mathbb{S}_q \left\{ t^{1/2} J_1^{(2)} \left(2\sqrt{at}; q \right); s \right\} = q^{-1} a^{1/2} s E_q(as), \\
 & \mathbb{S}_q \left\{ (qt)^{1/2} J_1^{(3)} \left(\sqrt{q^{-1}at}; q \right); s \right\} = q^{-1} a^{1/2} s e_q(-q^{-2}as),
 \end{aligned} \tag{47}$$

where $\text{Re}(s) > 0$.

Corollary 6. Similarly, if one chooses $\nu = 0$ in Corollary 4, respectively, one gets

$$\begin{aligned}
 & \mathbb{S}_q \left\{ J_0^{(1)} \left(2\sqrt{at}; q \right); s \right\} = e_q(-as), \\
 & \mathbb{S}_q \left\{ J_0^{(2)} \left(2\sqrt{at}; q \right); s \right\} = {}_0\Phi_1 \left[\begin{matrix} - \\ 0 \end{matrix}; q, -qas \right], \\
 & \mathbb{S}_q \left\{ J_0^{(3)} \left(\sqrt{q^{-1}at}; q \right); s \right\} = E_q(as), \\
 & \mathbb{S}_q \left\{ J_0^{(2)} \left(2\sqrt{at}; q \right); s \right\} = E_q(as), \\
 & \mathbb{S}_q \left\{ J_0^{(3)} \left(\sqrt{q^{-1}at}; q \right); s \right\} = e_q(-q^{-1}as),
 \end{aligned} \tag{48}$$

where $\text{Re}(s) > 0$.

Corollary 7. In Corollary 3, if one writes $\nu = 0$ and then $a = 0$, one finds

$$\begin{aligned}
 & \mathbb{S}_q \left\{ t^{\mu-1}; s \right\} = \Gamma_q(\mu)(1 - q)^{\mu-1} s^{\mu-1}, \\
 & \mathbb{S}_q \left\{ t^{\mu-1}; s \right\} = \frac{\Gamma_q(\mu)(1 - q)^{\mu-1}}{K(s; \mu)} s^{\mu-1},
 \end{aligned} \tag{49}$$

where $\text{Re}(s) > 0$ and $\text{Re}(\mu) > 0$.

Remark 8. The results in Corollary 7 are previously obtained in [9, page 418, (2.5)-(2.6)].

5. Concluding Remarks

We conclude this investigation by remarking that q -Sumudu transforms of many other q -Bessel functions can be evaluated in this manner by applying the above theorems and their various corollaries and consequences considered here. Also, the results obtained here will be used in the forthcoming paper where some q -difference equations are solved.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

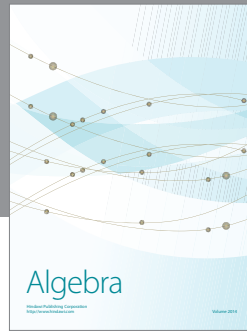
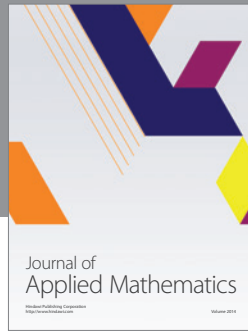
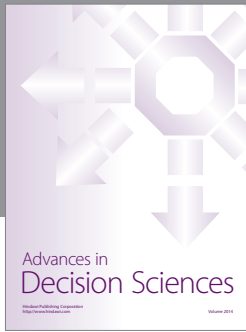
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